\( k(a, b) \) is \( \text{cof} \) of \( x^{2n+2b} \) in \( P_{2n+2a} \).

Coeff. of highest power in \( P_{2n}(x) \) is \( \frac{1.3 \ldots (2n-1)}{n!} \).

So in \( P_{2n+2a} \) is \( \frac{1.3 \ldots (4n+4a-1)}{(2n+2a)!} \).

Denominator is always of from \( k(c, c) \).

So \( \frac{k(a, a)}{k(c, c)} = \frac{(4n+4c+1)}{(2n+2c+1)} \cdot \frac{(4n+4a-1)}{(2n+2a)} \) if \( a > c \).

\( k(c, c) = \frac{(2n+2c+1)}{(2n+2c+1)} \cdot \frac{(2n+2a)}{(2n+2a)} \) is constant.

Coeff. of \( x^{2n+2b} \) is high est coeff. \( x \):

\[- \frac{(2n+1)}{2 \cdot (2n+2)(2n+2-1) \ldots (2n-2b+1)} \]

Now, for \( k(a, b) \) we have \( l = a - b \) \( v = 2n+2a \).

So \(- \frac{(a-b)}{2 \cdot (2a-2b) \ldots (2n+2b)} \cdot \frac{(2n+2a-1)}{(2n+2a+2b+1)} \)

To get \( H(a, b, c) \) we multiply these together.

\[ H(1, 0, 0) = \frac{(2n+1)(4n+3)}{(2n+1)(2n+2)} \cdot \frac{(2n+2)(2n+1)}{2 \cdot (4n+3)} \]

Could be shortened.
\[ l(q, a) = \frac{(4n+4c+1) \ldots (4n+4a-1)}{(2n+2a) \ldots (2n+2c+1)} \quad \text{if} \quad a > c. \]

\[ l(q, b) = \frac{(-1)^{a-b} (2n+2a) \ldots (2n+2b+1)}{2 \ldots (2a-2b) \ldots (4n+4a+2b+1)} \]

If \( a+b>c \), it will be possible to cancel all the factors of second denominator, leaving numerator
\[ (4n+4c+1) \ldots (4n+4a+2b-1) \]

Also, if \( b>c \) (which implies \( a+b>2c \)) we have
\[ \frac{(-1)^{a-b} (4n+4c+1) \ldots (4n+2a+2b-1)}{2 \ldots (2a-2b) \ldots (2n+2c+1) \ldots (2n+2b)} \]
\[
\sqrt{14} = 3 + \frac{(\sqrt{14} - 3)}{5} \\
\frac{5}{\sqrt{14} - 3} = \frac{\sqrt{14} + 3}{5} = 1 + \frac{\sqrt{14} - 3}{5} \\
\frac{5}{\sqrt{14} - 2} = \frac{\sqrt{14} + 2}{2} = 2 + \frac{\sqrt{14} - 2}{2} \\
\frac{5}{\sqrt{14} - 2} = \frac{\sqrt{14} + 2}{5} = 1 + \frac{\sqrt{14} - 3}{5} \\
\frac{5}{\sqrt{14} - 2} = \frac{\sqrt{14} + 3}{5} = 6 + (\sqrt{14} - 3) \\
\therefore \sqrt{14} = 3 + \frac{1}{2 + \frac{1}{6 + \frac{1}{1 + \frac{1}{2 + \frac{1}{6 + \ldots}}}}} 
\]
QUOTIENTS OF MOMENT FUNCTIONS.

W.W.SAWYER.

Abstract: A formula is found giving the quotient of two moment functions as a moment function.

1. Introduction. I have been trying to prove that $L(s)$, the maximum eigenvalue of the integral equation

$$L(s)\phi(x,s) = \int_0^1 \phi(y,s) (1-xy)^{-1} dy$$

is a moment function of the parameter $s$ in the sense that

$$L(s) = \int_0^1 m(t)(1-st)^{-1} dt$$

with the weight distribution, $m(t)$, non-negative. This stems from an earlier paper, (1) which considered an integral equation, differing from equation (1) above only in having the integration from -1 to 1. This paper contained the history of the problem, and presented a number of conjectures based on computer data. These conjectures are equally plausible for equation (1).

If the solution $\phi(x,s)$ is normalized so that $\phi(0,s)=1$ for all $s$, by putting $s=0$ in equation (1) we obtain

$$L(s) = \int_0^1 \phi(y,s) dy.$$ 

If we can express $\phi(y,s)$ as a moment function with weight $w(y,t)$, and if it is legitimate to reverse the order of integration, we shall be able to express $L(s)$ as a moment function.

