$h(a,b)$ is coeff of x^{2n+2b} in P_{2n+2a}.

Coeff. of highest power in $P_\nu(x)$ is $\dfrac{1.3\ldots(2\nu-1)}{\nu!}$

So in P_{2n+2a} is $\dfrac{1.3\ldots\ldots(4n+4a-1)}{(2n+2a)!}$

Denominator is always of form $h(c,c)$.

So $\dfrac{h(a,a)}{h(c,c)} = \dfrac{(4n+4c+1)\overset{(+2)}{\ldots}(4n+4a-1)}{(2n+2c+1)\underset{(+1)}{\ldots}(2n+2a)}$ if $a>c$

Coeff of $x^{\nu-2r}$ is highest coeff $\times$

$$(-1)^r \frac{\nu(\nu-1)\ldots(\nu-2r+1)}{2.4\ldots(2r)\cdot(2\nu-1)\ldots(2\nu-2r+1)}$$

Now, for $\dfrac{h(a,b)}{h(a,a)}$ we have $r = a-b$ $\nu = 2n+2a$.

So $$(-1)^{a-b}\frac{(2n+2a)(2n+2a-1)\ldots(2n+2b+1)}{2\ldots(2a-2b)\cdot(4n+4a-1)\underset{(-2)}{\ldots}(4n+2a+2b+1)}$$

To get $H(a,b,c)$ we multiply these together.

$H(1,0,0)$

$$= -\frac{(4n+1)(4n+3)}{(2n+1)(2n+2)}\cdot\frac{(2n+2)(2n+1)}{2\cdot(4n+3)}$$

$=$

Could be shortened.

C | Back

H.1 Back of C 2 Sat. Club

$$\frac{h(a,a)}{h(c,c)} = \frac{(4n+4c+1) \overset{(+2)}{\cdots} (4n+4a-1)}{(2n+2c+1) \overset{(+1)}{\cdots} (2n+2a)} \quad \text{if } a>c.$$

$$\frac{h(a,b)}{h(a,a)} = (-1)^{a-b} \frac{(2n+2a) \overset{(-1)}{\cdots} (2n+2b+1)}{2 \cdots (2a-2b) \cdot (4n+4a-1) \underset{(-2)}{\cdots} (4n+2a+2b+1)}$$

If $a+b>2c$, it will be possible to cancel all the factors of second denominator, leaving numerator

$$(4n+4c+1) \overset{(+2)}{\cdots} (4n+2a+2b-1)$$

Also, if $b>c$ (which $\Rightarrow$ earlier $a+b>2c$)

we have

$$(-1)^{a-b} \frac{(4n+4c+1) \overset{(+2)}{\cdots} (4n+2a+2b-1)}{2 \overset{(+2)}{\cdots} (2a-2b) \cdot (2n+2c+1) \overset{(+1)}{\cdots} (2n+2b)}$$

$$\sqrt{14} = 3 + (\sqrt{14} - 3)$$

C_3 back

$$\frac{1}{\sqrt{14}-3} = \frac{\sqrt{14}+3}{5} = 1 + \frac{\sqrt{14}-2}{5}$$

$$\cancel{\frac{5}{\sqrt{14}+3} = \frac{\sqrt{14}-3}{1}}$$

$$\frac{5}{\sqrt{14}-2} = \frac{\sqrt{14}+2}{2} = 2 + \frac{\sqrt{14}-2}{2}$$

$$\frac{2}{\sqrt{14}-2} = \frac{\sqrt{14}+2}{5} = 1 + \frac{\sqrt{14}-3}{5}$$

$$\frac{5}{\sqrt{14}-3} = \frac{\sqrt{14}+3}{1} = 6 + (\sqrt{14}-3)$$

$$\text{So } \sqrt{14} = 3 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{6+}\cdots$$

C_1 : $n=3$ gives

$$\begin{aligned}
\mathcal{E}_{03} \quad 0 &= m_{01}v_{12} + m_{02}v_{21} + m_{03}v_{30} - \lambda_2 \\
\mathcal{E}_{13} \quad 0 &= -v_{10}\lambda_3 + (m_{11}-\lambda_1)v_{12} + m_{12}v_{21} + m_{13}v_{30} - \lambda_2 v_{11} \\
\mathcal{E}_{23} \quad 0 &= -v_{20}\lambda_3 + m_{21}v_{12} + (m_{22}-\lambda_2)v_{21} + m_{23}v_{30} - \lambda_1 v_{22}
\end{aligned}$$

$$\therefore \frac{\lambda_3}{\begin{vmatrix} m_{01} & m_{02} & v_{30}m_{03} - \lambda_2 \\ m_{11}-\lambda_1 & m_{12} & v_{30}m_{13} - \lambda_2 v_{11} \\ m_{21} & m_{22}-\lambda_2 & v_{30}m_{23} - \lambda_1 v_{22} \end{vmatrix}} = \frac{1}{\begin{vmatrix} 0 & m_{01} & m_{02} \\ -v_{10} & m_{11}-\lambda_1 & m_{12} \\ -v_{20} & m_{21} & m_{22}-\lambda_2 \end{vmatrix}}$$

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May be Dad
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QUOTIENTS OF MOMENT FUNCTIONS.

W.W.Sawyer.

Abstract: A formula is found giving the quotient of two moment functions as a moment function.

1. Introduction. I have been trying to prove that L(s), the maximum eigenvalue of the integral equation

(1) $$L(s)\phi(x,s) = \int_0^1 \phi(y,s)\ (1-sxy)^{-1}\ dy$$

is a moment function of the parameter <u>s</u> in the sense that

(2) $$L(s) = \int_0^1 m(t)(1-st)^{-1}dt$$

with the weight distribution, m(t), non-negative. This stems from an earlier paper, (1) which considered an integral equation, differing from equation (1) above only in having the integration from -1 to 1. This paper contained the history of the problem, and presented a number of conjectures based on computer data. These conjectures are equally plausible for equation (1).

If the solution $\phi(x,s)$ is normalized so that $\phi(0,s)=1$ for all <u>s</u>, by putting s=0 in equation (1) we obtain

(3) $$L(s) = \int_0^1 \phi(y,s)\ dy.$$

If we can express $\phi(y,s)$ as a moment function with weight w(y,t), and if it is legitimate to reverse the order of integration, we shall be able to express L(s) as a moment function.

By a method involving iteration a sequence of functions, $\phi_n(x,s)$, can be found that increasingly approximate to a solution of equation (1). I have been able to express the earliest members, at any rate, of this sequence as moment functions. To obtain normalized functions it is necessary to consider $\phi_n(x,s)/\phi_n(0,s)$. It is then desirable to express this

2. A 4 x 6 Transportation problem is as follows:

$s_1 = 13$, $s_2 = 5$, $s_3 = 7$, $s_4 = 11$, Total 36

$D_1 = 3$, $D_2 = 7$, $D_3 = 10$, $D_4 = 5$, $D_5 = 5$, $D_6 = 6$, Total 36

(a) Find a feasible solution by N.W. Corner rule.
(b) What is its total cost?
(c) Can you find a better (lower cost) set of 9 basic shipments?
(d) Can you (i) calculate / (ii) guess the optimal solution?

N.B. Total cost of optimal solution = 125

Matrix of Unit Costs

8	5	7	3	3	6
5	6	3	2	5	4
2	4	5	6	4	3
5	3	6	7	8	4

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It is a great advantage to be able to formulate a problem in terms of the classical transportation model if possible, since computer programs for this case are usually very much faster than the corresponding one for general L.P. It has been estimated that about a third of all L.P. problems found in practice turn out to be of the transportation type.

Timings to compare with the above figures are given by a problem involving $M = 30$ warehouses and $N = 1200$ customers, with which I was concerned using the special C.E.I.R. TS/90 transportation program on the same computer. This took only 20 minutes to solve! (N.B. it corresponded to $M + N - 1 = 1229$ equations of L.P.).

It should be noted that Phase I can occupy a good deal of computer time in practice. Indeed, there are many problems in which the principal object is to find out whether the system is at all feasible, never mind optimal.

Illustrative Problems.

A. The system: Maximize $z = 3x_1 - 2x_2$

subject to $x_1 + x_2 \leq 1$

$2x_1 + 2x_2 \geq 4$

has no feasible solution, because constraints inconsistent.

B. The system: Maximize $z = x_1 + x_2$

subject to $x_1 - x_2 \geq 0$

$3x_1 - x_2 \leq -3$

has no feasible solution because of non-negative conditions.

C. The system: Maximize $z = 2x_1 + 2x_2$

subject to $x_1 - x_2 \geq -1$

$-.5x_1 + x_2 \leq 2$

is unbounded.

D. The system: Maximize $2x_2 - x_1 = z$

subject to $x_1 - x_2 \geq -1$

$-.5x_1 + x_2 \leq 2$

is bounded (z max = 4), but the values of x_1 and x_2 are unbounded.

Problems to try yourself.

1. Maximize $5x_1 + 3x_2$

subject to $3x_1 + 5x_2 \leq 15$

$5x_1 + 2x_2 \leq 10$

with $x_1, x_2 \geq 0$

Solve (a) graphically, (b) by Simplex method.

Thus we introduce y_1 into row (1) and y_2 into rwo (2), here. There is no need for an artificial variable in row (3), as the positive slack x_5 can be included in the initial basis. The successive canonical forms of the simplex steps are given in the following table in detached co-efficient form:-

Basis	x_1	x_2	x_3	x_4	x_5	y_1	y_2	b'	
y_1	1	3	-1			1		6	
y_2	2	1		-1			1	4	E
x_5	1	1			1			4	
w-10	-3	-4	1	1				w=10	
x_2	$\frac{1}{3}$	1	$-\frac{1}{3}$			$\frac{1}{3}$		2	
y_2	5/3		$\frac{1}{3}$	-1		$-\frac{1}{3}$	1	2	D
x_5	$\frac{2}{3}$		$\frac{1}{3}$		1	$-\frac{1}{3}$		2	
w-2	-5/3		$-\frac{1}{3}$	1		4/3		w=2	
x_2		1	-2/5	1/5		2/5	-1/5	8/5	
x_1	1		1/5	-3/5		-1/5	3/5	6/5	A
x_5			1/5	2/5	1	-1/5	-2/15	6/5	
w		End of Phase I				1	1	w=0	
$z + \frac{6}{5}$			1/5	-3/5				$z=\frac{-6}{5}$	
x_2		1	$-\frac{1}{2}$		$-\frac{1}{2}$			1	
x_1	1		$\frac{1}{2}$		3/2			3	B
x_4			$\frac{1}{2}$	1	5/2			3	
z			$\frac{1}{2}$		3/2			z=-3	

With reference to the above graph, we have gone E, D, A in Phase I and A, B in Phase II.

Timing on an Electronic Computer.

Generally speaking, only the number n of constraints affects the timing for most computer L.P. programs, and the number of activities n is usually irrelevant in this respect.

Very often, the timing turns out to be proportional to m^3, so that if for instance, on a fast computer, a 100-equation problem could be solved in two minutes, it would take 2×4^3 minutes, roughly, to solve a 400-equation problem on the same machine. (i.e. about 2 hours).

These times correspond in order of magnitude to what was realistically achievable when I was at C.E.I.R. by their LP/90 program.

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quotient of moment functions as a single moment function.

2. Theorem. If, for summable functions W(t) and w(t) we have

(4) $F(z) = \int_0^1 W(t)\ (z-t)^{-1}\ dt$

(5) $f(z) = \int_0^1 w(t)\ (z-t)^{-1}\ dt$

(6) $O(z) = F(z)/\{zf(z)\}$

then

(7)

Sines.

When we first learn trigonometry, we think of the angle θ as a simple idea, $\sin\theta$ and $\cos\theta$ as complicated functions of θ. Actually the situation is rather the reverse. We define x and y as the co-ordinates of the point on the unit circle at angle θ. θ is measured by the length of the arc ~~from (1,0)~~ to P, which involves integration.

$$ds^2 = dx^2 + dy^2, \text{ so } \frac{ds}{dx} = \sqrt{1+y'^2}.$$

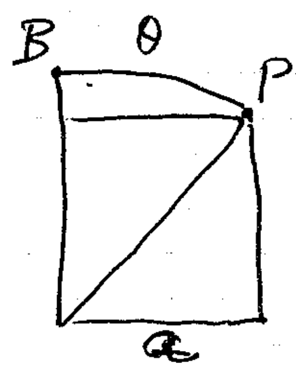
For a circle of unit radius $x^2+y^2=1$

Differentiating, $2x + 2yy' = 0$

$$\therefore \ y' = -\frac{x}{y}$$

$$\therefore \ \frac{ds}{dx} = \sqrt{1+\frac{x^2}{y^2}} = \sqrt{\frac{x^2+y^2}{y^2}} = \frac{1}{y} = \frac{1}{\sqrt{1-x^2}}$$

$$s = \int \frac{1}{\sqrt{1-x^2}}\,dx.$$



In the diagram, $a = \sin\theta$

$\theta = \sin^{-1}a$

and $\theta = \int_{x=0}^{x=a} \frac{1}{\sqrt{1-x^2}}\,dx$

Newton expanded $(1-x^2)^{-\frac{1}{2}}$ by the Binomial Theorem and thus arrived at an infinite series for θ. He managed to derive a series for $\sin\theta = a$ in terms of θ from this.

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Sequence sums.

r	t	Condition.	
2	$2a+1$	t odd	
3	$3a+3$	$t \equiv 0 \mod 3.$	$t \geqslant 3$
4	$4a+6$	$t \equiv 2 \mod 4$	$t \geqslant 6$
5	$5a+10$	$t \equiv 0 \mod 5$	$t \geqslant 10$
6	$6a+15$	$t \equiv 3 \mod 6$	$t \geqslant 15$
7	$7a+21$	$t \equiv 0 \mod 7$	$t \geqslant 21$
8	$8a+28$	$t \equiv 4 \mod 8$	$t \geqslant 28$
9	$9a+36$	$t \equiv 0 \mod 9$	$t \geqslant 36.$
10	$10a+45$	$t \equiv 5 \mod 10$	$t \geqslant 45.$

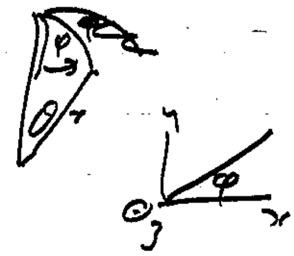
Angular momentum vectors in spherical polars.

C 49 back ①

$$\text{Momentum} = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},\ z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},\ x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$x = r\sin\theta\cos\varphi,\quad y = r\sin\theta\sin\varphi,\quad z = r\cos\theta$$

$$r = (x^2+y^2+z^2)^{1/2} \qquad \tan\varphi = \frac{y}{x} \qquad \varphi = \tan^{-1}\frac{y}{x}$$



$$\cos\theta = \frac{z}{r} = \frac{z}{\sqrt{x^2+y^2+z^2}}.$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi}$$

$$= \cancel{\frac{x}{r}\frac{\partial}{\partial r} - \frac{xz}{r^3}\frac{\partial}{\partial\theta}}$$

$$-\sin\theta\,\frac{\partial\theta}{\partial x} = -\frac{xz}{r^3} = -\frac{1}{r}\sin\theta\cos\varphi\cdot\cos\theta$$

$$\therefore\ \frac{\partial\theta}{\partial x} = +\frac{1}{r}\cos\varphi\cos\theta.$$

$$\varphi = \frac{\pi}{2} - \tan^{-1}\frac{x}{y} \qquad \frac{\partial\varphi}{\partial x} = -\frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{1}{y} = -\frac{y}{x^2+y^2}.$$

$$= -\frac{r\sin\theta\sin\varphi}{r^2(\sin^2\theta)} = -\frac{1}{r}\frac{\sin\varphi}{\sin\theta}.$$

$$\frac{\partial}{\partial x} = \sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}$$

$$-\sin\theta\frac{\partial\theta}{\partial y} = -\frac{z}{r^2}\cdot\frac{\partial r}{\partial y} = -\frac{yz}{r^3} = -\frac{1}{r}\sin\theta\sin\varphi\cos\theta$$

$$\frac{\partial\theta}{\partial y} = +\frac{1}{r}\cos\theta\sin\varphi. \qquad \frac{\partial\varphi}{\partial y} = \frac{x}{x^2+y^2} = \frac{1}{r}\frac{\sin\theta\cos\varphi}{\sin^2\theta}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}$$

<u>Check</u> $$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = 0\cdot\frac{\partial}{\partial r} + 0\cdot\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\left[r\sin\theta\cos^2\varphi + r\sin\theta\sin^2\varphi\right]\frac{\partial}{\partial\varphi}$$

$$= \frac{\partial}{\partial\varphi}\quad \text{as it should be}$$

C49 back

C 50 back

$$\psi = R(r)\, S(\theta, \varphi)$$

Take $m=1$, $\ell=1$, $n=1$.

$$P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) = x. = \cos\theta$$

$$P_{1,1} = (1-x^2)^{\frac{1}{2}}\left(\frac{d}{dx}\right)^1 P_1(x) = (1-x^2)^{1/2} = \sin\theta$$

$$S = e^{i\varphi}\sin\theta$$

$$R(r) = r\, e^{-r/2} L_2^3(r).$$

$$L_2^3(x) = \left(\frac{d}{dx}\right)^3 L_2(x)$$

$$L_2(x) = e^x\left(\frac{d}{dx}\right)^2 e^{-x}x^2$$

$$= (D-2)^2x^2 = (D^2-4D+4)x^2$$

$$= 4x^2 - 8x + 2.$$

So $L_2^3 = 0$.

In fact $n \geq \ell+1$ is required.

Try $m=1$, $\ell=2$, $n=4$.

$$P_{\ell,m}(x) = P_{21}(x) = (1-x^2)^{\frac{1}{2}}\left(\frac{d}{dx}\right)P_2(x)$$

$$= (1-x^2)^{\frac{1}{2}}\frac{d}{dx}\cdot\frac{3x^2-1}{2} = (1-x^2)^{\frac{1}{2}}\cdot 3x.$$

Thus $\underline{e^{i\varphi}\sin\theta\cos\theta.}$

C57 Back

What happens when spherically sym field is not Coulomb?

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) - \frac{\ell(\ell+1)}{r^2}f_\ell + \left(E + ar^{-\sigma}\right)f = 0$$

$$r^2 f'' + 2rf' + \left(Er^2 - \ell(\ell+1) + ar^{2-\sigma}\right)f = 0.$$

Try $\sigma = 2$, $a>0$

$$r^2 f'' + 2rf' + \left(Er^2 - \ell(\ell+1) + a\right)f = 0$$

$\cancel{rf' = \gamma r^2 f}$ If $r^2 f'' \sim Er^2 f$ $f'' \sim Ef$.

Let $f = e^{-\gamma r} v$. $f' = e^{-\gamma r}v' - \gamma e^{-\gamma r} v$

$f'' = e^{-\gamma r}v'' - 2e^{-\gamma r}\gamma v' + \gamma^2 e^{-\gamma r} v$

So

$$\left.\begin{aligned} 0 = r^2 v'' &- 2\gamma r^2 v' + \gamma^2 r^2 v \\ &+ 2rv' - 2r\gamma v \\ &+ v\left(Er^2 - \ell(\ell+1) + a\right)\end{aligned}\right\} = 0$$

Choose γ so that $E + \gamma^2 = 0$

$$0 = r^2 v'' + v'\left(2r - 2\gamma r^2\right) + v\left(+a - \ell(\ell+1) - 2r\gamma\right)$$

weights: 0 0 +1 0 +1

Indicial eqn $0 = \rho(\rho-1) + 2\rho - \ell(\ell+1) + a$

$= \cancel{\rho(\rho+1) - \ell(\ell+1)}$.

~~Take value $\rho = \ell$.~~ $\cancel{v = r^\ell u.\ v' = r^\ell u' + \ell r^{\ell-1}u}$

$v'' = r^\ell u'' + 2\ell r^{\ell-1}u' + \ell(\ell-1)r^{\ell-2}u$

$$\begin{aligned} 0 = {} & r^{\ell+2}u'' + \cancel{2\ell r^{\ell+1}u' + \ell(\ell-1)r^\ell u} \\ & + 2r^{\ell+1}u' + \cancel{2\ell r^\ell u} \\ & - 2\gamma r^{\ell+2}u' - 2\gamma\ell r^{\ell+1}u \\ & + r^\ell\,\cancel{\left[a - \ell(\ell+1)\right]}\end{aligned}$$

$$0 = \rho^2 + \rho - (\ell^2 + \ell + a)$$

$$\rho = \frac{-1 \pm \sqrt{1 + 4(\ell^2 + \ell - a)}}{2}.$$

Let ρ denote ~~the~~ this exp^n with + sign

back of C59

June - July 87 They raised question of calculation of log tables. Discussed log (1) via e^x (2) by def. $\ln a = \int_1^a \frac{dx}{x}$. Integral test for convergence. Euler's formula for m! from $\lim_{n\to\infty} \int_0^n x^m (1-\frac{x}{n})^n dx$.

Aug. Theory of vibrating piano-string. Soln $f(x-ct)+F(x+ct)$
Theory of pendulum for large oscillations.

Sept. 1987. D.E for falling raindrop. Electrical analogy. Electrical oscillations:

October. DES with constant coeffs
Circuit theory using complex numbers, $j\omega L$ for inductance etc.

November. Review of calculus. Chain rule etc: differentiation of inverse functions.

Jan. 1988. Coordinate geometry of conics. Diameter of parabola, ellipse. Polar co-ordinates for ellipse.
Proof that planets describe ellipses
Motion on a vertical circle.

Feb. Max and min in 2 dimensions

April Calculation of $\int_{-\infty}^{\infty} e^{-x^2} dx$ by via double integral.

Surfaces for which $\frac{df}{dt}$ is not $\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.

~~Change of~~ Partial diffn, change to polars.

Fi $\frac{\partial u}{\partial v}\frac{\partial v}{\partial w}\frac{\partial w}{\partial u} = -1$.

Power Series found by formation of differential equation

May.

back of C 60

Coefficients of Legendre Polynomials on (0,1), $L_n(x)$.

$$L_n(x) = \sum_{r=0}^{n} C_{nr} x^r \qquad C_{n0} = 1$$

$$C_{nr} = -C_{n,r-1}(n-r+1)(n+r)/r^2.$$

Integer arithmetic?

~~DIM C(60,60)~~

$$C_{nr} = (-1)^r \frac{(n-r+1)\ldots(n-1)\,n\,(n+1)(n+2)\ldots(n+r)}{(r!)^2}$$

$$= (-1)^r \frac{(n+r)!}{(n-r)!\,r!\,r!}$$

This is number of ~~choices for n+r things~~, n+r things, of which n-r are identical, also r and r sets of identical objects, hence always whole number. Use integer arithmetic.

```
DIM C(60,60)
FOR N = 0 TO 60
P. "M=";: INPUT M
DIM C#(M,M)
FOR N=0 TO M : C#(N,0)=1 : NEXT.
 N=1
 REPEAT
 FOR R=1 TO N
 C#(N,R) = -(N-R+1)(N+R) C#(N,R-1)/R/R.
 NEXT
 N=N+1
 UNTIL N>M
 END.
```

Choose M large, and see at what stage coeffs are too large. Then choose M below this.

back of C 68

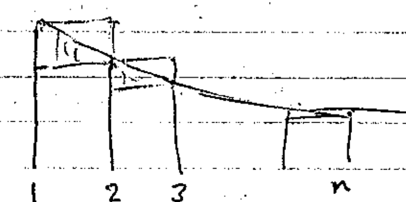
$$\sum_{r=1}^{n-1} \frac{1}{(n+\frac{1}{2}-r)\sqrt{r}}$$

$$\sum_{r=1}^{n-1} \frac{1}{(n+\frac{1}{2}-r)\,r} = \sum_{r=1}^{n-1}\left(\frac{\frac{1}{n+\frac{1}{2}}}{r} + \frac{\frac{1}{n+\frac{1}{2}}}{n+\frac{1}{2}-r}\right)$$

$$= \frac{1}{n+\frac{1}{2}} \sum_{r=1}^{n-1} \left(\frac{1}{r} + \frac{1}{n+\frac{1}{2}-r}\right)$$

$$= \frac{1}{n+\frac{1}{2}}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}$$
$$\frac{1}{1\frac{1}{2}} + \frac{1}{2\frac{1}{2}} + \cdots + \frac{1}{n-\frac{1}{2}}\Big)$$

$$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} < \int_1^n \frac{1}{x}\,dx$$
$$= \log n.$$



$\log n - \left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$ = sum of "triangles" This fn ↑ with n.