By a method involving iteration a sequence of functions, $\phi_n(x,s)$, can be found that increasingly approximate to a solution of equation (1). I have been able to express the earliest members, at any rate, of this sequence as moment functions. To obtain normalized functions it is necessary to consider $\phi_n(x,s)/\phi_n(0,s)$. It is then desirable to express this
2. A 4 x 6 Transportation problem is as follows:

\[ s_1 = 13, \quad D_1 = 3, \]
\[ s_2 = 5, \quad D_2 = 7, \]
\[ s_3 = 7, \quad D_3 = 10, \]
\[ s_4 = 11, \quad D_4 = 5, \]

Total \(36\)
\[ D_5 = 5, \]
\[ D_6 = 6, \]

Total \(36\)

N.B. Total cost of optimal solution = 125

Matrix of Unit Costs

<table>
<thead>
<tr>
<th></th>
<th>8</th>
<th>5</th>
<th>7</th>
<th>3</th>
<th>3</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>6</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>
It is a great advantage to be able to formulate a problem in terms of the classical transportation model if possible, since computer programs for this case are usually very much faster than the corresponding one for general L.P. It has been estimated that about a third of all L.P. problems found in practice turn out to be of the transportation type.

Timings to compare with the above figures are given by a problem involving $M = 30$ warehouses and $N = 1200$ customers, with which I was concerned using the special C.E.I.R. TS/90 transportation program on the same computer. This took only 20 minutes to solve! (N.B. it corresponded to $M + N - 1 = 1229$ equations of L.P.).

It should be noted that Phase I can occupy a good deal of computer time in practice. Indeed, there are many problems in which the principal object is to find out whether the system is at all feasible, never mind optimal.

**Illustrative Problems.**

A. The system: Maximize $z = 3x_1 - 2x_2$
   subject to $x_1 + x_2 \leq 1$
   $2x_1 + 2x_2 \geq 4$

   has no feasible solution, because constraints inconsistent.

B. The system: Maximize $z = x_1 + x_2$
   subject to $x_1 - x_2 \geq 0$
   $3x_1 - x_2 \leq 3$

   has no feasible solution because of non-negative conditions.

C. The system: Maximize $z = 2x_1 + 2x_2$
   subject to $x_1 - x_2 \geq -1$
   $-5x_1 + x_2 \leq 2$

   is unbounded.

D. The system: Maximize $2x_2 - x_1 = z$
   subject to $x_1 - x_2 \geq -1$
   $-5x_1 + x_2 \leq 2$

   is bounded ($z_{\text{max}} = 4$), but the values of $x_1$ and $x_2$ are unbounded.

**Problems to try yourself.**

1. Maximize $5x_1 + 3x_2$
   subject to $3x_1 + 5x_2 \leq 15$
   $5x_1 + 2x_2 \leq 10$
   $x_1 \geq 0$

   Solve (a) graphically, (b) by Simplex method.
Thus we introduce \( y_1 \) into row (1) and \( y_2 \) into row (2), here. There is no need for an artificial variable in row (3), as the positive slack \( x_5 \) can be included in the initial basis. The successive canonical forms of the simplex steps are given in the following table in detached co-efficient form:

<table>
<thead>
<tr>
<th>Basis</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
<th>( y_1 )</th>
<th>( y_2 )</th>
<th>( b' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_1 )</td>
<td>1</td>
<td>3</td>
<td>-1</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td>6</td>
</tr>
<tr>
<td>( y_2 )</td>
<td>2</td>
<td>1</td>
<td>-1</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td>4 E</td>
</tr>
<tr>
<td>( x_5 )</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td>4 W=10</td>
</tr>
<tr>
<td>w-10</td>
<td>-3</td>
<td>-4</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_2 )</td>
<td>1/3</td>
<td>1</td>
<td></td>
<td>1</td>
<td></td>
<td>1/3</td>
<td></td>
<td>2 D</td>
</tr>
<tr>
<td>( y_2 )</td>
<td>5/3</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td>1/3</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>( x_5 )</td>
<td>1/2</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td>1/2</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>w-2</td>
<td>-5/3</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td>1/2</td>
<td></td>
<td>4/3 W=2</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>1</td>
<td></td>
<td>-2/5</td>
<td>1/5</td>
<td></td>
<td>2/5</td>
<td>-1/5</td>
<td>8/5 A</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>1</td>
<td></td>
<td>1/5</td>
<td>-3/5</td>
<td></td>
<td>-1/5</td>
<td>3/5</td>
<td>6/5</td>
</tr>
<tr>
<td>( x_5 )</td>
<td>1/5</td>
<td></td>
<td>2/5</td>
<td>1</td>
<td></td>
<td>1</td>
<td>-2/15</td>
<td>6/5</td>
</tr>
<tr>
<td>w</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( z + \frac{6}{5} )</td>
<td></td>
<td>1/5</td>
<td>-3/5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( z = -\frac{6}{5} )</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>1</td>
<td></td>
<td>1/2</td>
<td>-1/2</td>
<td></td>
<td>1</td>
<td></td>
<td>B</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>1</td>
<td></td>
<td>1/2</td>
<td></td>
<td></td>
<td>3/2</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>1/2</td>
<td></td>
<td>1</td>
<td>5/2</td>
<td></td>
<td>3</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>( x )</td>
<td>1/2</td>
<td></td>
<td>3/2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( z = -3 )</td>
</tr>
</tbody>
</table>

With reference to the above graph, we have gone E, D, A in Phase I and A, B in Phase II.