Sum of "triangles" < sum of rectangles < 1.

$\therefore$ $\log n - \sum_2^n \frac{1}{x} \to$ limit ≤ 1

$\log n - \sum_1^n \frac{1}{x} \to$ limit ≤ 0.

so $\gamma = \lim \sum_1^n \frac{1}{x} - \log n$

$$\Psi(x+h) = \log x - \sum_{\nu=1}^{m} (-1)^\nu \frac{B_\nu(h)}{\nu x^\nu} + \int_0^\infty \frac{\bar{B}_m(z-h)\,dz}{(x+z)^{m+1}}$$

$m=1$ $$\Psi(x+h) = \log x + \frac{B_1(h)}{x} + \int_0^\infty \frac{\bar{B}_1(z-h)\,dz}{(x+z)^2}$$

$B_1(x) = x - \frac{1}{2}$ $\quad \bar{B}_1(z)$ is periodic

$$\int_k^{k+1} \frac{\bar{B}_1(z)\,dz}{(x+z)^2} = \int_0^1 \frac{B_1(z)\,dz}{(x+z-k)^2}$$

$$= \int_0^1 \frac{z-\frac{1}{2}}{(z+x-k)^2}\,dx$$

$$z - \tfrac{1}{2} = (z+x-k) + k - x - \tfrac{1}{2}$$

$$\int_0^1 \frac{dz}{z+x-k} + \int_0^1 \frac{k-x-\frac{1}{2}}{(z+x-k)^2}\,dz$$

$$= \log(z+x-k)\Big]_0^1 + \frac{(x+\frac{1}{2}-k)}{z+x-k}\Big]_0^1 = \log(x+1-k) - \log(x-k)$$
$$+ (x+\tfrac{1}{2}-k)\left[\frac{1}{x+1-k} - \frac{1}{x-k}\right]$$

The Phase I Procedure (Illustrated by Problem C).

Unlike the transportation problem, or the case when all the slacks are positive, an initial feasible solution may not be available immediately in certain problems. Indeed, a feasible solution may not exist. i.e. it is impossible to satisfy the constraints with non-negative values of the activities.

Consider the following system:-

Maximize x_1

subject to

$$\begin{aligned} x_1 + 3x_2 &\geq 6 \\ 2x_1 + x_2 &\geq 4 \\ x_1 + x_2 &\leq 4 \end{aligned}$$

and $x_1, x_2 \geq 0$.

When put into standard form, this becomes

Minimize $z = -x_1$ (0)

subject to

$$x_1 + 3x_2 - x_3 = 6 \qquad (1)$$

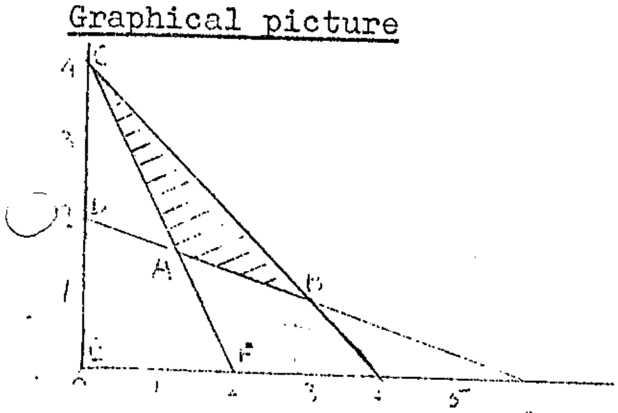
$$2x_1 + x_2 - x_4 = 4 \qquad (2)$$

$$x_1 + x_2 + x_5 = 4 \qquad (3)$$

and $x_j \geq 0 \quad j = 1$ to 5.

x_5, being a positive slack can go into the initial basis, but not so x_3 and x_4 which are negative slacks.

Graphical picture



z max = 3 at B(3, 1)

Introduction of artifical variables

Artifical variables y_1, y_2 are introduced into the rows that need them to give an initial basis.

A new objective function $w = y_1 + y_2 + \ldots\ldots$ is introduced and this is minimized (as for Phase II with z). Whilst w is positive, Phase I lasts, until finally Phase I ends with w = 0 and each artificial variable has been eliminated from the basis (i.e. has become non-basic). Unlike what happened to x_6 in Problem B, once an artificial is removed from the basis it cannot return. If w cannot be reduced to zero, the problem is infeasible.

back C75 6.

Problem B.

Consider the following problem in 2 variables:-

Maximize $z = 2x_1 + x_2$, (1)

$$\left.\begin{aligned} x_1 + 2x_2 &\leqslant 10 \\ x_1 + x_2 &\leqslant 6 \\ x_1 - x_2 &\leqslant 2 \\ x_1 - 2x_2 &\leqslant 1 \end{aligned}\right\} \qquad (2)$$

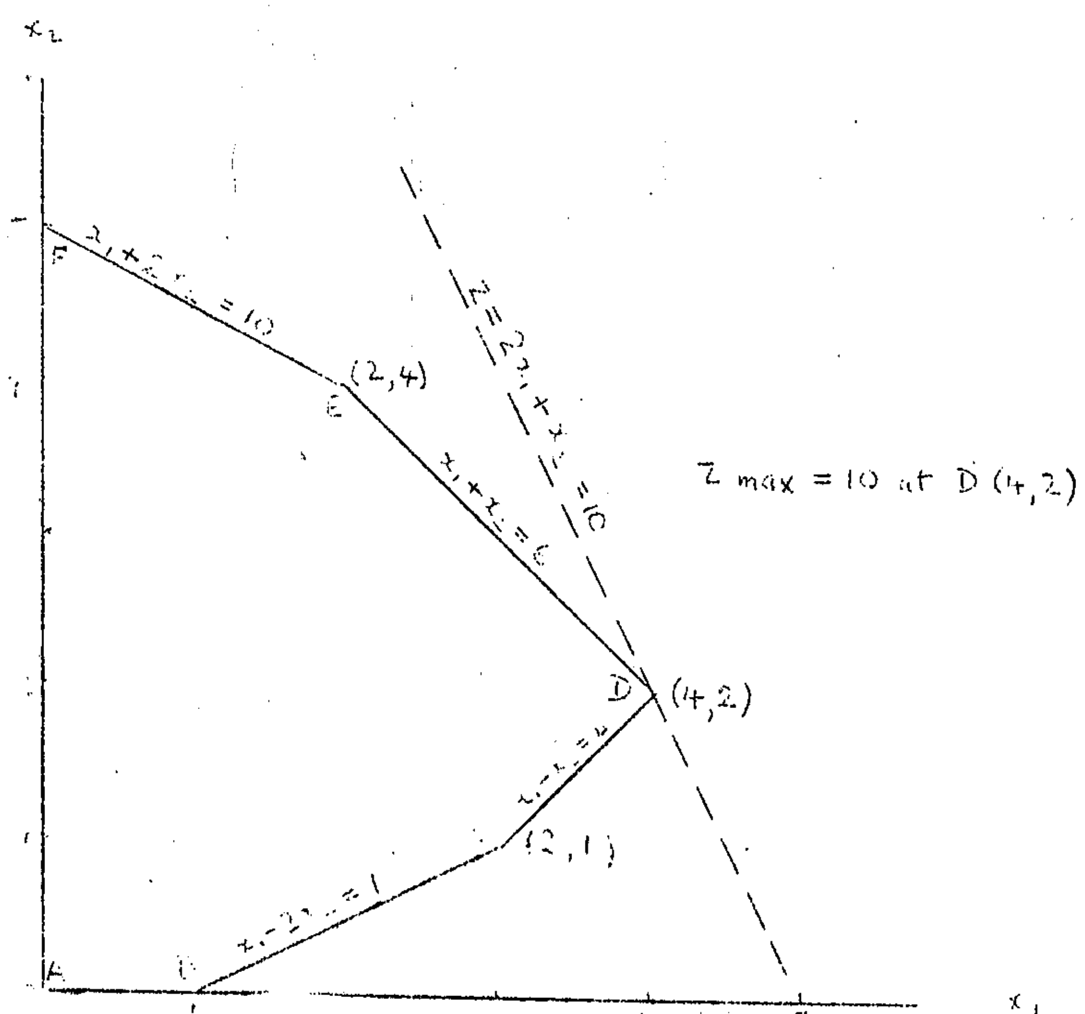
and $x_1, x_2 \geqslant 0$ (3)

It appears that here we have $m > n$ (i.e. $m = 4$, $n = 2$); but when we put in the four positive slacks x_3, x_4, x_5, x_6, we have $m = 4$, $n = 6$ with $m < n$, as it should be.

Graphical Interpretation of equations(2).

If we plot x_1 against x_2 on a graph, we need only concern ourselves with the positive quadrant, as $x_1 \geqslant 0$, $x_2 \geqslant 0$.

Also the constraints (2) above form four straight line boundaries:-



Types of Solution

1. A feasible solution.

Any set of values of the x_j which satisfy the constraints (2) and non-negative conditions (3). (e.g. in Problem B, any point (x_1, x_2) in or on polygon ABCDEF).

2. A basic solution.

A solution of equations (2) obtained by letting n - m of the variables be zero, and solving for the remaining m variables. The m chosen variables constitute what is called the basis.

A basic feasible solution is defined as a basic solution of equations (2), which also satisfies conditions (3).

For example, in Problem B, when the slacks are added, equations (2) become:-

$$\begin{aligned} x_1 + 2x_2 + x_3 \qquad\qquad &= 10 \\ x_1 + x_2 \qquad + x_4 \qquad &= 6 \\ x_1 - x_2 \qquad\qquad + x_5 \quad &= 2 \\ x_1 - 2x_2 \qquad\qquad\qquad + x_6 &= 1 \end{aligned} \qquad (2)$$

Here we can set 2 out of the 6 variables zero and solve for the other 4 in ${}^6C_2 = 15$ different ways, viz:-

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
x_1	0	0	0	0	0	10	6	2	1	2	14/3	11/2	4	13/3	3
x_2	0	5	6	-2	$-\frac{1}{2}$	0	0	0	0	4	8/3	9/4	2	5/3	1
x_3	10	0	-2	14	11	0	4	8	9	0	0	0	2	7/3	5
x_4	6	1	0	8	$6\frac{1}{2}$	-4	0	2	5	0	-4/3	-7/4	0	0	2
x_5	2	7	-4	0	$2\frac{1}{2}$	-8	-4	0	1	4	0	-5/4	0	-2/3	0
x_6	1	11	-11	-3	0	-9	-5	-1	0	7	5/3	0	1	0	0
	A	F							B	E			D		C

It is seen that nine out of the 15 basic solutions are non-feasible as they violate the non-negative conditions. (The 6 feasibles are marked A to F in table).

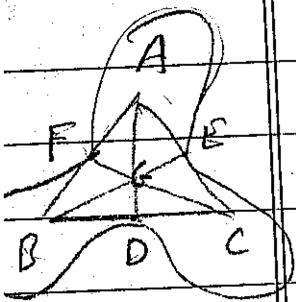
Also the 6 solutions which are feasible, correspond to the corners ABCDEF of the polygon (shaded on graph). This is a property of basic feasible solutions.

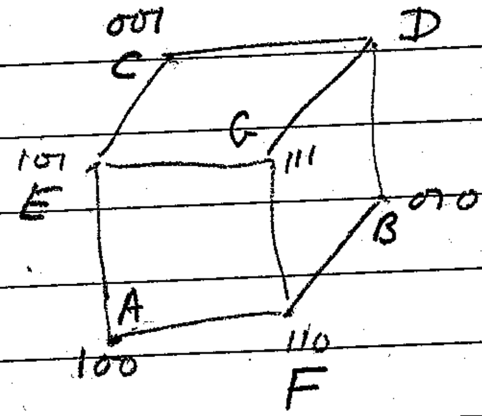
With 2 variables, we always get a convex polygon like ABCDEF;

With 3 variables, we always get a convex polygon, with basic solutions at corners;

Back of C85

? orientation. Some kind of network topology.





ABCEGFD.

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ABCE	E is 3rd pt of:	AC
BCEG	G – – –	BE
CEGF	F – –	CG

and so on.

TOPICS.

Lockstep. Grades 5-8. Wagon Wheels. Affects out-of-school work "enrichment" vs "acceleration". Taking wake off. Forbidding advanced work is capitulating to lockstep view. Create situation in which great variety of pace — some bright pupils fear to become conspicuous.
SMILE — devoted work. Need to go on to daily work with systematic textbook.
Occasional course for gifted — scale altogether wrong.

The escape from bad teaching
Gordon Thomas & SJ JD — going back to start
By meeting a good teacher.

back of C105

I.2.§9. $Q(z) = \sum_{-\infty}^{\infty} c_n z^n$. $P(z) = \sum_{0}^{\infty} c_n z^n$. $P_1(\frac{1}{z}) = \sum_{1}^{\infty} \frac{c_{-n}}{z^n}$

$P(z)$ cgt for $|z| < r$, $P_1(\frac{1}{z})$ for $|z| > r_1$. If $r > r_1$ we have a convergence ring of $Q(z)$, Laurent series.

$r_1 < \rho < r$. $M = \max |Q(z)|$ for $|z| = \rho$.

Then $|c_n| \rho^n \leq M$.

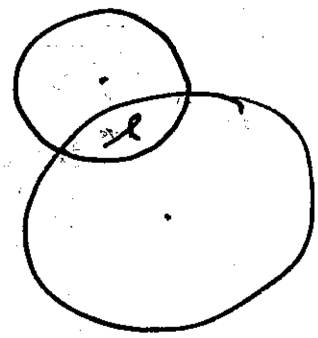
CHAPTER 3.

I.3§1. A collection of power series is called a monogenic system if 1) each power series is a continuation of any other, (2) every "remodelling" of any power series belongs to the system.

§2. $f(z)$ defined as follows. For z_0 we consider the power series of the system that have z_0 inside circle of convergence. We define $f(z_0)$ as

Back C 106

Direct continuation.

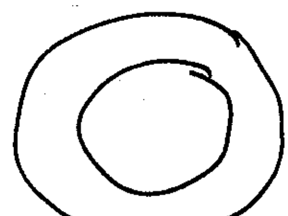


$\mathcal{P}(z|a)$ and $\mathcal{P}_1(z|a_1)$
and a region $\mathcal{S}$ lies inside both circles of convergence.
If $\mathcal{P}(z|a) = \mathcal{P}_1(z|a_1)$
for $z = b_1, b_2, b_3 \dots$ (points in $\mathcal{S}$)
where $b_r \to b$ b in $\mathcal{S}$.
then the series are equal for every z in $\mathcal{S}$.

I. 2. §9. Laurent series $\mathcal{Q}(z) = \sum_{-\infty}^{\infty} c_n z^n$

$\mathcal{P}(z) = \sum_0^{\infty} c_n z^n \quad \mathcal{P}_1(\frac{1}{z}) = \sum_1^{\infty} \frac{c_{-n}}{z^n}$

$\mathcal{P}(z)$ cgt for $|z| < r$
$\mathcal{P}_1(\frac{1}{z})$ cgt $|z| > r_1$.



Convergence ring if $r > r_1$
Suppose $r_1 < \rho < r$
$M = \max \mathcal{Q}(z)$ on the circle with $|z| = \rho$
Then $|c_n| \rho^n \le M \quad -\infty < n < +\infty$.

[We have met this from Cauchy $c_n = \frac{1}{2\pi i}\int \frac{f(z)\,dz}{z^{n+1}}$]

Chapter 3 The concept of an analytic function.
§1 Monogenic system. A system of power series :-
(1) any power series in it is a continuation of any other in it.
(2) any remodelling of a power series in it belongs to the system.
§2. We have a monogenic system. We define a function possibly "many valued" as follows. For a point z_0 we consider those power series in the system for which z_0 lies inside the circle of convergence. We associate with z_0 the value(s) given by these. We can define this: let $\mathcal{P}(z|z_0)$ belong to the system and be $\sum_0^{\infty} c_n (z-z_0)^n$. We take c_0 as a value of $f(z_0)$.

Back C107

$$f(z) = \sum_{0}^{\infty} a_n z^n$$

$$a_n = \frac{1}{2\pi i} \oint \frac{f(z)\,dz}{z^{n+1}}$$

Take C as circle of radius r

$$M = \max |f(z)|$$

$$|a_n| \leq \frac{1}{2\pi} \oint \left| \frac{f(z)\,dz}{z^{n+1}} \right|$$

$$= \frac{1}{2\pi} \oint \frac{|f(z)||dz|}{|z|^{n+1}}$$

$$\leq \frac{1}{2\pi} \frac{M \cdot 2\pi r}{r^{n+1}}$$

$$= \frac{M}{r^n}$$

$$\therefore \quad \underline{|a_n| r^n \leq M}$$

Back
D14

x^2 divided by $x-4$ is $x+4$ with remainder 16

$$\begin{array}{r} x+4 \\ x-4 \,\big)\, x^2 \qquad \\ \underline{x^2-4x} \\ 4x \qquad \\ \underline{4x-16} \\ 16 \end{array}$$

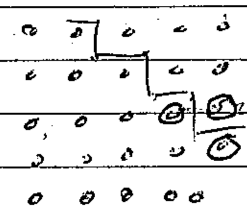
Put $x=5$: 25 divided by 1 is 9 with remainder 16.

Back
D 22

$Z = 5.\quad \frac{1}{2}(Z-2)(Z-3) = \frac{1}{2}\cdot 3\cdot 2 = 3.$

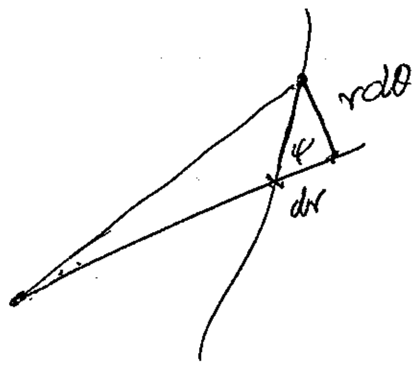
Take $m_{ij} = 1/i+j-1$

$$\begin{pmatrix} a & p & & & \\ p & b & q & & \\ & q & c & r & \\ & & r & d & s \\ & & & s & e \end{pmatrix}\begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & \frac{1}{5} \\ 1/2 & 1/3 & 1/4 & \frac{1}{5} & \frac{1}{6} \\ 1/3 & 1/4 & 1/5 & \frac{1}{6} & \frac{1}{7} \\ 1/4 & 1/5 & 1/6 & 1/7 & 1/8 \\ 1/8 & 1/6 & 1/7 & 1/8 & 1/9 \end{pmatrix}$$



	a	b	c	d	e	p	q	r	s
(12)	$-1/2$	$1/2$	0	0	0	$2/3$	$1/3$	0	0
* (13)	$-1/3$	0	$1/3$	0	0	$-1/4$	$1/2$	$1/4$	0
* (14)	$-1/4$	0	0	$1/4$	0	$-1/5$	0	$1/3$	$\frac{1}{5}$
* (15)	$-1/5$	0	0	0	$1/5$	$-1/6$	0	0	$1/6$
(23)	0	$-1/4$	$1/4$	0	0	$-1/3$	$2/15$	$1/5$	0
(24)	0	$-1/5$	0	$1/5$	0	$-1/4$	$-1/6$	$1/4$	$1/6$
(25)	0	$-1/6$	0	0	$1/6$	$-1/5$	$-1/7$	0	$1/5$
→ (34)	0	0	$-1/6$	$1/6$	0	0	$-1/5$	$2/35$	$1/7$
(35)	0	0	$-1/7$	0	$1/7$	0	$-1/6$	$-1/8$	$1/6$
(45)	0	0	0	$-1/8$	$1/8$	0	0	$-1/7$	$2/63$

Back D27



Polar coordinates

$$\tan\psi = r\frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}}$$

If $r = ae^{b\theta}$

$$\frac{dr}{d\theta} = bae^{b\theta}$$

$$\tan\psi = \frac{1}{b}.$$

$\boxed{\text{Back D 30}}$ $\mu = 1/r$

$$L = \frac{1}{r}\sqrt{\dot r^2 + r^2\dot\theta^2}$$

$$\frac{d}{dt}\left[\frac{\dot r}{r\sqrt{\dot r^2 + r^2\dot\theta^2}}\right] = -\frac{1}{r^2}\sqrt{\dot r^2 + r^2\dot\theta^2} + \frac{1}{r}\frac{r\dot\theta^2}{\sqrt{\dot r^2 + r^2\dot\theta^2}}$$

$$= \frac{-\dot r^2 - r^2\dot\theta^2 + r^2\dot\theta^2}{r^2\sqrt{\dot r^2 + r^2\dot\theta^2}} = \frac{-\dot r^2}{r^2\sqrt{\dot r^2 + r^2\dot\theta^2}}$$

If $r = e^{b\theta}$ $dr/d\theta = be^{b\theta} = br$.

$$-\frac{(dr/d\theta)^2\,\dot\theta}{r^2\sqrt{(dr/d\theta)^2 + r^2}} = \frac{-b^2r^2\,\dot\theta}{r^2\sqrt{b^2r^2 + r^2}} = \frac{-b^2\dot\theta}{r\sqrt{b^2+1}}$$

$$\frac{\dot r}{r\sqrt{\dot r^2 + r^2\dot\theta^2}} = \frac{dr/d\theta}{r\sqrt{(dr/d\theta)^2 + r^2}} = \frac{br}{r^2\sqrt{b^2+1}} = \frac{b}{r\sqrt{b^2+1}}$$

$$\frac{d}{dt}\frac{b}{r\sqrt{b^2+1}} = \frac{b}{\sqrt{b^2+1}}\,\frac{-dr/dt}{r^2}$$

$$= -\frac{b\dot\theta}{\sqrt{b^2+1}}\,\frac{dr/d\theta}{r^2} = \frac{-b\dot\theta}{\sqrt{b^2+1}}\cdot\frac{b}{r} \checkmark$$

So PE is correct.

Back of D 31

$$\cancel{\frac{A}{1-\sigma a^2}} \quad LF(x,\sigma) = \int \frac{1}{1-\sigma x y} F(y,\sigma)\, dw(y)$$

Weight $\frac{1}{2}$ at a, $\frac{1}{2}$ at b.

$$LF(x,\sigma) = \frac{\frac{1}{2}F(a,\sigma)}{1-\sigma a x} + \frac{\frac{1}{2}F(b,\sigma)}{1-\sigma b x}$$

$$\therefore F(x,\sigma) = \frac{A(\sigma)}{1-\sigma a x} + \frac{B(\sigma)}{1-\sigma b x}$$

$$F(a,\sigma) = \frac{A(\sigma)}{1-\sigma a^2} + \frac{B(\sigma)}{1-\sigma ab}$$

$$F(b,\sigma) = \frac{A(\sigma)}{1-\sigma ab} + \frac{B(\sigma)}{1-\sigma b^2}$$

Comparing $1-\sigma a x$ part

$$LA(\sigma) = \frac{\frac{1}{2}A(\sigma)}{1-\sigma a^2} + \frac{\frac{1}{2}B(\sigma)}{1-\sigma ab}$$

$$LB(\sigma) = \frac{\frac{1}{2}A(\sigma)}{1-\sigma ab} + \frac{\frac{1}{2}B(\sigma)}{1-\sigma b^2}$$

Let $\mu = 2L$.

$$\mu^2 - \mu\left[\frac{1}{1-\sigma a^2} + \frac{1}{1-\sigma b^2}\right] + \frac{1}{(1-\sigma a^2)(1-\sigma b^2)} - \frac{1}{(1-\sigma ab)^2} = 0$$

$$\Delta = \left[\frac{1}{1-\sigma a^2} - \frac{1}{1-\sigma b^2}\right]^2 + \frac{4}{(1-\sigma ab)^2}$$

$\Delta = 0$ if

$$\frac{1}{1-\sigma a^2} - \frac{1}{1-\sigma b^2} = \frac{2i}{1-\sigma ab}$$

$$\frac{\sigma(a^2-b^2)}{(1-\sigma a^2)(1-\sigma b^2)} = \frac{2i}{1-\sigma ab}$$

$$\sigma(a^2-b^2) - \sigma^2 ab(a^2-b^2) = 2i\left[1-\sigma(a^2+b^2)+\sigma^2a^2b^2\right]$$

$$0 = 2i + \sigma(b^2-a^2-2ia^2-2ib^2) + \sigma^2 ab(a^2-b^2+2iab)$$
$$= 2i + \sigma(b^2-a^2-2ia^2-2ib^2) + \sigma^2 ab(a+ib)^2$$

$$B^2 - 4AC = (b^2-a^2-2ia^2-2ib^2)^2 - 8i\,ab(a+ib)^2$$

$$[b^2(1-2i) - a^2(1+2i)]^2 = b^4(-3-4i) - 10a^2b^2 + a^4(-3+4i)$$
$$+8iab^3 \quad -8ia^3b$$
$$+16a^2b^2$$

$1-4-4i$

$\frac{1}{-3+4i} = \frac{-3-4i}{25}$

$$= b^4(-3-4i) + 8iab^3 + 6a^2b^2 - 8ia^3b + a^4(-3+4i)$$

$a=1 \quad b=.9$

$$\frac{\sigma(.19)}{(1-\sigma)(1-.81\sigma)} = \frac{2i}{1-\sigma(.9)}$$

$$.19\sigma - \sigma^2 .171 = 2i(1-1.81\sigma+.81\sigma^2)$$
$$= 2i - 3.62i\sigma + 1.62i\sigma^2$$

$$\sigma^2(.171+1.62i) - (.19+3.62i)\sigma + 2i = 0$$

$$\sigma = \frac{.19+3.62i \pm\sqrt{(.19+3.62i)^2 - 8i(.171+1.62i)}}{2(.171+1.62i)}$$

Back of E11

$\tau = ct,\quad \tau' = ct' = \beta\left(ct - \frac{v}{c}x\right) = \beta\left(\tau - \frac{v}{c}x\right)$

$x' = \beta(x - vt) = \beta\left(x - \frac{v}{c}\tau\right)$

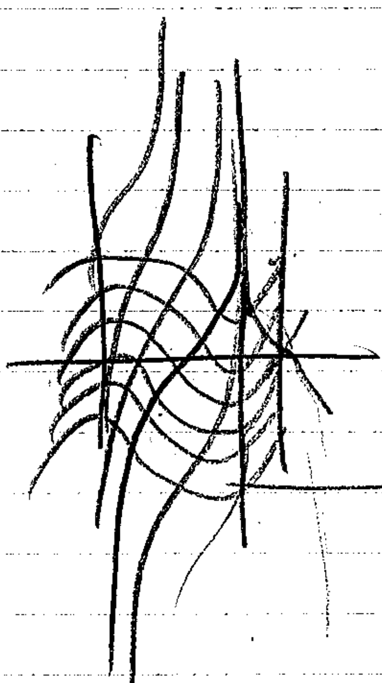
Take $c = 1$. $\quad x' = \dfrac{x - v\tau}{\sqrt{1-v^2}} \quad \tau' = \dfrac{\tau - vx}{\sqrt{1-v^2}}$

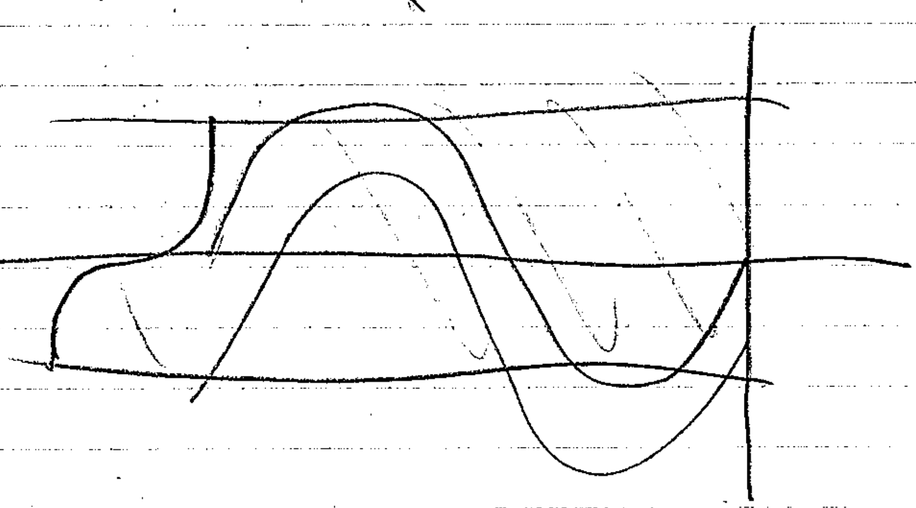
$x' + v\tau' =$

$y = \sin(x+2)$

$y' = \cos x$

$y'' = -\cos x$





Back of E27

```
5 T%=
9PRIN
10PRIN
20PRIN
30PRIN
40DIM
50PROC
90*KEY
100FOR
110PROC
120NEXT
990 END
1000DEFP
1010 FOR
1020 FOR
1030 A(0
1040NEXT
1050NEXT
1060A(0,
1070ENDP
2000 DEF
2010LOCA
2020FOR
2030FOR
2040SS=0
2050LL=0
2060REPE
2070SS=S
2080LL=L
2090UNTI
2095A(G+
2100NEXT
2110NEXT
2130ENDP
4000DEFP
4010LOCA
4020FOR
4030FOR
4040PRIN
4050NEXT
4060NEXT
4070 END
>
```

Back E25

$$\left(\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\frac{x^4}{5}+\frac{x^5}{6}+\frac{x^6}{7}+\cdots\right)^2$$

$$=\frac{x^2}{4}+\frac{x^3}{3}+x^4\left(\frac{1}{9}+\frac{1}{4}\right)+x^5\left(\frac{1}{5}+\frac{1}{6}\right)+x^6\left(\frac{1}{16}+\frac{2}{15}+\frac{1}{6}\right)+$$

$$x^7\left(\frac{1}{10}+\frac{1}{9}+\frac{1}{7}\right)$$

$\underline{a}$	5	$\underline{b}$	3	12
9	$\underline{c}$	$\underline{d}$	14	0
$\underline{f}$	$\underline{9}$	9	4	7
8	7	4	7	19
6	8	4	20	7

$a+b=25$

$c+d=25$

$f+9=$

Book of G4

played a decisive role. In analysis we are much concerned with questions about limits. For both real and complex numbers, a sequence of numbers z_n is tending to a limit L if the distance of z_n from L is tending to zero. He decided that, if it was possible to find a satisfactory definition of distance between two mathematical objects (of any kind), it would be possible to find theorems about these objects analogous to the theorems about real and complex numbers.

The first question then is - what is a satisfactory definition of distance? He looked at the traditional proofs and found the only properties of distance used were the following very simple ones;-

1. Distance is measured by a real number, which is never negative.
2. A distance is zero if, and only if, it is the distance between a point and itself.
3. The distance from A to B is the same as the distance from B to A.
4. You cannot shorten your journey by breaking it. If you go from A to C, and then from C to B, the total distance cannot be less than the distance from A to B. (It may of course be equal, if C lies on the direct route from A to B.) This is known as the triangle axiom. It corresponds to Euclid's remark, that the sum of the lengths of two sides of a triangle must exceed the length of the third side.

Frechet's investigation was extraordinarily fruitful. It was found possible to find a satisfactory definition for the distance between two matrices, two transformations, two functions, two operations that may involve differentiation and integration. At one blow, this opens the door to a whole series of results concerning the most varied situations.

It is often possible to find more than one definition of distance for given objects. For instance, on a chessboard we can define distance as the minimum number of moves a king needs to get from one square to another. We get a different definition if we consider a rook instead of a king. Options can equally well arise in more serious mathematical contexts.

The distance from A to B is the length of AB. We now consider defining length. Of the many possible definitions we shall here consider only those that lead to our usual geometry, or to a geometry very similar to it. In all the spaces now to be listed, Pythagoras Theorem is true in some sense. All these spaces are vector spaces.

The symbol $||\mathbf{u}||$ will be used for the length of the vector **u**. As **v-u** is the vector that goes from the point **u** to the point **v**, $||\mathbf{v-u}||$ gives the distance of the point **u** from the point **v**.

LENGTH.

1. Euclidean space of 2 dimensions.

If $\mathbf{u} = (u_1, u_2)$, we define length by

$$||\mathbf{u}||^2 = u_1^2 + u_2^2.$$

With the help of Pythagoras Theorem. we can define back of G5

$f(x_1)^2 + f(x_2)^2 + \ldots\ldots\ldots f(x_n)^2.$

The distances between the points so obtained for various functions might give us a useful way of measuring the distances between the functions. However, the method is rather rough and ready. How big should n be ? Why choose the mid-point of each interval?

Now a sum resembling that just written would appear if we were making an estimate of the value of $\int_p^q f(x)^2\,dx$.

This suggests that we might define the length of the function f by

$$||f||^2 = \int_p^q f(x)^2\,dx .$$

This leads to something quite novel. We can define a dot product for this space, and thus find a meaning for one function being perpendicular to another.

The work follows the same pattern as before, but with integrals instead of sums.

When we were finding the condition for **u** to be perpendicular to **v**, the length of the hypotenuse was $||(\mathbf{v}-\mathbf{u})||$, so the condition was

$$||(\mathbf{v}-\mathbf{u})||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 .$$

If **u** corresponds to f(x) and **v** to g(x), **v**−**u** will correspond to g(x) − f(x) . With the definition of length just found this condition will become

$c^2 = a^2 + b^2$ where

$$c^2 = \int_p^q [g(x)-f(x)]^2\,dx$$

$$a^2 = \int_p^q f(x)^2\,dx \quad , \quad b^2 = \int_p^q g(x)^2\,dx .$$

We now need to muliply out the bracket that appears in c^2. This gives

$$c^2 = \int_p^q g(x)^2 - 2f(x)g(x) + f(x)^2\,dx.$$

The integrals of $f(x)^2$ and $g(x)^2$ appear on both sides of the equation $c^2 = a^2 + b^2$, and cancel just as the squares did in the earlier examples. Again we divide by −2 to arrive at the definition

$$\mathbf{f.g} = \int_p^q f(x)g(x)\,dx .$$

We shall say that the functions are perpendicular if this dot product is zero.

"Orthogonal" is a synonym for "perpendicular" and it is the custom to-day to speak of orthogonal functions rather than perpendicular ones. I do not know the reason for this. Perhaps the idea of functions being perpendicular is felt to be rather shocking, and the more learned word is used to lessen the shock.

Back of G6

fact is extremely close. If we take $\mathbf{u}_n$ as the vector representing sin nx, equation (5) can be written $\mathbf{u}_m.\mathbf{u}_n = 0$, which means that the sines are represented by mutually perpendicular vectors, while equation (6) says

$$\mathbf{f}.\mathbf{u}_3 = c_3\ \mathbf{u}_3.\mathbf{u}_3,$$

which is of exactly the same form as equation (3) earlier.

Thus finding the coefficients in a Fourier series turns out to be just the problem of expressing a vector in a new system of perpendicular axes.

This immediately suggests a thought. There are many ways to choose a set of perpendicular axes. There must be many other systems of perpendicular functions, from which we could derive series by exactly the same procedure. One such system is much simpler, and could even be used to compose problems in beginning calculus; all the functions in it are polynomials.

ORTHOGONAL POLYNOMIALS.

With [-1,1] as the basic interval, let $F_0(x) = 1$,

$F_1(x) = x$, $F_2(x) = 3x^2 - 1$,

$F_3(x) = 5x^3 - 3x$, $F_4(x) = 35x^4 - 30x^2 + 3$.

You can check that these are orthogonal. The work can reduced by using the following observation; the dot product involves the vectors linearly. For example, $\mathbf{f}.(a\mathbf{u} + b\mathbf{v} + c\mathbf{w}) = a(\mathbf{f}.\mathbf{u}) + b(\mathbf{f}.\mathbf{v}) + c(\mathbf{f}.\mathbf{w})$. This means that, if $\mathbf{f}$ is perpendicular to $\mathbf{u}$, to $\mathbf{v}$ and to $\mathbf{w}$, it is bound to be perpendicular to $a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$, for any a,b,c. This holds for any number of vectors in the bracket; if $\mathbf{f}$ is perpendicular to each of them, it is perpendicular to any linear mixture of them.

So, if we check that $F_4(x)$, for example, is perpendicular to 1, to x, to x^2 and to x^3, it is bound to be perpendicular to any linear combination of these, hence to any polynomial with degree less than 4, hence in particular to $F_0(x)$, to $F_1(x)$, to $F_2(x)$ and to $F_3(x)$. The same idea can be applied to $F_3(x)$.

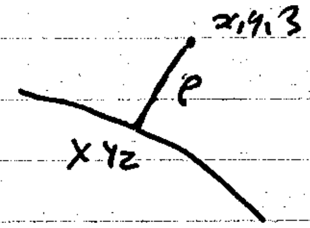
Further polynomials in this sequence can be obtained by taking $F_n(x) = (d/dx)^n\ (x^2 - 1)^n$. That $F_n(x)$ is perpendicular to x^m if m<n can be proved by integration by parts. Observe that $(d/dx)^s\ (x^2-1)^n$ is zero for x= -1 and x=1 if s<n ; do not expand any of the powers of x^2-1.

If multiplied by certain constants, these polynomials give the Legendre polynomials, which play an important part in electromagnetic theory and other branches of science. They can be used to build series in much the same way that sines are for Fourier series. Like Fourier series, they are capable of representing functions that have discontinuities.

curl $(l, m, n) = 0$ means that $\int l\,dx + m\,dy + n\,dz$ is independent of path. The surface $\int l\,dx + m\,dy + n\,dz$ = constant is $\perp$ rays.

§9. §10. If there is no system of surfaces $\perp$ rays, the problem of focussing is impossible.

If ray comes to $x\,y\,z$ after crossing a surface at X, Y, Z distant ρ from x, y, z



$$X - x = \rho l \quad Y - y = \rho m \quad Z - z = \rho n.$$

$$dX - dx = l\,d\rho + \rho\,dl \quad \text{etc.}$$

Multiply by l, m, n and add. $l\,dX + m\,dY + n\,dZ = 0$ by $\perp$.

$l\,dl + m\,dm + n\,dn = 0$ as before.

$$\therefore \quad -l\,dx - m\,dy - n\,dz = (l^2 + m^2 + n^2)\,d\rho = d\rho.$$

Thus O.E of eqn C can be put in the form $d\rho + d\rho' = 0$.

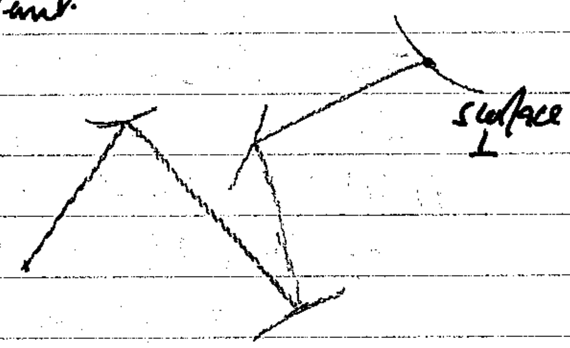
so $\rho + \rho'$ = constant.

This enables us to construct mirror surface. Choose points on the normals to surface such that sum of distances to required focus is constant. (Difference if rays diverge from focus)

III Surfaces of constant action

§11. We have shown that a system of rays can be brought to a point only if there are $\perp$ surfaces. If rays diverge from a point and are reflected by any kind of mirror there must be surfaces $\perp$ reflected rays. He shows that after any no. of reflections, a surface cutting rays $\perp^{ly}$ makes total length of each path constant.

§12 The argument seems to be that variation at each point of reflection leaves total path in that part stationary, and variation in $\perp$ surface leaves final length stationary, so total variation is zero



Back of H 2

$\delta\rho + \mu\,\delta\rho' = 0$ leads to $\int \mu\, ds$ as formula for action. Here again, rays are perpendicular to surfaces of equal action.

If ray has direction (l, m, n) ~~and $V = \Sigma \mu \Delta s$~~

$(l, m, n) = \text{grad } V.$

Here $V = \Sigma\, m \Delta s$, m being the ratio of refractive indices for refracted and incident rays (?)

If $I = \int \mu\, ds$, ~~grad I~~

$$l = \frac{1}{\mu}\frac{\partial I}{\partial x} \quad m = \frac{1}{\mu}\frac{\partial I}{\partial y} \quad n = \frac{1}{\mu}\frac{\partial I}{\partial z}.$$

(This agrees with Whittaker.)

I is the characteristic fn for curved rays, and will be denoted by V in future.

$$\mu^2 = \left(\frac{\partial I}{\partial x}\right)^2 + \left(\frac{\partial I}{\partial y}\right)^2 + \left(\frac{\partial I}{\partial z}\right)^2.$$

§18. For refracted rays $\Sigma \mu \rho$ = constant gives surface ⊥ rays. "if light be a material substance its velocity in uncrystallized mediums is proportional to the refractive power and is not altered by reflection" these surfaces will be called "surfaces of constant action."

Thus we have $\int v\, ds$ stationary, v being the velocity expected on particle th^y of light.

p.40. $\frac{\partial v}{\partial \alpha}, \frac{\partial v}{\partial \beta}, \frac{\partial v}{\partial \gamma}$ are obtained by expressing v as a homogeneous fn. of degree 1 in α, β, γ. where α, β, γ are l, m, n for the ray. The identity $\alpha^2 + \beta^2 + \gamma^2 = 1$ is used to make fn homogeneous in way described.

(nothing was on back of H3) Back of H4 26.1.79 (4)

$$\int_{-1}^{1}\int_{-1}^{1} \frac{x}{1-\alpha xy}\left\{\ln(1+x)-\ln(1-x)\right\} dx\,dy$$ (given on 31.1.79 (+))

$$= \int_{-1}^{1}\int_{-1}^{1}\left(1+\alpha xy+\alpha^2x^2y^2+\alpha^3x^3y^3+\alpha^4x^4y^4+\ldots\right)\times$$
$$2\left\{x^2+\frac{x^4}{3}+\frac{x^6}{5}+\ldots\right\} dx\,dy$$

$$= 8\left[\frac{1}{3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots\right]$$
$$+\alpha^2.2.\frac{2}{3}2\left[\frac{1}{1.5}+\frac{1}{3.7}+\frac{1}{5.9}+\ldots\right]$$
$$+\alpha^4.2.\frac{2}{5}.2\left[\frac{1}{1.7}+\frac{1}{3.9}+\frac{1}{5.11}+\ldots\right]$$
$$+$$

Sum by means of partial fractions.

$$= 4+\frac{8}{9}\alpha^2+\alpha^4\frac{8}{5}\left(\frac{1+\frac{1}{3}+\frac{1}{5}}{6}\right)+\alpha^6.\frac{8}{7}\left(\frac{1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}}{8}\right)$$
$$= 4\left\{1+\frac{2}{3}\alpha^2+\frac{1}{5}.\frac{23}{45}\alpha^4+\alpha^6.\frac{1}{7}.\frac{44}{105}+\ldots\right\}$$
$$= 4\mu_2.$$

So $\iiint\left(\frac{x}{1-\alpha xy}-\frac{z}{1-\alpha yz}\right)\frac{1}{x-z}\,dx\,dy\,dz$ (see top of 26.1.79 (3)).

$$= 2\iiint\frac{x}{(1-\alpha xy)(x-z)}\,dx\,dy\,dz = 8\mu_2.$$

back of H13

p 300 § 131. I think the result of this section can be obtained directly from § 128. Condition is

$\delta P_r dQ_r - \delta Q_r dP_r =$ (with tensor summation convention)

$= \delta p_r dq_r - \delta q_r dp_r$.

$$\delta P_r dQ_r - \delta Q_r dP_r = \left(\frac{\partial P_r}{\partial p_i}\delta p_i + \frac{\partial P_r}{\partial q_i}\delta q_i\right)\left(\frac{\partial Q_r}{\partial p_j}dp_j + \frac{\partial Q_r}{\partial q_j}dq_j\right) -$$
$$- \left(\frac{\partial Q_r}{\partial p_i}\delta p_i + \frac{\partial Q_r}{\partial q_i}\delta q_i\right)\left(\frac{\partial P_r}{\partial p_j}dp_j + \frac{\partial P_r}{\partial q_j}dq_j\right)$$

$\delta p_u dp_v$ does not occur on RHS. Coeff must be 0.

∴ $\cancel{\frac{\partial P_r}{\partial p_i}\frac{\partial Q_r}{\partial p_j}}$ — ~~Consider~~ $u = 1,\ v = 2$.

~~Owing to the summations we have~~

$\frac{\partial P_r}{\partial p_1}\frac{\partial Q_r}{\partial p_2} -$. Take $n = 2$.

$\cancel{\left(\frac{\partial P_1}{\partial p_1}\delta p_1 + \frac{\partial P_1}{\partial p_2}\delta p_2 + \frac{\partial P_1}{\partial q_1}\delta q_1 + \frac{\partial P_1}{\partial q_2}\delta q_2\right)\left(\frac{\partial Q}{\ }\right.}$

$$\frac{\partial P_r}{\partial p_1}\frac{\partial Q_r}{\partial p_2} - \frac{\partial Q_r}{\partial p_1}\frac{\partial P_r}{\partial p_2}$$

Back of H14

CB4

We thus find

$$\frac{A_{n,n+1}}{A_{nn}} = \sum_{r=0}^{n} \frac{L(r,r+1)}{L(r,r)} + \frac{L(n+1,n+1)}{L(n,n)}$$

Such ratios seem to offer the most convenient method of working.

Thus
$$\frac{L(n+1,n+2)}{L(n+1,n+1)} = \Delta\left[\frac{A_{n,n+1}}{A_{nn}} - \frac{L(n+1,n+1)}{L(n,n)}\right]$$

$$= \Delta\left[\frac{(n+1)^2(2n+1)^2}{(4n+3)^2}\right]$$

after straightforward calculations.

To some extent this sets the pattern for later work. ~~We find~~, when seeking the value of $L(n,n+c)$, we find

$$\frac{A_{n,n+c}}{A_{nn}} = \sum_{0}^{n} \frac{\lambda_{i,i+c}}{\lambda_{ii}} + \text{terms already known.}$$

However
$$\cancel{\frac{A_{n,n+c}}{A_{nn}} \sim (n/3)^{2c}}$$

$$A_{n,n+c}/A_{nn} \sim (n/2)^{2c}/c!$$

while it seems

$$L(n,n+c)/L(n,n) = O(n^c).$$

and many terms must cancel, so that this approach does not seem promising as a way of calculating $L(n,n+c)/L(n,n)$ or proving its properties.

Back of H21

A MATHEMATICIAN'S APOLOGY. G. H. Hardy. (Published 1940.)
(Opening paragraph) "It is a melancholy experience for a professional mathematician to find himself writing about mathematics. The function of a mathematician is to do something, to prove new theorems, to add to mathematics, and not to talk about what he or other mathematicians have done. Statesmen despise publicists, painters despise art-critics, and physiologists, physicists ~~and~~ or mathematicians have usually similar feelings ; there is no scorn more profound, or on the whole more justifiable, than that of the men who make for the men who explain. Exposition, criticism, appreciation, is work for second-rate minds. "

Probably it is a biological necessity for a research worker to feel like this. There would be serious trouble if one grafted the brain of an eagle into the body of an elephant. Hardy continued to work at the theory of numbers through several of the most disturbed decades in history. Compare Einstein's General Theory of Relativity, 1916 ; contrast Einstein's attitude. Both hated war. Not criticise either ; different genes.

p 88-9. "It is plain now that my life, for what it is worth, is finished, and that nothing I can do can perceptibly increase or diminish its value. It is very difficult to be dispassionate, but I count it a "success" ; I have had more reward and not less than was due to a man of my particular grade of ability. I have held a series of comfortable and "dignified" positions. I have had very little trouble with the duller routine of universities. I hate "teaching" and have had to do very little, such teaching as I have done having been almost entirely supervision of research ; I love lecturing, and have lectured a great deal to extremely able classes ; and I have always had plenty of leisure for the researches which have been the one great permanent happiness of my life.

p. 90 "I have never done anything "useful". No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world....Judged by all practical standards, the value of my mathematical life is nil."

p. 56 (defining "useful") "Mathematics may, like poetry or music 'promote and sustain a lofty habit of mind', and so increase the happiness of mathematicians and even of other people ; but to defend it on that ground would be merely to elaborate what I have said already. What we have to consider now is the 'crude' utility of mathematics. "

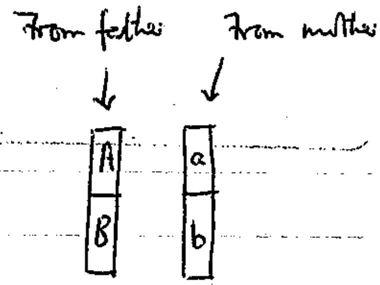
p71. "I was not thinking only of pure mathematicians. I count Maxwell and Einstein, Eddington and Dirac, among "real" mathematicians. The great modern achievements of applied mathematics have been in relativity and quantum mechanics , and these subjects are, at present at any rate, almost as "useless" as the theory of numbers. "

Written about five years before the atom bomb.

Back of H22

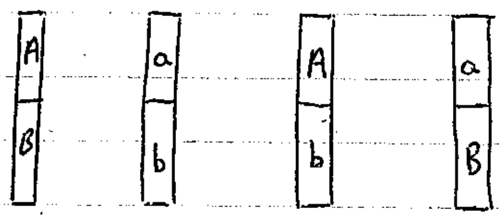
MEIOSIS

Body cell



At meiosis, a double division occurs: the diagram supposes a single crossover to occur.

Germ cells.



Any one of these is equally likely to be contributed to offspring. In male, each becomes a sperm, but only one sperm gets into egg. In female, three are discarded. One forms egg.

$$a_1 = a^1 \qquad - gt a^4$$
$$a_2 = a^2$$
$$a_3 = a^3$$
$$a_4 = -gta^1 \qquad + (-1+g^2t^2)a^4$$

$$(-1+g^2t^2)a_1 + gta_4 = a^1[-1+g^2t^2-g^2t^2] = -a^1$$
$$gta_1 + a_4 = a^4[-g^2t^2-1+g^2t^2] = -a^4$$

$$\therefore \quad g^{11} = 1-g^2t^2 \qquad g^{14} = -gt$$
$$g^{41} = -gt \qquad g^{44} = -1$$
$$g^{22} = 1 \qquad g^{33} = 1 \quad \text{Others} = 0.$$

If $\sigma \neq \lambda$, $g^{\sigma\lambda} = 0$ unless we have g^{14} or g^{41}.
If $\sigma = \lambda$, and $\lambda = 2$ or $\lambda = 3$, then $[\mu\nu, \lambda] = 0$.
So $\{\mu\nu, \sigma\} = 0$ if $\sigma = 2$ or 3.
It is also 0 if $\mu = 2$ or 3, or $\nu = 2$ or 3.
So only nonzero $\{\mu\nu, \sigma\}$ contain 1 and 4 only.
$[\mu\nu, \lambda]$ must contain two 4s. We may have $\sigma = 1, \lambda = 4$
but even so one of $\mu\nu$ must be 4.

$$\{14, 1\} = g^{11}[14,1] + g^{14}[14,4]$$
$$= 0 + (-gt)0 = 0$$
$$\{14,4\} = g^{41}[14,1] + g^{44}[14,4]$$
$$= 0 + (-1)0 = 0.$$
$$\{44,1\} = g^{11}[44,1] + g^{14}[44,4]$$
$$= (1-g^2t^2)(-g) - gt(2g^2t)$$
$$= -g - g^3t^2 = -g(1+g^2t^2)$$

$$\{44,4\} = g^{41}[44,1] + g^{44}[44,4]$$
$$= -gt(-g) - 2g^2t = -g^2t.$$

Equations of a geodesic.

In (28.5), $\{\mu\nu, \alpha\}$ must contain two 4s.

If $\alpha \neq 4$, $\mu = 4, \nu = 4$.

If $\alpha = 4$, μ, ν chosen from 1, 4 with at least one 4.

$$\frac{d^2 x_1}{ds^2} + \{44, 1\} \frac{dx_4}{ds}\frac{dx_4}{ds} = 0$$

i.e. $$\frac{d^2 x_1}{ds^2} - g(1 + g^2 r^2)\left(\frac{dx_4}{ds}\right)^2 = 0.$$

$$\frac{d^2 x_2}{ds^2} = 0 \qquad \frac{d^2 x_3}{ds^2} = 0.$$

$$\frac{d^2 x_4}{ds^2} + \{14, 4\} \frac{dx_1}{ds}\frac{dx_4}{ds} + \{41, 4\} \frac{dx_4}{ds}\frac{dx_1}{ds} + \{44, 4\} \frac{dx_4}{ds}\frac{dx_4}{ds} = 0$$

$$\frac{d^2 x_4}{ds^2} = - g^2 r \left(\frac{dx_4}{ds}\right)^2$$

Back of M 33

$$\begin{array}{cccc}
\cdot & v_{02} & v_{03} & \\
\cdot & v_{11} & v_{12} & \\
\boxed{v_{20}} & \lambda_1 & & \\
 & \boxed{v_{31}} & v_{32} & \\
 & & \boxed{v_{42}} & v_{43}
\end{array}$$

$\lambda = 2,$ $\mathcal{E}_{22}$ $\lambda_0 v_{20} = m_{222} v_{20}$.

$\mathcal{E}_{33}$ $\lambda_0 v_{31} = m_{323} v_{20}$

$\mathcal{E}_{44}$ $\lambda_0 v_{42} = m_{424} v_{20}$

$\mathcal{E}_{12}$ $m_{111} v_{11} + m_{122} v_{20} = 0$.

$\mathcal{E}_{02}$ $m_{000} v_{02} + m_{011} v_{11} + m_{022} v_{20} = 0$,

$\mathcal{E}_{13}$ $m_{111} v_{12} = \lambda_0 v_{11} - m_{101} v_{02} - m_{112} v_{11} - m_{123} v_{20}$

$\mathcal{E}_{03}$ $m_{011} v_{12} + m_{000} v_{03} = - m_{001} v_{02} - m_{012} v_{11} - m_{023} v_{20}$

There are only 2 rows above v_2 with its noughts.
$\mathcal{E}_{1t}$ and $\mathcal{E}_{0t}$ seem to give these.

$\mathcal{E}_{rs}$

	$s = 0$	1	2	3	4
$r = 0$	$\cdot$	$\cdot$	v_{02}	v_{03}	
1	$\cdot$	$\cdot$	v_{11}	v_{12}	
2	$\cdot$	$\cdot$	λ_0	λ_1	
3	$\cdot$	$\cdot$	$\cdot$	v_{31}	v_{32}
4	$\cdot$	$\cdot$	$\cdot$		v_{42}

If $r > 2$ and $r > s$, $\mathcal{E}_{rs}$ gives $0 = 0$.

$\mathcal{E}_{rs}$ is $(Mv)_r = \lambda v_r$, coeff of σ^s

$\lambda = \lambda_0 \sigma^2 + \ldots$ $\quad v_r = v_{r,r-2}\sigma^{r-2} \ldots$

$\lambda v_r \to \sigma^r$ $\quad s < r$ so 0 in λv_r

$(Mv)_r = \sum_s m_{rs} v_s$ As $m_{rs} \to \sigma^{\max(r,s)} = \sigma^r$ coeff of $\sigma^s = 0$.

Question. I had recollection that $c_{10} \sim n$. But this does not happen with $V_{n-1,1}$. It does.

Back of M 34

The formula for $V_{n-2,2}$.

$\mathcal{E}_{n-2,n}$: $0 = m_{n-2,n,n} V_{n0} + m_{n-2,n-1,n-1} V_{n-1,1} + m_{n-2,n-2,n-2} V_{n-2,2}$

$\mathcal{E}_{n-1,n}$: $0 = m_{n-1,n,n} V_{n0} + m_{n-1,n-1,n-1} V_{n-1,1}$

$$\frac{V_{n0}}{-m_{n-1,n-1,n-1}\, m_{n-2,n-2,n-2}} = \frac{-V_{n-1,1}}{-m_{n-1,n,n}\, m_{n-2,n-2,n-2}}$$

$$= \frac{V_{n-2,2}}{\begin{vmatrix} m_{n-2,n,n} & m_{n-2,n-1,n-1} \\ m_{n-1,n,n} & m_{n-1,n-1,n-1} \end{vmatrix}}$$

Det^{nt} under $V_{n-2,2}$ is $\begin{vmatrix} a_{n-2,n} b_{nn} & a_{n-2,n-1} b_{n-1,n-1} \\ a_{n-1,n} b_{nn} & a_{n-1,n-1} b_{n-1,n-1} \end{vmatrix}$

$$= b_{nn} \underline{b_{n-1,n-1}} \begin{vmatrix} a_{n-2,n} & a_{n-2,n-1} \\ a_{n-1,n} & a_{n-1,n-1} \end{vmatrix}$$

Denom^{r} of V_{n0} is

$- a_{n-1,n-1} \underline{b_{n-1,n-1}}\, a_{n-2,n-2}\, b_{n-2,n-2}$

$a_{n-2,x} / b_{x,n-2}$ is ind of x.

So $$\frac{V_{n-2,2}}{V_{n0}} = \frac{-b_{nn} \begin{vmatrix} b_{n,n-2} & b_{n-1,n-2} \\ b_{n,n-1} & b_{n-1,n-1} \end{vmatrix}}{b_{n-2,n-2}\; b_{n-1,n-1}\; b_{n-2,n-2}}$$

The determinant leads to $(n^2) - (n^2)$. I believe C_{20} is $O(n^2)$ so no cancelling needed. First part is

$$\frac{b_{nn} b_{n,n-2}}{(b_{n-2,n-2})^2}$$

$$\frac{[2n(2n-1)\dots 2]\,[2n(2n-1)\dots 6]}{[(4n+1)\overset{(2)}{\dots} 3]\,[(4n-3)\overset{(2)}{\dots} 7]} \left\{\frac{(4n-7)\overset{(2)}{\dots} 3}{(\cancel{})(2n-4)\overset{(1)}{\dots} 2}\right\}^2$$

$$= \frac{2n(2n-1)(2n-2)(2n-3) \cdot 2n(2n-1)(2n-2)(2n-3) \quad 3.5}{5.4.3.2 \quad (4n+1)(4n-1)(4n-3)(4n-5) \quad (4n-3)(4n-5)}$$

$$\sim \frac{n^2\, 2^8}{8.\, 4^6} = \frac{n^2 . 2^8}{2^{15}} = \frac{n^2}{2^7}$$

$$\mu_n = \frac{1}{(2n+1)(2n+1)} \sum_0^n \frac{1}{2q+1}$$ moment function? (from iteration Hilbert)

$$\mu_{n} - \mu_{n+1} = \frac{1}{(2n+1)(n+1)} \sum_0^n \frac{1}{2q+1} - \frac{1}{(2n+3)(n+2)} \sum_0^{n+1} \frac{1}{2q+1}$$

$$= \left(\sum_0^n \frac{1}{2q+1}\right)\left[\frac{1}{(2n+1)(n+1)} - \frac{1}{(2n+3)(n+2)}\right] - \frac{1}{(2n+3)(n+2)(2n+3)}$$

$$= \left(\sum_0^n \frac{1}{2q+1}\right) \frac{4n+5}{(n+1)(n+2)(2n+1)(2n+3)} - \frac{1}{(2n+3)(n+2)(2n+3)}$$

$$\psi(t) = \sum_{\nu=0}^{\infty} (2\nu+1)\lambda_\nu \int_0^t L_\nu(u)\,du$$

If term by term differentiation is permissible

$$w(t) = \psi'(t) = \sum_{\nu=0}^{\infty} (2\nu+1)\lambda_\nu L_\nu(t).$$

$|L_\nu(t)| \le 1$. We shall have u-cgnce if $\sum (2\nu+1)|\lambda_\nu|$ is cgt.

$L_0(t) = 1$ $\qquad \lambda_0 = \mu_0$

$L_1(t) = 1-2t$ $\qquad \lambda_1 = \mu_0 - 2\mu_1$

$L_2(t) = 1-6t+6t^2$ $\qquad \lambda_2 = \mu_0 - 6\mu_1 + 6\mu_2$

$L_3(t) = 1-12t+30t^2-20t^3$ $\qquad \lambda_3 = \mu_0 - 12\mu_1 + 30\mu_2 - 20\mu_3$

If $\mu_n = \frac{1}{(n+1)(2n+1)} \sum_0^n \frac{1}{2q+1}$ $\qquad \lambda_0 = 1.\quad \lambda_1 = 5/9$

$\lambda_2 = 7/25 \qquad \lambda_3 = 149/735 = .202721.$

$(2\nu+1)\lambda_\nu = \pi_\nu \qquad \pi_0 = 1 \qquad \pi_1 = \frac{15}{9} \qquad \pi_2 = \frac{35}{25} \qquad \pi_3 = \frac{149}{105}$

1.667 $\qquad$ 1.4 $\qquad$ 1.419

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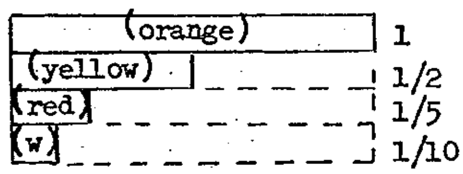
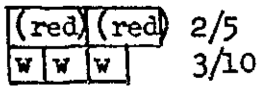
Divide the fraction curriculum into units and within any one unit (lasting at least two weeks and often three or more) limit the number of fractions the children are to work with and give each a very specific meaning.

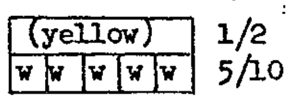
Within each unit, the fractions that we use are made specific by giving the whole number "one" a fixed and narrow meaning. The "ones" that we use are described below.

a) <u>Concrete "Ones"</u> Fractions can be given specific meanings by attaching them to some concrete material. For example, in our first unit we call the orange Cuisenaire rod "one". The yellow rod is thus named "one half", the red rod "one fifth" and the white rod "one tenth".



In this unit, which takes perhaps 3 weeks, these fractions and their multiples are the only ones to which the children are exposed. Furthermore, during this period, all symbolic work refers exclusively to the rods. Thus for example, the children are taught that the numeral 3/5 means three of the <u>fifths</u>, that is, three <u>red rods</u>. Even with this kind of narrow understanding they can discover that $2/5 > 3/10$ by comparing a "train" of two reds with a train of three whites. Similarly, it is easy for them to see that $5/10 = 1/2$. However, at least temporarily, the only meaning here is that a train of 5 white rods is just as long as a yellow.





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$$\frac{d}{dx}(x^2-1)y' - n(n+1)y = 0$$

$$(x^2-1)y'' + 2xy' - n(n+1)y = 0.$$

If $y = \Sigma a_s x^s$, from coeff of x^{s-2}

$$a_{s-2}\left[(s-2)(s-3) + 2(s-2) - n(n+1)\right] = a_s\, s(s-1)$$

$$[\ ] = (s-2)(s-1) - n(n+1) = (s-2-n)(s-1+n)$$

$$\therefore \quad a_s = a_{s-2}\,\cancel{(s\ldots)}\,\frac{(s-2-n)(s-1+n)}{s(s-1)}$$

If $a_0 = 1 \quad a_2 = \dfrac{-n(n+1)}{2.1}$

$$a_4 = \frac{\cancel{n(n+1)(n+2)(n+3)}}{\dfrac{(2-n)(-n)(n+1)(n+3)}{4!}}$$

$$\frac{a_s}{a_{s-2}} = \frac{s(s-1)}{(s-2-n)(s-1+n)}$$

$$= \frac{1-\frac{1}{s}}{\left(1-\frac{n+2}{s}\right)\left(1+\frac{n-1}{s}\right)}$$

$$\sim \left(1-\frac{1}{s}\right)\left(1+\frac{n+2}{s}\right)\left(1+\frac{1-n}{s}\right)$$

$$\sim 1 \quad \frac{2}{s}$$

$n = 4 \qquad a_s = \dfrac{(s-6)(s+3)}{s(s-1)}a_{s-2}$

$s=2 \qquad a_2 = \dfrac{-4.5}{2.1}a_0 \qquad a_4 = \dfrac{-2.5}{4.3}$

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§127. The bilinear covariant of a differential form.

$$\theta_d = \sum X_i\,dx_i \qquad \theta_\delta = \sum X_i\,\delta x_i$$

$$\delta\theta_d - d\theta_\delta = \sum\sum\left(\frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i}\right)dx_i\,\delta x_j$$

$a_{ij} = \dfrac{\partial X_i}{\partial x_j} - \dfrac{\partial X_j}{\partial x_i}$. This is the type of expression found in curl X. If $(y_1 \dots y_n)$ new variables, and $b_{ij} = \dfrac{\partial Y_i}{\partial y_j} - \dfrac{\partial Y_j}{\partial y_i}$ we have

$$\sum\sum a_{ij}\,dx_i\,\delta x_j = \sum\sum b_{ij}\,dy_i\,\delta y_j\,.$$

Then $x_1 = g_{11}x^{(1)} + g_{12}x^{(2)} = g_{1j}x^j$

$x_2 = g_{21}x^{(1)} + g_{22}x^{(2)} = g_{2j}x^j$

So $x_i = g_{ij}x^j$ (1)

Now $OP^2 = \underset{\sim}{x}\cdot\underset{\sim}{x} = x^i x_i = x^i g_{ij} x^j = g_{ij}x^i x^j$ (2)

~~as in equation (2)~~ Often g_{ij} is introduced in this way. It is supposed that s, the length of the vector $\underline{x}$, is given by

$$\cancel{s^2 = g_{ij}x^i x^j} \qquad \cancel{(2)}$$

~~This incidentally~~ where $g_{ij} = g_{ji}$.

Equations (1) and (2) for work in any number of dimensions.

The Polar Form.

Consider any two vectors $\underline{x}$ and $\underline{y}$. Let $|\underline{x}|$ denote the length of $\underline{x}$.

$$|x+y|^2 = g_{ij}(x^i+y^i)(x^j+y^j)$$

$$= g_{ij}x^i x^j + g_{ij}y^i x^j + g_{ij}x^i y^j + g_{ij}y^i y^j$$

$$= |x|^2 + 2g_{ij}x^i y^j + |y|^2$$

as g_{ij} is symmetric.

$$\therefore \quad 2g_{ij}x^i y^j = |x+y|^2 - |x|^2 - |y|^2$$

so $g_{ij}x^i y^j$ is invariant, i.e it is a scalar the same in every system.

From equation (1), ~~$x^i y_i$ is invariant~~ $y_i = g_{ij}y^j$

So $g_{ij}x^i y^j = x^i y_i$

Earlier, we obtained equation by considering a particular case in 2 dimensions. Here we have shown that equation (1) leads to a definition of y_i that makes $x^i y_i$ invariant, as it should be, and the proof holds in space of any number of dimensions where ~~|x|~~ s^2 is defined by (2).

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Continuation of p. 94. p 91 taken to further term.

We are concerned with $\sum_{i \neq j} \frac{\lambda_{i,i+1}}{\lambda_{ii}} \frac{\lambda_{j,j+1}}{\lambda_{jj}}$

$$\Delta \sum_{0 \le i < j \le n} = \frac{\lambda_{n+1,n+3}}{\lambda_{n+1,n+1}} \cdot \sum_{0}^{n} \frac{\lambda_{i,i+1}}{\lambda_{ii}}$$

$$\begin{vmatrix} a & b & c \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = a(y_1 z_2 - y_2 z_1) + b(z_1 x_2 - z_2 x_1) + c(x_1 y_2 - y_1 x_2)$$

If

$$\begin{matrix} a_1 & \cancel{a_2} & a_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{matrix} \qquad a_1 (x_2 y_3 - x_3 y_2)$$

$$\varepsilon_{ijk} a_i x_j y_k \qquad x_j = t_{js} X_s$$

$$y_k = t_{ku} Y_u$$

$$\varepsilon_{ijk} a_i t_{js} X_s t_{ku} Y_u$$

$$= \varepsilon_{ijk} A_i X_j Y_k$$

$$a(x_2 y_3 - y_2 x_3) +$$

$$x_i y_j - x_j y_i = t_{iu} X_u t_{jv} Y_v - t_{jv} X_v t_{iu} Y_u$$

$$= t_{iu} t_{jv} (X_u Y_v - X_v Y_u)$$

Can we prove $\lambda_{n,n+2}/\lambda_{nn} > 0$ by using simple overestimates for the subtracted terms?

$$\frac{\lambda_{n+1,n+1}}{\lambda_{nn}} = \frac{(2n+1)^2(2n+2)^2}{(4n+1)(4n+3)^2(4n+5)}$$

$$= \frac{16n^4+48n^3+52n^2+24n+4}{256n^4+768n^3+800n^2+336n+45}$$

$$= \frac{1}{16} + \frac{2n^2+3n+1\frac{3}{16}}{256n^4+768n^3+800n^2+336n+45}$$

Note: denominator $=$
$(16n^2+24n+5)(16n^2+24n+9)$
$=(16n^2+24n+7)^2-4$
$=\left[4\left(4n^2+6n+1\frac{3}{4}\right)\right]^2-4$
Numerator
$=(4n^2+6n+2)^2$
so fraction very close to $\frac{1}{16}$.

so $\dfrac{\lambda_{n+1,n+1}}{\lambda_{nn}} < \dfrac{1}{16} + \dfrac{1}{100n^2}$. or $< \dfrac{1}{16} + \dfrac{1}{128n^2}$ if preferred.

$\dfrac{\lambda_{n+2,n+2}}{\lambda_{n+1,n+1}} < \dfrac{1}{16} + \dfrac{1}{100(n+1)^2} < \dfrac{1}{16} + \dfrac{1}{100n^2}$ ~~$\frac{n+1}{n} < 2$ if $n \geq 1$~~

$\dfrac{\lambda_{n+2,n+2}}{\lambda_{nn}} \leq \dfrac{1}{256} + \dfrac{1}{800n^2} + \dfrac{1}{10^4n^4}$ ~~$\frac{1}{n+1} > \frac{1}{2n}$ so $\frac{\lambda_{n+2,n+2}}{\lambda_{n+1,n+1}} <$~~

$\dfrac{1}{800} + \dfrac{1}{10000n^2} < \dfrac{1}{800} +$

$n \geq 1$ $\quad \dfrac{1}{16} + \dfrac{1}{128n^2} < \dfrac{1}{16} + \dfrac{1}{128} = \dfrac{9}{128} < \dfrac{1}{14}$.

$\dfrac{\lambda_{n+1,n+1}}{\lambda_{nn}} < \dfrac{1}{14} \qquad \dfrac{\lambda_{n+2,n+2}}{\lambda_{nn}} < \dfrac{1}{196}$.

We may deal with these before taking Δ. | care needed.

$$\varphi(n) = \left\{\frac{(n+1)(2n+1)}{4n+3}\right\}^2$$

$$\frac{\lambda_{n+1,n+1}}{\lambda_{nn}} = \frac{1}{16} + \frac{2n^2+3n+19/16}{256n^4+768n^3+800n^2+336n+45}$$

$$\frac{1}{16} < \frac{\lambda_{n+1,n+1}}{\lambda_{nn}} < \frac{1}{16} + \frac{1}{128n^2}$$

$$\therefore \quad \Delta \frac{\lambda_{n+1,n+1}}{\lambda_{nn}} < \left[\frac{1}{16} + \frac{1}{128(n+1)^2}\right] - \frac{1}{16} = \frac{1}{128(n+1)^2}.$$

$$\Delta \frac{\lambda_{n+1,n+1}}{\lambda_{nn}} < \frac{1}{128n^2}.$$

$$\frac{1}{16} < \frac{\lambda_{n+2,n+2}}{\lambda_{n+1,n+1}} < \frac{1}{16} + \frac{1}{128(n+1)^2} < \frac{1}{16} + \frac{1}{128n^2}$$

$$\therefore \quad \frac{1}{256} < \frac{\lambda_{n+2,n+2}}{\lambda_{nn}} < \left(\frac{1}{16} + \frac{1}{128n^2}\right)^2 = \frac{1}{256} + \frac{1}{1024n^2} + \frac{1}{2^{14}n^4}$$

$$< \frac{1}{256} + \frac{1}{512n^2} \quad \text{for } n \geq 1.$$

$$\therefore \quad \Delta \frac{\lambda_{n+2,n+2}}{\lambda_{nn}} < \frac{1}{512n^2}.$$

In difference $\Delta F(n) = F(n+1) - F(n)$ we take overestimate of $F(n+1)$ and underestimate of $F(n)$.

$$\cancel{\frac{1}{16} < \frac{\lambda_{n+1,n+1}}{\lambda_{nn}} < \frac{1}{16} + \frac{1}{128(n)^2}}$$

$$\cancel{\therefore \quad \Delta \frac{\lambda_{n+1,n+1}}{\lambda_{nn}} <}$$

$$\Delta \frac{\lambda_{n+1,n+1}}{\lambda_{n-1,n-1}} < \frac{1}{512(n-1)^2}$$

$$\therefore \quad \Delta\left[\frac{\lambda_{n+2,n+2}}{\lambda_{n+1,n+1}} + \frac{\lambda_{n+1,n+1}}{\lambda_{nn}}\right] < \frac{1}{256(n-1)^2}$$

In general relativity, the object is to state laws in such a way that they have the same form in any (analytic) system of co-ordinates. The central theme of tensor analysis is seen here — to compare statements as they appear in different co-ordinate systems.

On the surface of the earth, a point can be specified by θ, the co-latitude and φ, the longitude. Distances are calculated using the equation $ds^2 = d\theta^2 + \sin^2\theta \cdot d\varphi^2$. Relativity uses coordinates of this type. A point may be specified by $(x_1, \ldots x_n)$, and distances are given by a quadratic expression

$$ds^2 = \sum_i \sum_j g_{ij}\, dx^{(i)} dx^{(j)}. \qquad (1)$$

Summation repeatedly occurs in tensor theory, and there is a convention — sum over any order number that occurs twice. Thus the equation above would be written $ds^2 = g_{ij}\, dx^{(i)} dx^{(j)}$.

Tensors can also be used in connection with different co-ordinate systems in a plane or other linear surface. These systems have the same origin. Distance will now be given by

(2) $s^2 = g_{ij} \cdot x^{(i)} x^{(j)}$ for the length of a vector $\underset{\sim}{x}$.

In different co-ordinate systems a point will have different specifications. We suppose that people in different systems are aware of the correspondence: it is known how to transform the numbers specifying a point in system A so as to obtain the numbers used in system B. The transformation is assumed linear. If (x^i) in A specifies the same point as (ξ^i) in B we have

(3) $x^i = t^i_{\ j} \xi^j$ (summation convention).

When we say that something is a vector, we imply that its co-ordinates are known in every system and that these are connected by the transformations that apply to points, (3).

Vector and Tensor Theory.

This is essentially concerned with geometrical and physical situations, as recorded in different co-ordinate systems.

In a plane for instance, in one system a displacement may be specified by x^1, x^2, in another by ξ^1, ξ^2. (1 and 2 are labels, not powers)

The systems are supposed to be related by linear equations.

$$x^1 = a\xi^1 + b\xi^2$$
$$x^2 = c\xi^1 + d\xi^2$$

These express algebraically that the vector called $(x^1 x^2)$ in the first system is geometrically the same as that called $(\xi^1 \xi^2)$ in the other.

In general if in n dimensions $x^1 \ldots x^n$ is geometrically identified with $\xi^1 \ldots \xi^n$, if

$$x^i = \sum_{j=1}^{n} t^i_{\cdot j} \xi^j \qquad i = 1 \ldots n \qquad (I)$$

Tensor convention When a letter such as j occurs twice, once above and once below, it is understood that we sum 1 to n, as the work abounds in such work. Thus we write $x^i = t^i_{\cdot j} \xi^j$

A number is called a scalar if it the same in all systems, e.g. electric potential. If f is such a quantity

$$df = \frac{\partial f}{\partial x^i} dx^i$$
$$= \frac{\partial f}{\partial x^i} t^i_{\cdot j} d\xi^j$$

so $$\frac{\partial f}{\partial \xi^j} = \frac{\partial f}{\partial x^i} t^i_{\cdot j}$$

If we write $s_i = \dfrac{\partial f}{\partial x^i}$, $\sigma_i = \dfrac{\partial f}{\partial \xi^i}$ we have

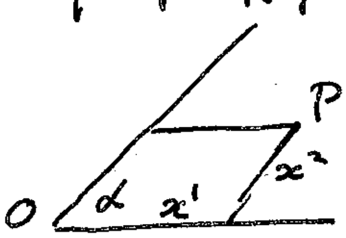
$$\sigma_j = s_i t^i_{\cdot j} \qquad (II)$$

Note that (I) and (II) are not the same transformation. ~~In matrix notation $x = T\xi$, $\sigma = sT$.~~

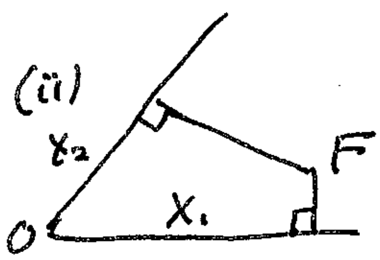
Covariant and Contravariant.

The distinction between v_i and v^i is often referred to as that between covariant and contravariant vectors. I find this terminology confusing. There are not two different kinds of vectors: I would rather speak of the covariant and contravariant methods of specifying a vector.

(i)



(ii)



Consider a plane with axes taken at angle α. A displacement OP can be specified by (x^1, x^2) as in (i). ~~A force (X_1, X_2) should be specified so that~~ A force (X_1, X_2) is most conveniently specified so that $X_1 x^1 + X_2 x^2$ represents the work done by the force in the displacement. To achieve this, we must take X_1, X_2 as the projections of OF on the axes, as in (ii)

$(X_1 X_2)$ is called covariant, $(x^1 x^2)$ contravariant

The difference lies in the form of the specification. Nothing stops us specifying OP by its projections on the axes, or OF by its components parallel to the axes. In fact there is a definite procedure for obtaining the representation (x_1, x_2) from $(x^1 x^2)$.

The work done by a force in a displacement is a scalar product, and appears as $X_i x^i$.

If we wanted to find the length of OP we would need to take the scalar product of OP with itself. $OP^2 = x_i x^i$.

$x_1 = x^1 + x^2 \cos A$, $x_2 = x^1 \cos A + x^2$, so

$x_i x^i = x^1(x^1 + x^2 \cos A) + x^2(x^1 \cos A + x^2)$

$= (x^1)^2 + 2x^1x^2 \cos A + (x^2)^2$

If we write $x_1 = g_{11}x^1 + g_{12}x^2$
$x_2 = g_{21}x^1 + g_{22}x^2$
this will agree with our result above if
$g_{11} = 1 \quad g_{12} = \cos A$

$g_{21} = \cos A \quad g_{22} = 1$
and we have
$OP^2 = g_{11}(x^1)^2 + g_{12}x^1x^2 + g_{21}x^2x^1 + g_{22}(x^2)^2.$
These equations may be written concisely
$x_i = g_{ij}x^j \qquad OP^2 = g_{ij}x^ix^j.$

g_{ij} can be generalized to apply to any case in which a length is defined as the square root of a quadratic form.

If, in this general case, we have another coordinate system $\{\xi^i\}$, with $x^i = t^i_{.j}\xi^j$, we expect $OP^2 = \gamma_{ij}\xi^i\xi^j$, and in fact we find this is so, for
$OP^2 = g_{ij}x^ix^j = g_{ij}(t^i_{.u}\xi^u)(t^j_{.v}\xi^v)$
$= g_{ij}t^i_{.u}t^j_{.v}\xi^u\xi^v$
and
$\gamma_{uv} = g_{ij}t^i_{.u}t^j_{.v}$

Thus we have a procedure for going from one system to another when dealing with something like g_{ij} with two subscripts.

$$\frac{A_{r+2}}{A_r} = \sum_{i=0} \frac{\lambda_{i,i+1}}{\lambda_{ii}} + \frac{\lambda_{r+2,r+2}}{\lambda_{rr}} + \frac{\lambda_{r,r+1}}{\lambda_{r-1,r-1}} + \frac{\lambda_{r,r+1}}{\lambda_{rr}} \quad (g) \qquad (4)$$

$$(d)\ + \frac{\lambda_{r+1,r+1} + \lambda_{r,r+1}}{\lambda_{rr}} \cdot \sum_0^{r-1} \frac{\lambda_{i,i+1}}{\lambda_{ii}} + \sum\sum_{0\le i<j\le r-1} \frac{\lambda_{i,i+1}}{\lambda_{ii}} \frac{\lambda_{j,j+1}}{\lambda_{jj}} \quad (e)$$

Take Δ.

$$(a)\ \Delta \frac{A_{r+2}}{A_r} \quad (b)\ \boxed{\frac{\lambda_{r+1,r+3}}{\lambda_{r+1,r+1}}} \quad (c)\ \frac{\lambda_{r+3,r+3}}{\lambda_{r+1,r+1}} - \frac{\lambda_{r+1,r+1}}{\lambda_{r-1,r-1}}$$

$$(f)\ \frac{\lambda_{r+2,r+3}}{\lambda_{r+1,r+1}} - \frac{\lambda_{r+1,r+2}}{\lambda_{rr}}$$ covers top row.

$$\Delta(a_r b_r) = a_{r+1}b_{r+1} - a_r b_r$$
$$= b_{r+1}(a_{r+1} - a_r) + a_r(b_{r+1} - b_r)$$
$$= b_{r+1}\Delta a_r + a_r \Delta b_r$$

$$\varphi(r) = \sum_{i=0}^{r} \frac{\lambda_{i,i+1}}{\lambda_{ii}}$$

$$\Delta \frac{\lambda_{r+1,r+1} + \lambda_{r,r+1}}{\lambda_{rr}} \varphi(r-1) = \frac{\lambda_{r+1,r+1} + \lambda_{r,r+1}}{\lambda_{rr}} \Delta\varphi_{r-1} + \varphi_r \Delta$$

$$(d)\ = \frac{\lambda_{r+1,r+1} + \lambda_{r,r+1}}{\lambda_{rr}} \frac{\lambda_{r,r+1}}{\lambda_{rr}} + \varphi(r)\Delta \frac{\lambda_{r+1,r+1} + \lambda_{r,r+1}}{\lambda_{rr}}$$

$$\Delta \sum_{0\le i<j\le r-1} \frac{\lambda_{i,i+1}}{\lambda_{ii}} \frac{\lambda_{j,j+1}}{\lambda_{jj}}$$

If r increases, the only new terms are those with $j = r$

So $$\frac{\lambda_{r,r+1}}{\lambda_{rr}} \cdot \sum_0^{r-1} \frac{\lambda_{i,i+1}}{\lambda_{ii}} = \frac{\lambda_{r,r+1}}{\lambda_{rr}} \varphi(r) \quad (e)$$

$$\frac{\lambda_{r+1,r+1}}{\lambda_{rr}} = \frac{(2r+1)^2(2r+2)^2}{(4r+1)(4r+3)^2(4r+5)} \qquad \frac{\lambda_{r+2,r+2}}{\lambda_{r+1,r+1}} = \frac{(2r+3)^2(2r+4)^2}{(4r+5)(4r+7)^2(4r+9)}$$

$$\frac{\lambda_{r+2,r+2}}{\lambda_{rr}} = \frac{(2r+1)^2(2r+2)^2(2r+3)^2(2r+4)^2}{(4r+1)(4r+3)^2(4r+5)^2(4r+7)^2(4r+9)}$$

$$(c)\ \begin{cases} + & \dfrac{\lambda_{r+3,r+3}}{\lambda_{r+1,r+1}} = \dfrac{(2r+3)^2(2r+4)^2(2r+5)^2(2r+6)^2}{(4r+5)(4r+7)^2(4r+9)^2(4r+11)^2(4r+13)} \\ - & \dfrac{\lambda_{r+1,r+1}}{\lambda_{r-1,r-1}} = \dfrac{(2r-1)^2(2r)^2(2r+1)^2(2r+2)^2}{(4r-3)(4r-1)^2(4r+1)^2(4r+3)^2(4r+5)} \end{cases}$$

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Part d (d).

$$\frac{\lambda_{n+1,n+1}}{\lambda_{nn}}\Delta\varphi(r) = \frac{(2r+1)^2(2r+2)^2}{(4r+1)(4r+3)^2(4r+5)}\Delta\varphi(r)$$

None
New A [1] 1 2
New B [1] 2 2
* $(2n+1)(2n+2)$ in C
C to A
C to B
* $[(2r+1)(2r+2)]^2$ in C
C to A
New B [5] 1 -2 -28 16 160 128
*
C to S numerator in S.

None. New A [1] 1 4
New B [1] 3 4
*
None C to A
New B [1] 3 4
*
None C to A [~~4~~] 8
New B [1] 5 4
*
None C to A
New B [4] 9 -48 -32 256 256
* denom^r in C.
None S to A
C to B
%

Checking of results used in study of λ_{ij} from characteristic eqn

1. $\frac{\lambda_{n,n+1}}{\lambda_{nn}} = \frac{(n+1)^2(2n+1)^2}{(4n+3)^2} - \frac{n^2(2n-1)^2}{(4n-1)^2}$ confirmed.

$n=0$ $\quad \frac{\lambda_{01}}{\lambda_{00}} ?= \frac{1}{9}$ ✓

$n=1$ $\quad \frac{\lambda_{12}}{\lambda_{11}} = \frac{36}{49} - \frac{1}{9} = .623\ 582\ 767.$

$n=2$ $\quad \frac{\lambda_{23}}{\lambda_{22}} = \left(\frac{15}{11}\right)^2 - \frac{36}{49} = 1.124\ 810\ 255?$

From computed results $\lambda_{12}/\lambda_{11} = .623\ 582\ 767$
$\lambda_{23}/\lambda_{22} = 1.124\ 810\ 937.$

The most significant term in $A_{n,n+1}/A_{nn}$ is

$$z = \sum_{0\le i<j\le n-1} \frac{\lambda_{i,i+1}}{\lambda_{ii}} \frac{\lambda_{j,j+1}}{\lambda_{jj}}$$

$$\Delta z = \frac{\lambda_{n,n+1}}{\lambda_{nn}} \sum_0^{n-1} \frac{\lambda_{i,i+1}}{\lambda_{ii}}$$

$$= \Delta\varphi(n) \cdot \varphi(n) \quad \text{with } \varphi(n) = \left[\frac{n(2n-1)}{4n-1}\right]^2$$

$$= [\varphi(n+1) - \varphi(n)]\varphi(n).$$

~~$n=1$. $\Delta z = \sum_{0\le i<j<1} \frac{\lambda_{i,i+1}}{\lambda_{ii}}\frac{\lambda_{j,j+1}}{\lambda_{jj}} = \frac{\lambda_{01}}{\lambda_{00}}\cdot\frac{\lambda_{12}}{\lambda_{11}}$~~

$= \frac{1}{9}\cdot$ ~~$0.623\ 582\ 767 = .069\ 286\ 974$~~

~~$\varphi(0)[\varphi(1) - \varphi(0)] =$~~

$z(2) = \frac{\lambda_{01}}{\lambda_{00}}\cdot\frac{\lambda_{12}}{\lambda_{11}} = \frac{1}{9}\cdot 0.623\ 582\ 767 = .069\ 286\ 974$

~~$z(2) = \frac{\lambda_{01}}{\lambda_{00}}$~~ $\quad \varphi(1)[\varphi(2) - \varphi(1)] = \frac{1}{9}\left[\frac{36}{49} - \frac{1}{9}\right] = \frac{1}{9}\ 0.623\ 582\ 767.$

As $z(1) = 0$, $\quad z(2) = \Delta z(1).$

$$z(3) = \frac{\lambda_{01}}{\lambda_{00}}\frac{\lambda_{12}}{\lambda_{11}} + \frac{\lambda_{01}}{\lambda_{00}}\frac{\lambda_{23}}{\lambda_{22}} + \frac{\lambda_{12}}{\lambda_{11}}\frac{\lambda_{23}}{\lambda_{22}}$$

$$z(3) - z(2) = \frac{\lambda_{23}}{\lambda_{22}}\left(\frac{\lambda_{01}}{\lambda_{00}} + \frac{\lambda_{12}}{\lambda_{11}}\right) = (02)\left[\frac{1}{9} + (01)\right] = .826\ 391\ 208$$

$$[\varphi(3) - \varphi(2)]\varphi(2) = \left[\left(\frac{15}{11}\right)^2 - \frac{36}{49}\right]\cdot\frac{36}{49} = (02)\cdot\frac{36}{49} = .826\ 391\ 207$$

So $\Delta \sum_{0\le i<j\le n-1} = [\varphi(n+1) - \varphi(n)]\varphi(n)$ is confirmed.

Back of M53

$\int x(x-3)^5 dx$ $\quad z = x-3$

$dz = dx$

$x - 3 = z$

$x = z+3$

$\int = \int (z+3) z^5 dz$

$= \int z^6 + 3z^5 dz$

$= \frac{z^7}{7} + \frac{3z^6}{6}$

$= \frac{(x-3)^7}{7} + \frac{(x-3)^6}{2}$

$= (x-3)^6 \left[\frac{x-3}{7} + \frac{1}{2} \right]$

$[\ ] = \frac{2x-6 + 7}{14} = \frac{2x+1}{14}$

$\frac{6z^7 + 21z^6}{42}$

$\frac{z^6(6z+21)}{42}$ $\quad \frac{z^6(2z+7)}{14}$

$2x-6+7 = 2x+1$

If $i=0$, we are simply using the series $f(x,\sigma)$.
What about $i=1$, $k=0$, $j=n-1$?
For $n=3$, $C_{20}\,\sigma^2 x^4 \cdot \frac{\sigma}{9}$

$$\varphi_2(x,\sigma) = 1 + \sigma\left(\frac{x^2}{3}+\frac{1}{9}\right) + \sigma^2\left(\frac{x^4}{5}+\frac{x^2}{15}+\frac{1}{25}\right) + \sigma^3\left(\frac{x^6}{7}+\frac{x^4}{21}+\frac{x^2}{35}+\frac{1}{49}\right) + \cdots$$

$$\varphi_2(0,\sigma) = 1 + \frac{\sigma}{9} + \frac{\sigma^2}{25} + \frac{\sigma^3}{49} + \cdots$$

$$\frac{\varphi_2(x,\sigma)}{\varphi_2(0,\sigma)} = 1 + \frac{\sigma x^2}{3} + \sigma^2\left(\frac{x^4}{5} + \underline{x^2\left(\frac{1}{15}-\frac{1}{27}\right)}\right) \cancel{+\sigma^3\left(\frac{x^6}{7}+\frac{x^4}{21}+\frac{x^2}{35}+\frac{1}{49}\right)}$$
$$+\sigma^3\left(\frac{x^6}{7} + \underline{x^4\left(\frac{1}{21}-\frac{1}{45}\right)}\right) \cdots$$

The underlined terms represent an increase on 0 in $f(x,\sigma)$

$$f(x,\sigma) = 1 + \frac{\sigma x^2}{3} + \frac{\sigma^2 x^4}{5} + \frac{\sigma^3 x^6}{7} + \cdots$$

$$\frac{\varphi_2(x,\sigma)}{\varphi_2(0,\sigma)} - f(x,\sigma) = \underline{\sigma^2 x^2\left(\frac{1}{15}-\frac{1}{27}\right)} + \sigma^3\left[x^4\underline{\left(\frac{1}{21}-\frac{1}{45}\right)} + \cdots\right]$$

$$\varphi_2(x,\sigma) - \varphi_2(0,\sigma)\,f(x,\sigma)$$
$$= 1 + \sigma\left(\frac{x^2}{3}+\frac{1}{9}\right) + \sigma^2\left(\frac{x^4}{5}+\frac{x^2}{15}+\frac{1}{25}\right) + \sigma^3\left(\frac{x^6}{7}+\frac{x^4}{21}+\frac{x^2}{35}+\frac{1}{49}\right) - \cdots$$
$$- \left[1+\frac{\sigma}{9}+\frac{\sigma^2}{25}+\frac{\sigma^3}{49}\cdots\right]\left[1+\frac{\sigma x^2}{3}+\frac{\sigma^2 x^4}{5}+\frac{\sigma^3 x^6}{7}+\cdots\right]$$

$\sigma^2 x^2$ comes from $\sigma^2\frac{x^2}{15}$ in $\varphi_2(x,\sigma)$, less $\frac{\sigma^2 x^2}{27}$ from $\frac{\sigma}{9}\cdot\frac{\sigma x^2}{3}$
There could be a term $\sigma^2 x^2$ in f, but is not at this stage

back of M61

From Professor W.W.Sawyer,
34, Pretoria Road,
Cambridge CB4 1HE.
19th September 1989.

Mr. R. Mirchandani,
Penguin Books Limited,
27 Wrights Lane
W8 5TZ.

Dear Mr. Mirchandani,

Thankyou for your letter of September 5th.

You say something about the possibility of a new preface. At present of course it does not have a preface but goes straight to Chapter 1, which was designed as an opening, and I should like it to stay that way. You may have in mind the fact that it was written nearly 50 years ago. I think you might deal with this in the publisher's blurb, something along the following lines ; this book first appeared in 1943 in the Forces Book Club. It was designed to help those in the forces who needed to become engineers in a hurry. However the fact that it has stayed in print until the present indicates that it has been useful to readers other than those for whom it was originally planned. Since 1943 there have been changes in money, weights and measures. References to the old system are only incidental, and it does not seem necessary to change these.

I think you have my biography up to 1976, when I retired from the University of Toronto and came, with my wife, to live in Canbridge (England). Since then I have written a book showing in detail some practical applications of modern mathematics - a thing the "Modern Mathematics" campaign in U.S.A. notably failed to do. I have also written two booklets with problems and topics to keep the quickest pupils busy when they have finished the regular stint of exercises. Since 1977 I have been meeting a small group of interested secondary school pupils on Saturday mornings.

I hope you find this useful,

With best wishes,

Yours sincerely,

$$\frac{\lambda_{n+1,n+1}}{\lambda_{nn}} = \frac{(\frac{1}{2}x+1)^2(\frac{1}{2}x+2)^2}{(x+1)(x+3)^2(x+5)} \quad \text{where } x = 4n$$

$$= \frac{1}{16}\,\frac{(x+2)^2(x+4)^2}{(x+1)(x+3)^2(x+5)}$$

Partial fractions: $x=-1$ gives $\frac{9}{16}$. $x=-5$ gives $-\frac{9}{16}$ in fraction without $1/16$

$$\frac{9/16}{x+1} - \frac{9/16}{x+5} = \frac{9/4}{(x+1)(x+5)}$$

$$(x+2)^2(x+4)^2 - \frac{9}{4}(x+3)^2 = (x^2+6x+8)^2 - \left(\frac{3x+9}{2}\right)^2$$

4 times this $= (2x^2+12x+16)^2 - (3x+9)^2$

$$= (2x^2+15x+25)(2x^2+9x+7)$$

$$= (x+5)(2x+5)(x+1)(2x+7)$$

so $$\frac{(x+2)^2(x+4)^2}{(x+1)(x+3)^2(x+5)} - \frac{9/4}{(x+1)(x+5)} = \frac{(2x+5)(2x+7)}{(x+3)^2}$$

$$= 4 - \frac{1}{(x+3)^2}$$

Hence $$\frac{\lambda_{n+1,n+1}}{\lambda_{nn}} = \frac{1}{16}\left\{\frac{9/16}{x+1} - \frac{9/16}{x+5} - \frac{1/4}{(x+3)^2} + 1\right\}$$

$$= \frac{1}{16}\left\{1 + \frac{9/16}{4n+1} - \frac{9/16}{4n+5} - \frac{1/4}{(4n+3)^2}\right\}$$

$$= \frac{1}{16}\left\{1 + \frac{9/4}{(4n+1)(4n+5)} - \frac{1/4}{(4n+3)^2}\right\}$$

A detailed analysis of (c) may be unnecessary, as this part is of low order

$$\Delta\frac{\lambda_{n+1,n+1}}{\lambda_{nn}} = \frac{1}{16}\left\{\begin{array}{l} \frac{9/16}{4n+5} - \frac{9/16}{4n+9} - \frac{1/4}{(4n+7)^2} \\ -\frac{9/16}{4n+1} + \frac{9/16}{4n+5} + \frac{1/4}{(4n+3)^2} \end{array}\right\}$$

$$= \frac{1}{16}\left\{\frac{-32}{(4n+1)(4n+5)(4n+9)} + \frac{32n+40}{(4n+7)^2(4n+3)^2}\right\}$$

$$= \frac{2n+2.5}{(4n+3)^2(4n+7)^2} - \frac{2}{(4n+1)(4n+5)(4n+9)}$$

$$= O\left(\frac{1}{n^3}\right)$$

$\varphi(n+1)$ is $O(n^2)$ so $\varphi(n+1)\Delta\frac{\lambda_{n+1,n+1}}{\lambda_{nn}} = O\left(\frac{1}{n}\right)$ and does not make any contribution to the integral part.

$$\frac{\lambda_{n+1,n+3}}{\lambda_{n+1,n+1}} = \overset{a}{\Delta} \frac{A_{n,n+2}}{A_{nn}} - \overset{e}{\Delta} \sum\sum_{0\le i<j\le n-1} \frac{\lambda_{i,i+1}}{\lambda_{ii}} \frac{\lambda_{j,j+1}}{\lambda_{jj}} - \overset{c}{\Delta}\left\{ \frac{\lambda_{n+2,n+2}}{\lambda_{nn}} + \frac{\lambda_{n+1,n+1}}{\lambda_{n-1,n-1}} \right\} +$$

$$+ \overset{d_1}{-} \frac{\lambda_{n+1,n+1}}{\lambda_{nn}} \Delta\varphi(n) \overset{d_2}{-} \left(\Delta\varphi(n)\right)^2 \overset{d_3}{-} \varphi(n+1)\Delta \frac{\lambda_{n+1,n+1}}{\lambda_{nn}} \overset{d_4}{-} \varphi(n+1)\Delta^2\varphi(n)$$

$$\overset{f}{-} \Delta \frac{\lambda_{n+1,n+2}}{\lambda_{nn}}.$$

$+\frac{285}{512} + \frac{55}{64}n + \frac{1}{2}n^2$ is quotient part of (a) – (e).

$$c = \frac{\lambda_{n+3,n+3}}{\lambda_{n+1,n+1}} - \frac{\lambda_{n+1,n+1}}{\lambda_{n-1,n-1}}$$

$$\frac{\lambda_{n+1,n+1}}{\lambda_{nn}} = \frac{1}{16}\left\{1 + \frac{9/4}{(4n+1)(4n+5)} - \frac{1/4}{(4n+3)^2}\right\} = \frac{1}{16}\left\{1 + O\left(\frac{1}{n^2}\right)\right\}$$

$$\therefore \frac{\lambda_{n+2,n+2}}{\lambda_{nn}} = \frac{\lambda_{n+2,n+2}}{\lambda_{n+1,n+1}} \cdot \frac{\lambda_{n+1,n+1}}{\lambda_{nn}} = \frac{1}{256}\left\{1 + O\left(\frac{1}{n^2}\right)\right\}$$

$\therefore \frac{\lambda_{n+3,n+3}}{\lambda_{n+1,n+1}} - \frac{\lambda_{n+1,n+1}}{\lambda_{n-1,n-1}} = O\left(\frac{1}{n^3}\right)$ so omitted from quotient study.

d_3: $\Delta \frac{\lambda_{n+1,n+1}}{\lambda_{nn}} = O\left(\frac{1}{n^3}\right)$ $\varphi(n+1) = O(n^2)$ $\varphi_{n+1}\Delta\frac{\lambda_{n+1,n+1}}{\lambda_{nn}}$ no contrib^n to quotient.

d_1 See 33, ~~x~~

d_2 See 32.

d_4 See 31. $\frac{7}{128} + \frac{3}{16}n + \frac{1}{8}n^2$

Check on p.49.
For $S_3!$, $\chi_0^{[3]}=1$ $\chi_0^{[21]}=2$ $\chi_0^{[111]}=1$.

The squares of these are involved in $A^{\lambda,n\in 3}/A_{rr}$.
We require $P^{n[3]}/P_n$, $P^{n[21]}/P_r$, $P^{n[111]}/P_r$.

p.656. Partition [edcba] gives $\frac{1}{n}$ coeff.
$$\tfrac{3}{2}(a^2+b^2+c^2+d^2+e^2)-12a-9b-6c-3d.$$
I think we get correct results by taking $a=0$, $b=0$ etc.
[3]. $e=3$, rest $=0$. $\frac{3}{2}(3^2)=27/2=13\frac{1}{2}$ agrees with p.48.
(21) $e=2, d=1$, rest $=0$. $\frac{3}{2}(4+1)-3=\frac{15}{2}-3=4\frac{1}{2}$ agrees.
[111] $e=1, d=1, c=1$. $\frac{3}{2}(1+1+1)-3-6=4\frac{1}{2}-9=-4\frac{1}{2}$ agrees.
$$\left(1+\frac{13\frac{1}{2}}{n}\cdots\right)+4\left(1+\frac{4\frac{1}{2}}{n}\cdots\right)+\left(1-\frac{4\frac{1}{2}}{n}\cdots\right)$$
$$=6+\frac{27}{n}=6\left(1+\frac{4\frac{1}{2}}{n}\cdots\right)$$ in agreement w. p.49.

P.92 suggests that it may be necessary to consider several terms for asymptotic behaviour. Consider $\sum_{i\le k}$ process with estimates of accuracy.

p.88 shows $\displaystyle\sum_{0}^{n} \frac{\lambda_{i,i+1}}{\lambda_{ii}} = \frac{(n+1)^2(2n+1)^2}{(4n+3)^2}$

$$= \frac{n^2}{4} + \frac{3}{8}n + \frac{7}{64} + \left[\frac{1}{8(4n+3)}\right]^2$$

Argument of p. 87.

$$\frac{\lambda_{n+1,n+2}}{\lambda_{n+1,n+1}} = \Delta(\ ) = \frac{n}{2} + \frac{5}{8} + O\left(\frac{1}{n^3}\right)$$

$$\Delta \sum_{i,j} = \left(\frac{n}{2} + \frac{5}{8} + O\left(\frac{1}{n^3}\right)\right)\left(\frac{n^2}{4} + \frac{3}{8}n + \frac{7}{64} + O\left(\frac{1}{n^2}\right)\right)$$

$$= \frac{n^3}{8} + \frac{11}{32}n^2 + \frac{37}{128}n + \frac{35}{512} + O\left(\frac{1}{n}\right).$$

$$\therefore \sum_{0\le i<j\le n} \frac{\lambda_{i,i+1}}{\lambda_{ii}}\frac{\lambda_{j,j+1}}{\lambda_{j,j}} = \frac{n^4}{32} + \frac{5}{96}n^3 + \frac{1}{256}n^2 - \frac{29}{1536}n + O(\ln n).$$

$a(4n^3+6n^2+4n+1)$
$+b(3n^2+3n+1)$
$+c(2n+1)$
$+d$

$4a = \frac{1}{8}$ $a = 1/32$

$3b = \frac{11}{32} - \frac{6}{32} = \frac{5}{32}$ $b = \frac{5}{96}$

$2c = \frac{37}{128} - \frac{4}{32} - \frac{5}{32} = \frac{1}{128}$

$c = 1/256$

$d = \frac{35}{512} - \frac{1}{32} - \frac{5}{96} - \frac{1}{256}$

Back of M 70

2 $n=4$ $(x_1\ x_2\ x_3\ x_4.)$ 4-vector. $\sum x_i^2$ simplest scalar.
Quadratic form $\sum a_{ik} x_i x_k$. ~~is called~~ Coefficients called 10-tensor. Invariants, the 4 coeffs in $\det(a_{ik} - \lambda\delta_{ik})$
An antisymmetric bilinear form $\sum \lambda_{ik}(x_i y_k - x_k y_i)$ is called a 6-tensor. The two simplest scalars are $\sum \lambda_{ik}^2$ and $\Lambda = \lambda_{12}\lambda_{34} + \lambda_{13}\lambda_{42} + \lambda_{14}\lambda_{23}$.

Note. Terminology has changed since spread of general relativity theory.

back of O3

Change of axes.

A matrix is involved whenever we change axes. The point (x,y) in our original axes is associated with the vector $\begin{pmatrix}x\\y\end{pmatrix}$, which is equal to $x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix}$. This equation brings out the fact that $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ are the unit vectors along the axes. When we change axes, other vectors are made to play these roles, so we interpret the point (X,Y) as corresponding to the vector $X\begin{pmatrix}a\\b\end{pmatrix} + Y\begin{pmatrix}c\\d\end{pmatrix}$. Here of course the co-ordinates (a,b) and (c,d) refere to our original graph paper. In the new system they will be $(1,0)$ and $(0,1)$. Accordingly the connection between the old and new co-ordinates is given by the equation $\begin{pmatrix}x\\y\end{pmatrix} = X\begin{pmatrix}a\\b\end{pmatrix} + Y\begin{pmatrix}c\\d\end{pmatrix}$ i.e. $x=aX+cY$, $y=bX+dY$, which is equivalent to the matrix equation $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}a & c\\b & d\end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix}$.

If the co-ordinates in the old system are the components of the vector v, and in the new system of the vector V, these will be related by some matrix S, so that $v=SV$. We shall use $V=S^{-1}v$ when we need to express the new co-ordinates in terms of the old ones.

It is important to notice that a matrix expressing a transformation and a matrix defining a conic **behave differently under change of axes.**

For the transformation $v \to v^*$ we have $v^* = Mv$. When the axes are changed by the matrix S we have $v=SV$ and $v^*=SV^*$, so $SV^* = MSV$, which means $V^*=S^{-1}MSV$, and the matrix for the transformation in the new axes is $S^{-1}MS$.

On the other hand, when the matrix M defines the conic $v^TMv = c$, $v=SV$ and $v^T = V^TS^T$. Accordingly the conic $v^TMv=c$ appears in the form $V^TS^TMSV = c$, and the matrix appearing here is S^TMS, which is not, as a rule, the same as $S^{-1}MS$, which we had for the transformation.

Later we shall meet an inportant exception in which the matrices still agree in the new system.

Back of O4

If we write Q for the matrix $S^T S$, we want $U^T QV$ to be the same as $U^T V$, whatever U and V are. Let U and V be the column vectors with elements U_i and V_i, where i goes from 1 to n. Then $U^T V = \sum U_i V_i$, the sum being from 1 to n, while $U^T QV = \sum\sum U_i Q_{ij} V_j$, i and j being summed from 1 to n. Here the coefficient of $U_i V_j$ is Q_{ij}. In $U^T V$ the coefficient of $U_i V_j$ is 1 when $i=j$ and 0 when i and j are unequal. This means that Q must equal the identity matrix, that is, that $S^T S = I$.

We have shown that the transformation S preserves scalar products if, and only if, $S^T S = I$. This means that $S^T = S^{-1}$; the inverse and the transpose must be the same. This means that, when we are dealing with an S of this type, we need not bother whether a matrix M is being used to define a transformation or a conic. Both will behave in the same way when they undergo S.

A transformation, S, that satisfies this condition is said to be **orthogonal**. A matrix that specifies a change of position by rotation or reflection is clearly orthogonal; we know that lengths and angles are not changing. The same applies if we are changing axes by rotating or reflecting them.

Usually finding the inverse of a matrix is liable to be awkward. In the case of an orthogonal matrix this difficulty disappears. We simply transpose it.

The determinant of an orthogonal transformation.

A matrix and its transpose have the same determinant. Also the determinant of the product of two matrices is the product of their determinants. If we consider the equation $S^T S = I$, and take determinants, we see that the square of the determinant of S must be 1. Accordingly, the determinant itself of S must be +1 or -1, and thus orthogonal transformations fall into two classes. Those with determinant +1 turn out to be rotations; they can be achieved by a continuous motion. Those with determinant -1 cannot. We can **imagine** a right-hand glove being transformed into a left-hand glove, but there is no way in which we can actually turn the one to become the other.

Back of Pg 3   Pl. 3.

A problem I was working on involved the principal integral $\int_0^1 \frac{-\ln t\,dt}{k-t}$ with $0 < k < 1$.
Infinity occurs near $t = k$

```
K=C/N WITH N=?20
C=?7
F(0.142857143)=9.98577425
F(0.285714286)=9.21034037
F(0.428571429)=9.48559992
F(0.571428571)=10.7295861
F(0.714285714)=13.8629436
F(0.857142857)=24.0794561
F(1.14285714)=-18.3258146
F(1.28571429)=-7.98507696
F(1.42857143)=-4.6209812
F(1.57142857)=-2.989185
F(1.71428571)=-2.04330249
F(1.85714286)=-1.43594305
F(2)=-1.01907127
F(2.14285714)=-0.719205181
F(2.28571429)=-0.495874558
F(2.42857143)=-0.325037859
F(2.57142857)=-0.191564574
F(2.71428571)=-0.0854888241
```

The Vital Roles
of Maths Clubs.

Back of P13

I regard maths clubs as the decisive instrument for the improvement of mathematical education.

Teacher Training

Teaching is a very complicated process. You have three things simultaneously in mind — the mathematics itself, the difficulties ~~children~~ pupils will have in understanding it, and keeping order. When I was training teachers, I tried to separate these. First we would get clear about the mathematics. Then the students would take one or two children and try to help them with their difficulties. After that, they would run a club for 9–10 year olds, who voluntarily stayed after school for mathematical stimulation. Children of that age still show the enthusiasm that ought to be the norm in a mathematics class.

In Canada, I used to ask future teachers to pick something they regarded as a defect in the educational system and run a small club in an attempt to remedy it. Some felt that bright students, if being ~~were~~ held back, ~~and~~ became bored and delinquent. They offered more advanced work to these. Others were struck by the plight of the physically active ~~and~~ child, who dislikes sitting at a desk. They started a club for six restless boys, selected by the principal of an elementary school. The idea was to have athletic competitions, with elaborate scoring, records of averages etc and see how much mathematics they could bring in without the boys noticing it.

Improvement of Education.

A club then is not simply a luxury or an entertainment. Rather it represents a situation in which you strive to provide something that is lacking in existing schools, or is not done as well as you think it should be.

Mathematics, like any other subject, flourishes only when taught by those who both love it and are able to communicate their enthusiasm to their classes. There have always been some such teachers, who have preserved the tradition of mathematics as a living subject. But their numbers have been limited: many teachers in their childhood met [illegible] lifeless husk of dead

Let $c_i(\sigma) = \sum_{j=0}^{\infty} w_{ij}\sigma^j$

Then $F(x,\sigma) = \sum x^i \sum w_{ij}\sigma^j$

$= \sum_j \sigma^j (\sum_i x^i w_{ij})$

Coeff of σ^j is of degree $j+q$.

$\therefore \sum_j x^i w_{ij}$ is of degree $j+q$.

$\therefore w_{ij} = 0$ if $i > j+q$

i.e. if $j < i-q$.

Example. For $q=2$,

$$\begin{array}{cccc} w_{00} & w_{01} & w_{02} & w_{03} \\ w_{10} & w_{11} & w_{12} & w_{13} \\ w_{20} & w_{21} & w_{22} & w_{23} \\ \cdot & w_{31} & w_{32} & w_{33} \\ \cdot & \cdot & w_{42} & w_{43} \\ \cdot & \cdot & \cdot & w_{53} \end{array}$$

Here $c_5(\sigma) = w_{53}\sigma^3 + \ldots$

So has factor σ^3, which is $i-q$ power of σ.

Let $d_i(\sigma) = \sigma^q c_i(\sigma)$.

Then $d_i(\sigma)$ has factor σ^i. As σ^q does not depend on x, $d_i(\sigma)$ is still an eigenvector of (F)

$\lambda d = A^2 H d$. $AHA . A^{-1}d = \lambda . A^{-1}d$

Let ~~$b = A^{-1}d$~~ $b = A^{-1}d$ $d_i(\sigma)$ has factor σ^i

$b_i = a^i \sum b_{ij}\sigma^j$ $b_i(\sigma) = a^{-i} d_i(\sigma)$

so has fact $a^{-i}\sigma^i = a^i$

Let $b_i = a^i \sum b_{ij} a^{2j}$

$Tb = Mb$. $\sum_s T_{rs} b_s = M b_r = b_r \sum M_i a^{2i}$

The commuting matrix T has

$T_{ii} = -(1+\alpha^2)\, i(i+1)$

$T_{i,i+1} = \alpha(i+1)^2$

$T_{i+1,i} = \alpha(i+1)^2$ $T_{r,r-1} = \sigma r^2$

Back of P 20 (i)

$$\lambda(\sigma) F(x,\sigma) = \int_0^1 \frac{F(y,\sigma)\,dy}{1-\sigma x y}$$

1) Let $F(x,\sigma) = \sum_0^\infty \sigma^t f_t(x)$.

$$\sum \lambda_r \sigma^r \sum \sigma^t f_t(x) = \int_0^1 \sum \sigma^i x^i y^i \sum \sigma^j f_j(y)\,dy$$

Coeff of σ^n

$$\sum_{r=0}^{n} \lambda_r f_{n-r}(x) = \sum_{i=0}^{n} x^i \int_0^1 y^i f_{n-i}(y)\,dy$$

If $\lambda_0 \dots \lambda_{q-1}$ are 0, $\lambda_q \neq 0$ $n=q$ gives

$$\lambda_q f_0(x) = \sum_{i=0}^{q} x^i \int_0^1 y^i f_{q-i}(y)\,dy.$$

$\therefore$ $f_0(x)$ is of degree q.

$f_n(x)$ of degree $q+n$.

2) Let $F(x,\sigma) = \sum c_i(\sigma) x^i$

$$\lambda(\sigma) \sum c_i(\sigma) x^i = \int_0^1 \sum_i \sigma^i x^i y^i \cdot \sum_k c_k(\sigma) y^k\,dy$$

$$= \sum x^i \sum_k \frac{c_k(\sigma)\,\sigma^i}{k+i+1}$$

$$\therefore \lambda(\sigma) c_i(\sigma) = \sum_k \frac{\sigma^i c_k(\sigma)}{k+i+1}$$

Thus $\lambda(\sigma)$ is eigenvalue of

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \dots \\ \frac{\sigma}{2} & \frac{\sigma}{3} & \frac{\sigma}{4} & \dots \\ \frac{\sigma^2}{3} & \frac{\sigma^2}{4} & \frac{\sigma^2}{5} & \dots \end{pmatrix} \qquad (I)$$

$= A^2 H$ where $H = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \end{pmatrix}$

$A = \mathrm{diag}(1, \alpha, \alpha^2 \dots)$, $\alpha^2 = \sigma$.

Back of P24 *

$q = 2$. $\mathcal{E}(2,0)$ gives $M_0 b_{20} = -2.3\, b_{20}$ $\therefore M_0 = -6$.

$\mathcal{E}(0,2)$ gives $M_0 = b_{11}$ $\therefore b_{11} = -6$.

$\mathcal{E}(1,1)$ $\quad M_0 b_{11} + M_1 b_{10} = b_{01} - 2[b_{11} + b_{10}] + 4 b_{20}$

$b_{10} = 0$ since $1+0 < 2$. Also b_{01}.

$$-6 b_{11} = -2 b_{11} + 4 b_{20}$$

$$-4 b_{11} = 4 b_{20} \quad b_{20} = -b_{11} = +6$$

$$\begin{array}{ccc} 0 & 0 & 1 \\ 0 & -6 & \\ 6 & & \end{array}$$

$\mathcal{E}(30)$

$M_0 b_{30} = 9 b_{20} - 12[b_{30} + \cancel{b_{21}}] + \cancel{}$

$\therefore 6 b_{30} = 9 b_{20} \quad b_{30} = \dfrac{9 \times 6}{6} = 9$.

$\mathcal{E}(0,3)$. $M_1 = b_{12}$

$\mathcal{E}(2,1)$ $\quad M_0 b_{21} + M_1 b_{20} = 4 b_{11} - 6[b_{21} + b_{20}] + 9 b_{30}$

$\mathcal{E}(q,1)$

$$M_0 b_{q1} + M_1 b_{q0} = q^2 b_{q-1,1} - q(q+1)[b_{q1} + b_{q0}] + \\ + (q+1)^2 b_{q+1,0}$$

By this stage we know
M_0, b_{q0}, ~~b_{q1}~~, $b_{q-1,1}$, $b_{q+1,0}$

$\underline{q=2}$ $\quad 6 M_1 = -24 - 36 + 81$

$$= 21 \quad M_1 = \frac{21}{6} = \frac{7}{2}.$$

$M_1 = \frac{1}{2}(q^2 + q + 1) = \frac{1}{2}(4 + 2 + 1) = \frac{7}{2}$ checks.

Back of P 30

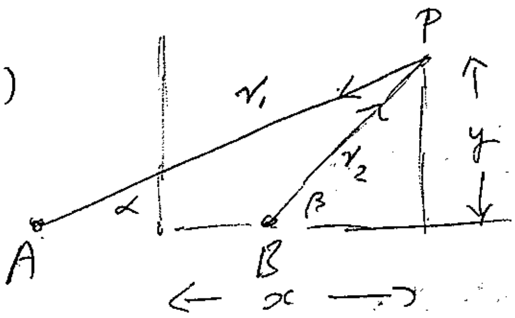
Field of a bar magnet.

$+m$ at $(a,0)$; $-m$ at $(-a,0)$

Force in BP is m/r_2^2

Horizontal component is

$$\frac{m}{r_2^2}\cos\beta = \frac{m(x-a)}{r_2^3}$$



Horizontal component of force in PA is $-\frac{m}{r_1^2}\cos\alpha = -\frac{m(x+a)}{r_1^3}$

Total horizontal component is $\frac{m(x-a)}{r_2^3} - \frac{m(x+a)}{r_1^3}$

If $y=0$, ~~$r_2 = y-a$, $r_1 = y$~~ $r_2 = x-a$, $r_1 = x+a$

$$H_x = \frac{m(x-a)}{|(x-a)^3|} - \frac{m(x+a)}{|(x+a)^3|} = \cancel{\frac{m}{(x-a)^2} - \frac{m}{(x+a)^2}}$$

$$= \cancel{\frac{m}{(x^2-a^2)^2}}\left[x+a. \right.$$

$$= \frac{m\,\mathrm{sgn}(x-a)}{(x-a)^2} - \frac{m\,\mathrm{sgn}(x+a)}{(x+a)^2}$$

If $x>a$, $H_x = \frac{m}{(x-a)^2} - \frac{m}{(x+a)^2} = \frac{m \cdot 4ax}{(x^2-a^2)^2}$

Back of P 34

$$I = \iiiint \frac{yz}{(z-x)(y-r)} \frac{1}{1-\alpha yz} \, dx\,dy\,dz\,dr$$

$$\text{pric. } \int_{-1}^{1} (z-x)\,dx = \ln|1+z| - \ln|1-z|$$

$$\int_{-1}^{1} \frac{1}{y-r}\,dr = \ln(1+y) - \ln(1-y).$$

$$\iint [\ln(1+z) - \ln|1-z|][\ln|1+y| - \ln|1-y|] \frac{yz}{1-\alpha yz} \, dy\,dz$$

If y, z both change signs, integrand is unchanged

$$\therefore = 2\int_{z=0}^{1}\int_{y=-1}^{1} \{\ \}\cdot[\ ]\, \frac{yz}{1-\alpha yz}\, dy\,dz$$

Let $u = yz \quad v = z \quad \frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} z & y \\ 0 & 1 \end{vmatrix} = z.$

$$I = 2\int_{v=0}^{1}\int_{u=-v}^{v} [\ln(1+v) - \ln(1-v)][\ln|1+\tfrac{u}{v}| - \ln|1-\tfrac{u}{v}|] \frac{u/v}{1-\alpha u}\, du\,dv$$

$$= 2\int_{v=0}^{1}\int_{u=-v}^{v} \left[\frac{\ln(1+v) - \ln(1-v)}{v}\right][\ln(u+v) - \ln(v-u)] \frac{u}{1-\alpha u}\, du\,dv$$

$$= 2\int_{u=-1}^{1}\int_{|u|=v}^{1} = 2\int_{u=0}^{1}\int_{v=u}^{1} + 2\int_{u=-1}^{0}\int_{v=-u}^{1}$$

Latter part

$$= 2\int_{U=1}^{0}\int_{v=U}^{1} \frac{\ln(1+v) - \ln(1-v)}{v} \cdot [\ln(v-U) - \ln(v+U)] \frac{-U}{1+\alpha U} (-dU)\,dv$$

$$= 2\int_{U=0}^{1}\int_{v=U}^{1} \frac{\ln(1+v) - \ln(1-v)}{v} \cdot [\ln(v+U) - \ln(v-U)] \frac{U}{1+\alpha U}\, dU\,dv$$

(4 changes of sign)

Whole

$$= 4\int_{u=0}^{1}\int_{v=u}^{1} \frac{\ln(1+v) - \ln(1-v)}{v} \cdot [\ln(v+u) - \ln(v-u)] \frac{u}{1-\alpha^2 u^2}\, du\,dv.$$

Back of S 9

Singularities at $-1, 1, \infty$
Put $x = 1+t$
Let

Back of S_{10}

Let $\mu = \cos t$ $\quad (1-\mu^2)^{m/2} \, z$

Becker's method

[2] Coefficient of $\sigma^{(0+1+\dots+n)+2}$.

$p_0+p_1+\dots+p_n = (0+1+\dots+n)+2$

$p_0<p_1<\dots<p_n$.

If we start with $0,1,\dots n$ and change i to $i+1$, we must increase by 1 at least all the numbers from i to n. Hence there can be at most 2 such numbers. Accordingly the only possibilities are

$p_{n-1}=n,\ p_n=n+1$ and $p_n=n+2$

the others having $p_i=i$.

$r_i\le p_i$ so for all the ~~change~~ "unchanged" numbers it follows that $r_i=i$.

$p_n=n+2$ allows $r_n=n$, $r_n=n+1$, $r_n=n+2$.

$\underline{p_n=n+2,\ r_n=n}$ we have $\begin{vmatrix} a_{00} & & \\ & \ddots\, a_{n-1,n-1} & \\ & & a_{n,n+2}\end{vmatrix}$

with zeros below diagonal. These zeros occur in the other cases as well. Thus we get

$h_{00}\dots h_{n-1,n-1}\ a_{00}a_{11}\dots \underline{a_{n-1,n-1}}(a_{n,n+2}\overset{b_{\cdot\cdot}}{\wedge}+a_{n+1,n+2}\overset{b_{\cdot\cdot}}{\wedge}+a_{n+2,n+2}\overset{b_{\cdot\cdot}}{\wedge})$

$p_{n-1}=n,\ p_n=n+1.$ $\quad r_{n-1}=n-1$ or n.

If $r_{n-1}=n-1$, then $r_n=n$ or $n+1$.

If $r_{n-1}=n$ then $r_n=n+1$.

Factor $a_{00}\dots a_{n-2,n-2}$ in all cases here.

$\begin{vmatrix} a_{n-1,n} & a_{n-1,n+1}\\ a_{n,n} & a_{n,n+1}\end{vmatrix}\begin{vmatrix} b_{n,n-1} & b_{n,n}\\ b_{n+1,n-1} & b_{n+1,n}\end{vmatrix}$

$\begin{vmatrix} a_{n-1,n} & a_{n-1,n+1}\\ a_{n+1,n} & a_{n+1,n+1}\end{vmatrix}\begin{vmatrix} b_{n,n-1} & b_{n,n+1}\\ b_{n+1,n-1} & b_{n+1,n+1}\end{vmatrix}$

$\begin{vmatrix} a_{n,n} & a_{n,n+1}\\ a_{n+1,n} & a_{n+1,n+1}\end{vmatrix}\begin{vmatrix} b_{n,n} & b_{n,n+1}\\ b_{n+1,n} & b_{n+1,n+1}\end{vmatrix}$

Differences lead to a coefficient of n that is

$$\sum_{s=0}^{k} \sum_{i=1}^{a_s} (i-s).$$

$$\sum_{i=1}^{a_s} (i-s) = \frac{a_s(1+a_s)}{2} - s a_s$$
$$= \tfrac{1}{2} a_s^2 - (s-\tfrac{1}{2}) a_s$$ checked.

Sum, $s=0$ to k. We get $\frac{1}{2}\sum_0^k a_s^2 - \sum_0^k (s-\frac{1}{2}) a_s$ ✓

Together we have $\frac{1}{2}\sum_0^k a_s^2 + \sum_0^k (-s+\frac{1}{2}) a_s$

$$\frac{1}{4}\sum_0^k a_s^2 + \sum_0^k \left(\frac{k}{2} - \frac{s}{2}\right) a_s$$
$$+ \sum_0^k \left(-\frac{k}{2}\right) a_s$$
$$= \frac{3}{4}\sum_0^k a_s^2 + \sum_0^k \left(\frac{1}{2} - \frac{3}{2}s\right) a_s$$

Squaring doubles this. ✓ Checked. p. 65b.

$$\sum 7^2 \to \frac{3}{2}\sum_0^k a_s^2 + \sum_0^k (1-3s) a_s.$$

Diagonals. $\prod_{s=0}^{k} \frac{(4n-4s+1)}{(4n-4s+4a_s+1)}$ (f)

(g) gives $\sum_{s=0}^{k} \left(\frac{1}{4}-s\right) - \sum_{s=0}^{k} \left(a_s + \frac{1}{4} - s\right) = -\sum_{s=0}^{k} a_s.$

Together we finally have for the coefficient of $1/n$

$$\frac{3}{2}\sum_0^k a_s^2 - 3\sum_0^k s a_s.$$

First part, $2k+2ys+1$ is

$$\frac{\prod_{s=0}^{k}\prod_{i=1}^{a_s}(2n-2s-1+2i) \quad (b)}{\prod_{s=0}^{k}\prod_{i=1}^{a_s}(4n-2s-2k-1+2i) \quad (c)}$$

Taking out factors 2 and 4, we are led to

$$\sum_{i=1}^{a_s}\left(i-s-\tfrac{1}{2}\right) - \sum_{i=1}^{a_s}\left(\frac{i}{2}-\frac{s}{2}-\frac{k}{2}-\frac{1}{4}\right)$$

$$= \sum_{i=1}^{a_s}\left(\frac{i}{2} - \frac{1}{4} - \frac{s}{2}+\frac{k}{2}\right)$$

$$= \frac{1}{2}a_s\left(\frac{1+a_s}{2}\right) + a_s\left(\frac{k-s}{2} - \frac{1}{4}\right)$$

$$= \frac{1}{4}a_s^2 + a_s\frac{k-s}{2}$$ Sum over s. Checked against p. 65b.

Second part: $\prod_{i=s+1}^{k}(4n-2s-2i+1) \Big/ \prod_{i=s+1}^{k}(4n+2a_s+2a_i-2s-2i+1)$ (d) (e)

Checked against p65b.

We have $\sum_{i=s+1}^{k}\left(\frac{1}{4}-\frac{s}{2}-\frac{i}{2}\right) - \sum_{i=s+1}^{k}\left(\frac{1}{4}-\frac{s}{2}-\frac{i}{2}+\frac{a_s}{2}+\frac{a_i}{2}\right)$

Note: $s=k$ gives no sum, so later sum for $s=0$ to $k-1$.

Above $= -\sum_{i=s+1}^{k}\left(\frac{a_s}{2}+\frac{a_i}{2}\right)$, summed from $s=0$ to $s=k-1$

Consider as example $\sum_{s=0}^{3}\sum_{i=s+1}^{4}(a_s+a_i)$

It gives
$$\begin{array}{llllll}
a_0 + a_1 & & & & & \\
+a_0 & + a_2 & & & & \Bigg\} s=0\\
+a_0 & & +a_3 & & & \\
+a_0 & & & +a_4 & & \\
& a_1 + a_2 & & & & \\
& a_1 & +a_3 & & & \Big\} s=1\\
& a_1 & & +a_4 & & \\
& & a_2 + a_3 & & & \Big\} s=2\\
& & a_2 & +a_4 & & \\
& & a_3 & +a_4 & & \} s=3
\end{array}$$

$= 4(a_0+a_1+a_2+a_3+a_4)$.

So sum above $= -\frac{k}{2}\sum_0^k a_s$

Back of 518

Approximation to $n!$

$$n! = \int_0^\infty x^n e^{-x}\,dx$$

$$\ln(x^n e^{-x}) = n\ln x - x$$

$$\frac{d}{dx}\ln(x^n e^{-x}) = \frac{n}{x} - 1 = \frac{n-x}{x}$$

This is $+$ve $x < n$; 0 $x = n$; $-$ve $x > n$ $\therefore$ maximum at $x = n$.

Let $x = y + n$.

$$n! = \int_{-n}^{\infty} (y+n)^n e^{-n-y}\,dy$$

$$= e^{-n} n^n \int_{-n}^{\infty} \left(1+\frac{y}{n}\right)^n e^{-y}\,dy$$

$$\left(1+\frac{y}{n}\right)^n = e^{n\ln\left(1+\frac{y}{n}\right)}$$

$$= e^{n\left[\frac{y}{n} - \frac{y^2}{2n^2} + \frac{y^3}{3n^3} \cdots\right]} \quad \text{for } |y| < n$$

$$= e^{y - y^2/2n + \frac{y^3}{3n^2} \cdots}$$

$$= e^{y - \frac{y^2}{2n} + \frac{y^3}{3n^2}}$$

~~So $n! = n^n e^{-n}\int_{-n}^{\infty} e^{-\frac{y^2}{2n} + \frac{y^3}{3n^2}}$~~

$$\int_0^k x^n e^{-x}\,dx < \int_0^k x^n\,dx = \left(\frac{x^{n+1}}{n+1}\right)_0^k = \frac{k^{n+1}}{n+1}$$

If $k = n(1-\alpha)$

$$k^{n+1} = n^{n+1}(1-\alpha)^{n+1}$$

If $\alpha = \frac{1}{n+1}$

Back of S 22

$$a\,\frac{e^{at}-1}{e^{a}-1} = \sum_{0}^{\infty} P_r(t)\,a^r$$

If $t=0$, LHS $= \dfrac{a(1-1)}{e^a-1} = 0.$

$\therefore\ P_r(0) = 0$ for all r.

If $t=1$, LHS $= \dfrac{a(e^a-1)}{e^a-1} = a$

$\therefore\ P_r(1) = 0$ except for $r=1$.

$\therefore\ \underline{P_r(1) = 0 \text{ for } r \geq 2}$

Consider beginning of series

$$a\,\frac{e^{at}-1}{e^a-1} = a\,\frac{at + \frac{1}{2}a^2t^2 + \ldots}{a + \frac{1}{2}a^2 + \ldots}$$

$$= \left(1 - \tfrac{1}{2}a \ldots\right)\left(at + \tfrac{1}{2}a^2t^2\right)$$

$$= at + a^2\left(\tfrac{1}{2}t^2 - \tfrac{1}{2}t\right)$$

$\therefore\ P_0(t) = 0 \quad P_1(t) = a \quad P_2(t) = \frac{1}{2}t^2 - \frac{1}{2}t$

$\therefore\ P_2'(t) = t - \frac{1}{2}$

Back of S 32

$$1/z = (1/2)z^{-2} + (z+1)^{-2} + (z+2)^{-2} + \ldots\ldots\ldots\ldots$$
$$-B_2 z^{-3} - B_4 z^{-5} + \ldots\ldots\ldots -B_{2r} z^{-2r-1}$$
$$+R \qquad (7)$$

where R stands for a remainder term involving an infinite sum. The top line here, if we ignore the first term, is equal to L"(z). Solving for L"(z) we find

$$L''(z) = 1/z - 1/(2z^2) + B_2 z^{-3} + B_4 z^{-5} + \ldots . B_{2r} z^{-2r-1} - R \qquad (8)$$

Integrating gives

$$L'(z) = \ln(z) + 1/(2z) - B_2/(2z^2) - B_4/(4z^4) \ldots - B_{2r}/(z^{2r+1}) - R^* \qquad (9)$$

where R* is a remainder found by integrating R.

Integrating again, we have

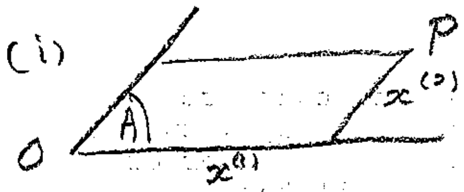
$$\ln(z!) = C + (z+0.5)\ln(z) - z + B_2/(1.2z) + B_4/(3.4z^3) + \ldots \qquad (10)$$

This equation is to be understood in the sense that, if we take a finite number of terms, this will differ from the left-hand side by less than the magnitude of the first term that is neglected.

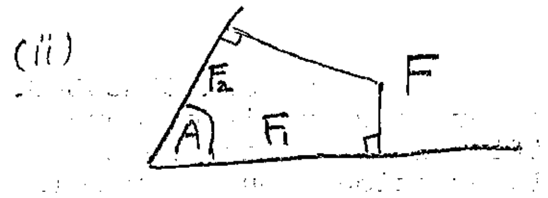
It can be shown that $C = (1/2)\ \ln(2\pi)$. The terms at the beginning give the usual Stirling approximation.

Back of T_1

(i)



(ii)



If OP is a displacement and OF represents a force, the work done by the force in the displacement is

$x^{(1)}F_1 + x^{(2)}F_2$, which we shall write x^iF_i .

Sums continually occur in tensor theory, and the <u>tensor convention</u> stipulates that we sum over any index that occurs twice, as "i" does here. *– once above, once below*

The raised index i in x^i indicates that we are specifying <u>x</u> in the manner (i), known as contravariant specification. The lowered index i in F_i indicates that <u>F</u> is being represented in the manner (ii), covariant specification.

Books often refer to "covariant vectors" and "contravariant vectors". This language I think is likely to confuse learners. There are two different forms of representation, not two different kinds of vectors – it is hard to see how there could be.

Note that x^iF_i is the scalar product, x.F in vector notation.

We are perfectly free to represent <u>x</u> in the covariant manner. If A is the angle between the axes, by dropping perpendiculars we find

$$x_1 = x^{(1)} + x^{(2)} \cos A,$$
$$x_2 = x^{(1)} \cos A + x^{(2)} .$$

Back of T2

Change of Variables.

Thus $x^i y_i$ is invariant. What does this tell us about the way y_i is transformed when axes are changed?

Suppose we bring in new variables, X^i, defined by

$$x^i = t^i_j X^j \ldots\ldots(3)$$

Then $x^i y_i = t^i_j X^j y_i$. In the new system this should be $X^j Y_j$ and we can make this so for every X^j only by taking

$$Y_j = t^i_j y_i \quad \ldots.(4)$$

The transformations (3) and (4) differ. (3) gives $\underline{x}$ in terms of $\underline{X}$, (4) gives $\underline{Y}$ in terms of $\underline{y}$. Thus we have different rules of transformation for vectors shown in covariant form and in contravariant form.

Note that when perpendicular axes are being used, the two modes of representation coincide, as is evident when the angle A in Figure 1 is 90^o.

Now g_{ij} too must be recognized as a tensor, for it tells us how to calculate the length of a vector, a quantity on which all systems are agreed. How does it transform? As before we introduce new co-ordinates X^i by $x^i = t^i_j X^j$.

Then $g_{ij}x^i x^j = g_{ij} t^i_p t^j_q X^p X^q$, which ought to be $G_{pq}X^p X^q$.

Accordingly $G_{pq} = t^i_p t^j_q g_{ij} \quad \ldots(5)$

gives the transformation for g_{ij}. Note that the transformation is the same as the one we would use if g_{ij} was the product of two vectors $x_i y_j$. This property is sometimes used to define a tensor, by saying that a tensor transforms like the product of vectors. I have never found this approach particularly helpful.

A tensor with two subscripts does very much the same job in this symbolism as a matrix in ordinary vector-matrix notation. If we write equation (2) as $OP^2 = x^i g_{ij} x^j$ so that summation is over adjacent pairs of symbols it is seen that, in matrix notation, we would have an expression of the form $\underline{x}^T M \underline{x}$ with a matrix M. This is the usual way to express a quadratic of two variables in matrix symbolism.

It is also possible to define a tensor corresponding to the matrix, M, for a mapping $\underline{y} = M\underline{x}$.

Oops 2nd page with # T1 0 Back of T1 Thermodynamics 2.

the velocity of a particle lies in the interval u, u+du. The chance that the velocity of a particle has components within u,u+du ; v,v+dv ; w,w+dw is then

$f(u)f(v)f(w)\,du\,dv\,dw$. If we have a small region, of volume ΔV in the (u,v,w) space, at or around the position (u,v,w), the chance of the velocity lying in this region is $f(u)f(v)f(w)\,\Delta V$. The reason for bringing in ΔV is that this quantity, being an element of volume, does not depend on the choice of axes of co-ordinates. Since no direction is singled out for special privileges in a gas, this probability should not depend in any way on the direction of the velocity, but only on its magnitude. Thus

$f(u)f(v)f(w)$ should be a function of $u^2+v^2+w^2$ alone. The same will be true of its logarithm. Accordingly we shall have

$$\ln f(u) + \ln f(v) + \ln f(w) = \phi(u^2+v^2+w^2)$$

for some function ϕ. If we differentiate partially with respect to u we obtain

$$\frac{f'(u)}{f(u)} = 2u\phi'(u^2+v^2+w^2) .$$

Similarly we have $$\frac{f'(v)}{f(v)} = 2v\phi'(u^2+v^2+w^2)$$

and $$\frac{f'(w)}{f(w)} = 2w\phi'(u^2+v^2+w^2)$$

it will be seen that $$\frac{f'(u)}{uf(u)} = \frac{f'(v)}{vf(v)} = \frac{f'(w)}{wf(w)} .$$

This can only be so if $f'(u)/[uf(u)]$ is a constant, C. Then

$f'(u)/f(u) = Cu$, and $\ln f(u) = (C/2)u^2 + \text{constant}$.

Hence $f(u) = A\exp[(C/2)u^2]$ for some constant A. Clearly C must be negative; otherwise the velocity would be overwhelmingly likely to be infinite. If $C/2 = -a$ we have

$$f(u) = A e^{-au^2} .$$

To obtain A in terms of a, we use the fact that the probability of the particle having <u>some</u> velocity is 1. Accordingly

$$1 = \int_{-\infty}^{\infty} A\exp(-au^2)\,du$$

A standard result is $\int_{-\infty}^{\infty} \exp(-au^2\,du = \pi^{1/2}a^{-1/2}$ (10)

Hence

$A = \pi^{-1/2}a^{1/2}$ and

$$f(u) = \pi^{-1/2}A^{1/2}\; e^{-au^2} . \qquad (11)$$

a

" To previous page" in pencil

back of T2

ie: calculating pressure

To previous page

(1/2)kT.)

The kinetic energy of a molecule is $(m/2)(u^2+v^2+w^2)$. As u^2, v^2 and w^2 all have the same value, it follows that the average kinetic energy of a molecule is (3/2)kT. As a mole contains N molecules, the kinetic energy for a mole is (3/2)NkT. As Nk=R, it follows that the kinetic energy per mole of ideal gas is given by

$$E = (3/2)RT.$$

The results here are in accordance with a general principle known as the equipartition of energy, that the average energy of a particle has (1/2)kT for each degree of freedom.

Line Integrals. Exact Differentials.

The role of line integrals can be seen by considering the calculation of the work done on a mass, which moves along a specified path in a field of force. The work done by a force (X,Y) when the mass moves through (dx,dy) is Xdx+Ydy. X and Y we suppose to be known functions of x,y. A convenient way to specify the path followed is to take x and y as functions of a parameter t; x=x(t), y=y(t). Then X and Y will be definite functions of t. The work done will then be

$$\int X(t)\, x'(t) + Y(t)\, y'(t)\, dt,$$

taken between t_o and t_1 corresponding to the beginning and the end of the path. The integral will normally be written as $\int Xdx+Ydy$, with an indication of the path to be followed.

For instance, if we wanted the path to go round the triangle with corners (0,0), (1,1), (2,0) we might take

from t=0 to t=1   x=t, y=t
from t=1 to t=2   x=t, y=1-t
from t=2 to t=4   x=4-t, y=0.

This rather elementary example is given, so that there is no sense of vagueness or mystery in the concept of a line integral. Curved paths of course require only the introduction of higher powers of t.

If the field of force should happen to be one that has a potential, the work done in going from A to B will equal the change in the value of the potential from A to B, whatever path may be taken, so the value of the integral will not depend on the path. The integral taken around a closed curve will give zero. In this case we say that Xdx+Ydy is an exact differential.

When the value of the integral depends on the path taken, Xdx+Ydy is said not to be an exact differential, and the integral around a closed path may fail to be zero. This happens, for example, if X=y, Y=0, as can be verified by calculating the integral for the path specified above. It is obvious that the value of this integral must depend

(T3 begins with "on the path")

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equal.

The First Law allowed us to define a new function of state, the energy, E. The Second Law leads to the definition of a new function of state, the entropy, S, though this is far from obvious. Clausius, who introduced the concept of entropy in 1854, is regarded as the founder of physical chemistry. (Pledge, p.145.)

There are two lines of thought that suggest the existence of entropy. I find it surprising that Clausius was able to reach this new concept, if he did it from the first of these, which runs as follows. Let ΔQ_1 be the amount of heat taken in the part AB of the Carnot cycle, and ΔQ_2 the amount taken in in CD (this of course is negative). We have

$$\Delta Q_1 = RT_1(\ln V_2 - \ln V_1)$$

$$\Delta Q_2 = - RT_2(\ln V_2 - \ln V_1)$$

It follows that $\Delta Q_1/T_1 - \Delta Q_2/T_2 = 0.$

Now the left-hand side of this last equation is what we would get if we found the line integral around ABCD of dQ/T (if for once we allow ourselves to write dQ for the infinitesimal amount of heat taken in). This means that we are dealing with an exact differential; there is a function of state, S, such that, in an infinitesimal change, $\Delta Q = TdS$. S is called the entropy.

This result has been found only for a Carnot cycle. There is then a lengthy argument to show that any reversible cycle can be approximated by a combination of Carnot cycles.

The second approach demonstrates the existence of such a function of state for an ideal gas, and thus suggests that such a function may exists in other situations. For an ideal gas,

$$\Delta Q = C\,dT + (RT/V)dV$$

which is clearly not an exact differential. However, if we divide by T we get

$$\Delta Q/T = CdT/T + RdV/V$$

which clearly is an exact differential, so entitled to be called dS with

$$S = C \ln T + R \ln V + \text{constant.}$$

(T3 begins with "on the path")   back of T3

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(T4 face has Thermodynamics) Back of T4

It is interesting that we can reach the main point of Carnot's argument without any detailed calculations about the behaviour of gases or other substances. His theorem is this; - no engine can ever be more efficient than a reversible engine.

Suppose the reversible engine works between the temperatures **T** and **t.** No engine, working between these temperatures, can produce more work for a given quantity of heat. For suppose the reversible engine can take in H units of heat at temperature **T** and use it to produce W units of work at temperature **t.** Being reversible, it could take in W units of work at temperature **t** and use this to give out H units of work at temperature **T.** Now suppose some other engine could do better and change H units of heat at temperature **T** to W' units of work at temperature **t,** with W' > W. Carnot argued as follows;-

First, let the improved engine change H units of heat at **T** to W' units of work at **t.** Use W'-W to do some useful job. This leaves W units, which the reversible engine can change to H units of heat at **T.** Repeat this process. Each time we get W'-W units of work, and end back where we started. **We have perpetual motion** which Carnot believed was impossible.

Here we have an essential theorem of thermodynamics, freed of technical details.

The Carnot cycle.

The mathematical implications of this theorem are found by considering a particular series of operations, known as the Carnot cycle, involving the expansion and contraction of gas, with heating and cooling. We suppose the gas to obey the equations for an ideal gas. This might seem to bring in an element of unreality. However the behaviour of actual gases at sufficiently low pressures approximates very closely to that of the hypothetical perfect gas, so that in principle it would be possible to carry out an actual experiment as close as one may wish to a Carnot cycle.

All the changes in the volume of the gas are to be carried out extremely slowly. Some of them are to take place at constant temperature, such as might be achieved by having the gas in a metal container in contact with a large mass of water. These are known as **isothermal.** Others are to be done in such a way that no heat can enter or leave, the container being surrounded by insulation; these are called **adiabatic.**

As the Carnot cycle involves both these processes, we need to calculate the work, heat and pressure changes involved in each of them, before we can consider the Carnot cycle itself.

Adiabatic expansion.

Theory suggests and experiment verifies that the internal energy of a gas depends only on the temperature. For an ideal gas we assume E=CT, where E is the energy and T the absolute temperature.

front has "For adiabatic..."   Back of T5

from this decrease of energy.

Work in an isothermal process.

This is a process in which the temperature does not change so T is constant. As pV=RT for a mole of gas, p=RT/V and p dV = (RT).dV/V. Integrating this leads to logarithms and we find

$$W = RT.\ln(V_2/V_1) \qquad (6)$$

The energy of the gas, being a function of temperature, does not change, so , in an isothermal expansion, this work can only have come from that amount of heat being absorbed.

If the gas is compressed, $V_2 < V_1$, and the work done by the gas is negative and that amount of heat comes out of the system.

Graphing the Carnot cycle.

In the Carnot cycle there are two isothermal processes and two adiabatic processes. . In textbooks the cycle i usually shown graphically with co-ordinates p and v. This has certain advantages. However it seems interesting to take T and v as the variables. In the usual graph, the curves for the two types of process do not look vary different. If we take T as one of the variables, it immediately becomes obvious which processes are isothermal, for in them T is constant and the graph is a level straight line. The equation for an adiabatic process was given in euation (3) as pV^g = constant. As pV=RT, p=RT/V and the equation becomes TV^{g-1} = constant. As g=(C+R)/C, g-1=R/C. Thus we have $TV^{R/C}$ = constant, and so

$$\ln T + (R/C)\ln V = \text{constant.}$$

Accordingly if we take x =lnV and y=lnT, adiabatic changes will be shown by a descending straight line with gradient -R/C.

The nature of the cycle can be read off from the diagram, Figure 1.

```
Figure 1.    A ------>----- B
              \             \
               \             \
                \             \
                 C ------<----- D
```

If we begin the cycle with the isothermal change, at temperature T_1, from A to B, the lines AC and BD represent adiabatics through A and B respectively. C and D are points on these at temperature T_2. In the cycle we go from A to B and then B to D; in both of these the volume increases. We then go from D to C and from C to A; in both of these the volume decreases.

We now consider the total work done and the heat changes in the cycle.

As we saw in equation (5), the work done in BD is $C(T_1-T_2)$, as we go from temperature T_1 to T_2. In CA we do exactly the opposite; we go from T_2 to T_1, so the face has Thermodynamics 3 | back of T 8 |

walls are perfectly reflecting or not, since the radiation is in equilibrium with the material of the container. It does not matter whether the radiation is reflected or whether some of it is absorbed, raising the temperature of the container, and later radiated. He shows that the conservation of energy would be violated if this were not so (p.200).

Black Body.

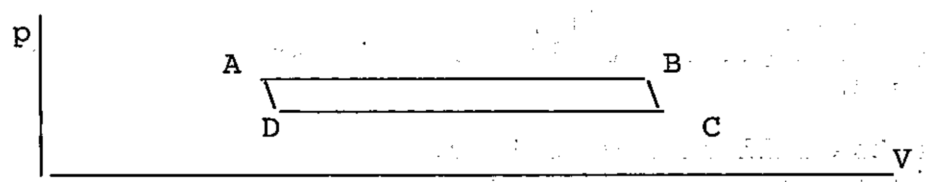
Above we referred to a black body as one that absorbed all radiation falling onto it. There is no natural object that has this property. However, if a cavity in a body has a very small opening, so that light entering through this opening is reflected and partially absorbed many times in the interior, very little light eventually finds its way out through the opening, and this is found to give an effectively black body. This arrangement can also be used in the opposite direction, the material being heated and radiation allowed to pass out through the narrow opening. This has certain theoretical advantages which will not be discussed here. Such a source of radiation is used in the first and third stages of the cycle described in the next section, which follows the argument as presented by Richtmeier in his "Introduction to Modern Physics".

The Stefan-Boltzmann Law.

In 1879 Stefan suggested, on the basis of experimental observations, a law equivalent to saying that **u**, the intensity of radiation inside a container, is proportional to the 4th power of the temperature. In 1884, Boltzmann showed that this result could be deduced by considering an "ether engine" on the analogy of the Carnot cycle.

As in the Carnot cycle, there are two stages in which the temperature is held constant and two in which heat is not allowed to enter or leave.

The constant temperature stages differ markedly from those in the Carnot cycle, for it can be shown that **u**, the radiation density, is a function of temperature T alone. As the pressure is u/3, it follows that pressure is constant in an isothermal change, so such changes appear as horizontal lines on a p,V diagram.



The container of the radiation is a cylinder, inside which a piston can move. The piston has unit area.
<u>Stage 1.</u> We start at A, with temperature T, pressure p_1 and volume V_1. A black body is held at temperature T, and radiation coming out of its small opening passes into a small opening in the cylinder. The

face has "external pressure   back of T9

Equating the two expressions for the efficiency we find

$$dT/T = du/(4u) \ .$$

Integrating this gives $4 \ln(T) = \ln(u) + \text{constant}$, so

$$u = aT^4 \qquad (19)$$

Note. The fact that du and dT have been chosen to respresent decreases does not invalidate the argument. If you have any doubts on this score, it is quite simple to carry through the argument with du,dp and dT regarded as increases with negative values.

<u>Question 1.</u> We have seen that isothermal curves have the equation p = constant on the p,V diagram. What equation do adiabatics satisfy?

<u>Question 2.</u> Verify that the approximation treating ABCD as a parallelogram does not lead to an error.

Planck's Version.

It is instructive to compare and contrast the proof of this result given in Planck's book, pp. 200-201. He imagines a black body with variable volume, capable of doing work. This might be a cylinder with a piston. There is a small aperture at the end remote from the piston, through which radiation is emitted, so the cylinder constitutes a black body. Let U represent the energy of the radiation inside the cylinder, so that U replaces E in the universal equation (16). We have

$$dU = TdS - pdV. \qquad (20)$$

U=uV. All the quantities U,p,V,S,T are functions of state, so black body radiation has a definite temperature, namely the temperature of the enclosing wall. We wish to work with T and V as our variables. Now u is a function of T alone, so we may write

$$dU = d(Vu) = Vdu+udV = V(du/dT)dT + udV.$$

Now, from equation 20, $TdS = dU + pdV = dU + (1/3)udV$ since $p=u/3$. Hence we have

$$TdS = V(du/dT)dT + (4/3)udV.$$ Dividing by T,we obtain

$$\left(\frac{\partial S}{\partial T}\right)_V = \frac{V}{T}\frac{du}{dT} \quad \text{and} \quad \left(\frac{\partial S}{\partial V}\right)_T = \frac{4u}{3T}$$

Applying $\partial/\partial V$ to the first expression gives the same result as applying $\partial/\partial T$ to the second, provided certain continuity conditions are satisfied. We find

$$\frac{1}{T}\frac{du}{dT} = \frac{4}{3T}\frac{du}{dT} - \frac{4}{3}\frac{u}{T^2}$$

On simplifying, this gives $4dt/T = du/u$ and integrating leads to $u=aT^4$ as before.

This may not be my fathers.... ? Doesn't sound like his voice but..?

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Tenets of a naturalistic mathematics

I shall now simply state the conclusions of my long inquiry into the possibility of approaching mathematics naturalistically. It sounds dreadfully pompous to say this, but since I have spent the better part of forty years reaching these conclusions, and have anyway already published many fragments of my thinking (which have puzzled and dismayed some readers), it would be foolish of me not to publish the overall result: the key which may enable those same bemused readers to see how it all makes sense.

The first tenet is important, because it may serve to curb any tendency to the kind of triumphalism which affected foundationalism, and led to its collapse, in the 1960s. We begin with the truism that mathematics is valued in society because of its predictive powers. This is reflected within mathematics by the relative importance of the idea of a sequence, and especially of the idea of a sequence which continues indefinitely in accordance with some rule. But there is a limit to the 'finding of rules' for sequences which continue indefinitely, because we can conceptualise a 'perverse random' sequence which will defeat any rule we try to use to predict it beforehand. Thus:

TENET 1 Indefinable sequences of events can be clearly and distinctly visualised, and they can, a fortiori, occur in experience. Such sequences lie outside the proper area of mathematics. Mathematics is fundamentally concerned with *well-defined* patterns, structures and objects. But indefinable sequences defeat every attempt at definition, indefinitely. They can never be incorporated into mathematics for this reason. Consequently there is a limit, which we can clearly and distinctly visualise, to the applicability of mathematics.

The second tenet is concerned with the possibility of applying mathematics to itself. This possibility, of applying mathematics to itself, is the great advance of Twentieth Century mathematics. It is used in the century's most significant result, Godel's Theorem; it was used by Turing in his conceptualisation of a Universal Machine; and it was celebrated by Hofstadter in his book *Godel, Escher, Bach.* (1979). But here, too, for all the new power opened-up by self-application, we eventually run into a *limit.* When one tries to formulate a statement which will assert its own falsity, one ends-up with a paradox. This is caused by the fact that one's provisional assessment of the statement's truth oscillates between 'true' and 'false'.

TENET 2 Once we legitimise the application of mathematics to itself, we open-up the possibility of *oscillations* of inconsistent partial meaning. This is a new, fundamental, species of contradiction, which may be called 'dynamic contradiction'. The paradoxes of set theory exhibit dynamic back of W 8 *

If we plot e_λ / T^5 against λT we get the same curve whatever T so $e_\lambda / T^5 = f(\lambda T)$; $e_\lambda = T^5 f(\lambda T) = \lambda^{-5} \varphi(\lambda T)$

ψ_λ the energy density inside the enclosure is proportional to e_λ. We have $\psi_\lambda = \lambda^{-5} \Phi(\lambda T)$

Planck, (for convenience of calculation, supposed energy to have finite packets ε) and found

$$\psi_\lambda = \frac{8\pi}{\lambda^4} \frac{\varepsilon}{e^{\varepsilon / k_0 T} - 1}$$

If we take $\varepsilon \propto 1/\lambda$ we get agreement. As frequency, ν, proportional to $1/\lambda$, we have result — an oscillator of frequency ν can have energies only multiples of $h\nu$. ($h = 6.55 \times 10^{-27}$ erg sec.)

In Bohr model of atom, it is required that $\oint p\,dq$ is a multiple of h. There are certain discrete orbits allowed by this. In such an orbit, radiation does not occur. Radiation involves a jump from one state to another.

* Please note that the writing at the top of this page suggests to me that it was written after one of his several mini-strokes which he had during his last six to eight years. Hopefully the content is valid. June Feb 2009

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$$ds^2 = 2\overline{T}(q_k, \dot{q}_k)\, dt^2 \qquad (3)$$

As $dq_k = \dot{q}_k dt$ this does not depend on time. We can now express angle between lines, $\perp$, divergence and curl, grad φ, Δ = div grad, and use these as simply as in Euclidean 3-D, with slightly more complicated analytic expressions.

From now on, all geometric statements refer to this space.

Important to distinguish between covariant and contravariant. Difficulty no greater than for oblique Cartesian axes.

dq_k are prototypes for contravariant vector. The coefficients of dq_k in $2\overline{T}$ give the fundamental covariant tensor. $2T$ is the contravariant form corresponding to $2\overline{T}$, for the momenta, as is well known, are ~~contravariant~~ covariant vectors corresponding to $\dot{q}_k$, so the impulse is the covariant form of the velocity.

The LHS of (1') is the contravariant fundamental form with variables $\partial W/\partial q_k$. The $\partial W/\partial q_k$ are components of grad W, which by its nature is covariant. Thus

(1'') $(\text{grad } W)^2 = 2(E-V)$.

(1''') $|\text{grad } W| = \sqrt{2(E-V)}$

[The expression of K.E. in terms of the momenta has the significance that covariant components can only be put into a contravariant form to give a sensible, i.e. invariant result]

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Schrödinger.

Consider particle $T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$ Potential $V(x,y,z)$

Define $ds^2 = \frac{1}{2}m(dx^2+dy^2+dz^2)$

$$\frac{\partial W}{\partial t} + T\left(q\frac{\partial W}{\partial q}\right) + V(q) = 0.$$

Let $W = -Et + S(q)$

$$-E + T\left(q\frac{\partial W}{\partial q}\right) + V(q) = 0.$$

$$T\left(q\frac{\partial W}{\partial q}\right) = E - V$$

$$2T\left(q\frac{\partial W}{\partial q}\right) = 2(E-V)$$

Grad W has rate of change of W component.

Distance in x direction is $\sqrt{\frac{m}{2}}\,dx$.

Rate of change of W is $\sqrt{\frac{2}{m}}\frac{\partial W}{\partial x}$

$$(\text{grad } W)^2 = \frac{2}{m}\left\{\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2\right\}$$

$$= \frac{2}{m}\left\{\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right\}$$

$$T = \frac{p_1^2 + p_2^2 + p_3^2}{2m}$$

$$\frac{1}{m}2\cancel{T}\left\{\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2\right\} = 2(E-V)$$

$g_{ij} = \frac{m}{2}\delta_{ij}$ $\quad ds^2 = \frac{m}{2}\delta_{ij}(dx^i dx^j)$

Covariant is $g_{ij}dx^j = \frac{m}{2}dx^i$

So $\frac{\partial W}{\partial q_k} =$

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Schrödinger p'le says v propor'l to $1/u$.

Here u and v are constants, so no deduction possible

When $V \neq 0$ $\quad E - V = T = \frac{1}{2}mv^2$

$$\frac{E}{\sqrt{2(E-V)}} = \frac{E}{\sqrt{mv^2}} = \frac{E}{\sqrt{m}} \cdot \frac{1}{v}$$

in accord with stated result.

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The construction can clearly be replaced by Huygen's elementary waves, with μ proportional to $\frac{1}{u}$ from (6). The q-space is inhomogeneous but isotropic (same phase velocity in all directions).

W plays the role of phase. Fermat's principle

$$0 = \delta \int \mu \, ds = \delta \int \frac{ds}{u} = \delta \int \frac{\sqrt{2(E-V)}}{E} ds = \delta \int \frac{2T}{E} dt. \quad (7)$$

is Maupertuis' Least Action principle. The "rays", i.e the orthogonal trajectories of the wave-surfaces, are the paths of the system for energy E, in agreement with $p_k = \frac{\partial W}{\partial q_k}$. (8)

The concept of rays belongs to geometrical optics: that is the only analogy we have established so far. The wavefronts are only loosely related so far to mechanics, for the representative point of the mechanical system in no way moves with the velocity u, but on the contrary the velocity is proportional to $1/u$. From (3)
$v = \frac{ds}{dt} = \sqrt{2T} = \sqrt{2(E-V)}$. This discrepancy is illuminating. From (8), the velocity is large where grad W is large, i.e where the surfaces are crammed

p 495 close together i.e where u is small.

Secondly, $W = \int L \, dt$. This naturally changes during the motion (by $(T-V)dt$ in dt) so it is impossible for the representative point to stay on the same W surface

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Note on Action

$$S = \int 2T\,dt = \int \Sigma p_i\,dq_i$$

Consider mass moving under no force, with energy E.

If r is distance from $(x_0\,y_0\,z_0)$ to $(x_1\,y_1\,z_1)$, as momentum $mv = m\sqrt{\frac{2E}{m}} = \sqrt{2mE}$

$$\text{So} \quad \int mv.\,dr = \sqrt{2mE}.r = S.$$

$$\frac{\partial S}{\partial E} = r\sqrt{2m}.\,\tfrac{1}{2}E^{-\frac{1}{2}} = r\sqrt{\frac{m}{2E}}$$

$$\text{as } E = \tfrac{1}{2}mv^2 \quad \frac{m}{2E} = \frac{1}{v^2}$$

$$\therefore \frac{\partial S}{\partial E} = \frac{r}{v} = t.$$

as stated by Hamilton (p107. $t_0 = 0$).

It may also happen that

$a > c,\ b < c,\ a + b < 2c$ The factors $(2n+2b+1) \overset{+1}{\ldots} (2n+2c)$ are as in previous section. However there are now more factors in denominator of ② than in numerator of ①. We have

$$\mathrm{lf}(a,b,c) = \frac{(2n+2b+1) \overset{+1}{\ldots\ldots} (2n+2c)^{\,2(c-a-b)\text{ factors}}}{\underbrace{2 \overset{+2}{\ldots} (2a-2b)}_{(a-b)\text{ factors}} \circ \underbrace{(4n+2a+2b+1) \overset{+2}{\ldots\ldots} (4n+4c-1)}_{(2c-a-b)\text{ factors.}}}$$

Thus we seem to have independent effects when $a > c$.

(α) If $b > c$ then $(2n+2c+1) \overset{+1}{\ldots} (2n+2b)$ in denominator.
If $b < c$ Then $(2n+2b+1) \overset{+1}{\ldots} (2n+2c)$ in numerator.

(β) If $a+b > 2c$ then $(4n+4c+1) \overset{+2}{\ldots} (4n+2a+2b-1)$ in numerator
If $a+b < 2c$ then $(4n+2a+2b+1) \overset{+2}{\ldots} (4n+4c-1)$ in denom^{r}.

If $b = c$, then $\times 1$ in (α) : if $a+b = 2c$ then $\times 1$ in (β).
Easy to arrange by PROCs.

(γ) $(-1)^{ab}\, 2.4 \ldots (2a-2b)$ in denom^{r} if $b < a$.

$A_{n,n+q}$ ~~is~~ equals the coefficient of the $(0+1+\ldots+n+q)$th power of s in the sum of products of the form

$$\prod_{r=0}^{n} L[i(r)].$$

Now $$L[i(r)] = \sum_{j(r)=i(r)}^{\infty} L[i(r), j(r)] s^{j(r)}.$$

Accordingly $A_{n,n+q}$ ~~= $\sum L[i(r), j(r)]$~~ contains all the products $\prod_{r=0}^{n} L[i(r)] L[j(r)]$ that satisfy the following conditions.

(1) $\sum j(r) = 0+1+\ldots+n+q$.

(2) The $i(r)$ are distinct.

(3) For each r, $i(r) \leq j(r)$.

~~At first the equations are simple.~~

~~$P(n) = A(n,n) =$~~

At first the equations are simple. ~~If $q=1$,~~ ~~we can form the sum for $A_{n,n+1}$ by from taking the product $L(0,0) L(1,1) \ldots L(n,n)$ by changing $L(r,r)$ to $L(r,r+1)$ and summing~~ We can form the sum for $A_{n,n+1}$ by the following procedure

1. In the product $L(0,0) L(1,1) \ldots L(n,n)$ replace $L(r,r)$ by $L(r,r+1)$. Do this for each r and sum ~~from~~ for $r=0$ to n.

2. Replace $L(n,n)$ by $L(n+1,n+1)$ and add to the sum already found.

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So we have $\frac{1}{z}\left\{\ln(u+z) - \ln(u-z)\right\}$ for $|u|>|z|$

$\frac{1}{z}\left\{\ln(z+u) - \ln(z-u)\right\}$ for $|z|>|u|$.

If $|u|>|z|$, $u+z$ and $u-z$ have the same sign, so $\frac{u+z}{u-z}>0$ $\therefore$ $\frac{u+z}{u-z} = \left|\frac{u+z}{u-z}\right|$ and $\ln|u+z| - \ln|u-z|$,

Same argument applies in second case

$\therefore$ We have

$\frac{1}{z}\left\{\ln|z+u| - \ln|z-u|\right\}$ for all cases

$$\therefore I = \int_{u=-1}^{1}\int_{z=-1}^{1} \frac{u}{1-\alpha u}\left\{\ln(1-z) - \ln|z-|u||\right\} \frac{\ln|z+u|-\ln|z-u|}{z}\, dz\, du.$$

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Lagrange's equations $\cancel{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = 0}$

$\mu \to \sqrt{E-V}$ so E must be specified

NB red pencil is on original: noted by ⊙

corrects $\alpha_1 \beta_1$ ⊙

We have

$$\frac{\lambda_{n+1,n+3}}{\lambda_{n+1,n+1}} + \frac{a_{n,n+1} b_{n+1,n}}{a_{nn}\, b_{nn}} \left[\frac{a_{n+1,n+2} b_{n+2,n+1}}{\cancel{a_{nn} b_{nn}}\; a_{n+1,n+1} b_{n+1,n+1}} - \frac{a_{n,n+1} b_{n+1,n}}{a_{nn}\, b_{nn}} \right]$$

$$+ \frac{a_{n+2,n+2} b_{n+2,n+2}}{a_{n+1,n+1} b_{n+1,n+1}} \left[\frac{a_{n+2,n+3} b_{n+3,n+2}}{a_{n+2,n+2} b_{n+2,n+2}} \text{①⊙} - \frac{a_{n+1,n+2} b_{n+2,n+1}}{a_{n+1,n+1} b_{n+1,n+1}} \right]$$

$$- \frac{a_{n+1,n+1} b_{n+1,n+1}}{a_{nn}\, b_{nn}} \left[\frac{a_{n+1,n+2} b_{n+2,n+1}}{a_{n+1,n+1} b_{n+1,n+1}} \text{②⊙} - \frac{a_{n,n+1} b_{n+1,n}}{a_{nn}\quad b_{nn}} \right]$$

$$= \frac{a_{n+1,n+3} b_{n+3,n+1}}{a_{n+1,n+1} b_{n+1,n+1}} - \frac{a_{n,n+2} b_{n+2,n}}{a_{nn}\, b_{nn}}$$

$$+ \frac{a_{n+2,n+3} b_{n+3,n+2}}{a_{n+1,n+1} b_{n+1,n+1}} \text{①⊙} - \frac{a_{n+1,n+2} b_{n+2,n+1}}{a_{nn}\, b_{nn}} \text{②⊙}$$

$$+ \frac{(\overset{\alpha_1 ⊙}{a_{n,n+1} a_{n+1,n+2}} - \overset{\alpha_2 ⊙}{a_{n+1,n+1} a_{n,n+2}})(\overset{\beta_1 ⊙}{b_{n+1,n} b_{n+2,n+1}} - \overset{\beta_2 ⊙}{b_{n+1,n+1} b_{n+2,n}})}{a_{n+1,n+1} b_{n+1,n+1}\, a_{nn}\, b_{nn}}$$

$$- \frac{(\overset{\alpha_3 ⊙}{a_{n-1,n} a_{n,n+1}} - \overset{\alpha_4 ⊙}{a_{nn} a_{n-1,n+1}})(\overset{\beta_3 ⊙}{b_{n,n-1} b_{n+1,n}} - \overset{\beta_4 ⊙}{b_{nn} b_{n+1,n-1}})}{a_{nn} b_{nn}\, a_{n-1,n-1}\, b_{n-1,n-1}}$$

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DEFPROCC

FOR I = 1 TO Z-1 PROCCAB

FOR I = 1 TO Z-1

FOR K = I+1 TO Z

IF K >< I+1 THEN PROCCGEN (general)

NEXT : NEXT

DEFPROCCGEN.

~~CER~~ PROC R

C~~(Z+I-1)=H~~ ~~Entered as before~~.

~~C(R,~~

C(R, Z+I-1) = H(I-1, K)

C(R, I) = H(I, K)

C(R, Z+I) = H(I+1, K)

C(R, Z+K-1) = -H(I, K-1)

C(R, K) = -H(I, K)

C(R, Z+K) = -H(I, K+1)

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PROCC. I have run into a lot of trouble with this and think a sledgehammer method may be quickest.

$$\sum a_{ij} h_{jk} - \sum h_{is} a_{sk} = 0 \qquad \text{omit if } k=3$$

a_{ij} tridiagonal. This becomes

$$0 = a_{i,i-1} h_{i-1,k} + a_{ii} h_{ik} + a_{i,i+1} h_{i+1,k} -$$
$$- h_{i,k-1} a_{k-1,k} - h_{ik} a_{kk} - h_{i,k+1} a_{k+1,k}$$

$a_{ii} = x_i \quad i = 1..3. \qquad a_{i,i+1} = x_{3+i} = a_{i+1,i}$

$a_{i-1,i} = x_{3+i-1} = a_{i,i-1}$

So

$$0 = h_{i-1,k} x_{3+i-1} + h_{ik} x_i + h_{i+1,k} x_{3+i} -$$
$$- h_{i,k-1} x_{3+k-1} - h_{ik} x_k - h_{i,k+1} x_{3+k}.$$

We suppose $k > i$, $i = 1 \dots 3-1$, $k = i+1 \dots 3$.

As $k > i$, a repeated element can occur only if $k-1 = i$

Then $0 = h_{i-1,i+1} x_{3+i-1} + h_{i,i+1} x_i + h_{i+1,i+1} x_{3+i} -$

$- h_{ii} x_{3+i} - h_{i,i+1} x_{i+1} - h_{i,i+2} x_{3+i+1}$

$h_{i+1,i+1} - h_{ii}$ for $i=4$ is $\frac{1}{9} - \frac{1}{7}$, which checks.

Here for I = 1 to Z−1, PROCR.

* C(R, Z+I−1) = H(I−1, I+1)

* C(R, I) = H(I, I+1)

C(R, Z+I) = H(I+1, I+1) − H(I,I)

* C(R, I+1) = − H(I, I+1)

* C(R, Z+I+1) = −H(I, I+2).

Call this PROCCAB. (AB for A, A+1)

If Z = 5 I = 4 Z+I+1 = 10 outside range.

$h_{i,k+1} a_{k+1,k}$

$k = 5$ this is outside range

PROCCABBIE

Back of W 30

Balmer 1885. The 9 lines then known in the spectrum of hydrogen fitted

wave length $\lambda = k \dfrac{m^2}{m^2 - 4}$

$m = 3, 4, 5, \ldots$

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Examples on Lagrange
in later pages
DP1,2 Double pendulum
PS1 particle on sphere

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