**Timing on an Electronic Computer.**

Generally speaking, only the number \( n \) of constraints affects the timing for most computer D.P. programs, and the number of activities \( n \) is usually irrelevant in this respect.

Very often, the timing turns out to be proportional to \( n^3 \), so that if for instance, on a fast computer, a 100-equation problem could be solved in two minutes, it would take \( 2 \times 4^3 \) minutes, roughly, to solve a 400-equation problem on the same machine. (i.e. about 2 hours).

These times correspond in order of magnitude to what was realistically achievable when I was at C.E.I.R. by their LP/90 program.
quotient of moment functions as a single moment function.

2. **Theorem.** If, for summable functions \( W(t) \) and \( w(t) \) we have

\[
\begin{align*}
(4) & \quad F(z) = \int_0^1 W(t) (z-t)^{-1} \, dt \\
(5) & \quad f(z) = \int_0^1 w(t) (z-t)^{-1} \, dt \\
(6) & \quad 0(z) = F(z) / \{zf(z)\}
\end{align*}
\]

then

(7)
When we first learn the trigonometric functions of \( \theta \) we think of the angles as a simple idea, \( \sin \theta \) and \( \cos \theta \) as complicated functions of \( \theta \). Actually the situation is rather the reverse. We define \( x \) and \( y \) as the coordinates of the point on the unit circle at angle \( \theta \). \( \theta \) is measured by the length of the arc \( \theta \) to \( P \), which involves integration.

\[
\int dx = x, \quad \text{so } \frac{dx}{dy} = \sqrt{1 + y^2}
\]

For a circle of unit radius, \( x^2 + y^2 = 1 \)

Differentiating, \( 2x + 2y \frac{dy}{dx} = 0 \)

\[
\frac{dy}{dx} = -\frac{x}{y}
\]

\[
\int \sqrt{1 + \frac{x^2}{y^2}} \, dx = \sqrt{\frac{x^2 + y^2}{y^2}} = \frac{1}{y} \sqrt{1 - x^2}
\]

In the diagram, \( \theta = \sin^{-1} a \)

\[
\theta = \sin^{-1} a \quad \text{and} \quad \theta = \int_{x=0}^{x=a} \frac{1}{\sqrt{1-x^2}} \, dx
\]

Newton expanded \((1-x^2)^{-\frac{1}{2}}\) by the Binomial Theorem and thus arrived at an infinite series for \( \theta \). He managed to denote a series for \( \sin \theta = a \) in terms of \( \theta \) from this.
<table>
<thead>
<tr>
<th>Sequence</th>
<th>Sum</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$2a + 1$</td>
<td>$t \equiv 0 \mod 3$, $t \equiv 3 \mod 3$</td>
</tr>
<tr>
<td>3</td>
<td>$3a + 3$</td>
<td>$t \equiv 2 \mod 4$, $t \equiv 6$</td>
</tr>
<tr>
<td>4</td>
<td>$4a + 6$</td>
<td>$t \equiv 0 \mod 5$, $t \equiv 10$</td>
</tr>
<tr>
<td>5</td>
<td>$5a + 10$</td>
<td>$t \equiv 3 \mod 6$, $t \equiv 15$</td>
</tr>
<tr>
<td>6</td>
<td>$6a + 15$</td>
<td>$t \equiv 0 \mod 7$, $t \equiv 21$</td>
</tr>
<tr>
<td>7</td>
<td>$7a + 21$</td>
<td>$t \equiv 4 \mod 8$, $t \equiv 28$</td>
</tr>
<tr>
<td>8</td>
<td>$8a + 28$</td>
<td>$t \equiv 0 \mod 9$, $t \equiv 36$</td>
</tr>
<tr>
<td>9</td>
<td>$9a + 36$</td>
<td>$t \equiv 5 \mod 10$, $t \equiv 45$</td>
</tr>
<tr>
<td>10</td>
<td>$10a + 45$</td>
<td></td>
</tr>
</tbody>
</table>
Angular momentum vector in spherical polar.

Momentum: \( \frac{\hbar}{i} \left( y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) \left( 3 \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \)

\[ r = \sqrt{x^2 + y^2 + z^2}, \quad \tan \theta = \frac{y}{x}, \quad \tan \phi = \frac{z}{\sqrt{x^2 + y^2}} \]

\[ \cos \theta = \frac{1}{r} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \]

\[ \frac{2}{dx} = \frac{dx}{d\theta} \frac{d\theta}{dx} + \frac{d\theta}{d\phi} \frac{d\phi}{d\theta} = \frac{dx}{d\theta} \frac{d\theta}{dx} + \frac{d\theta}{d\phi} \frac{d\phi}{d\theta} \]

\[ -\sin \theta \frac{\partial \theta}{\partial x} = -\frac{x}{r^2} \frac{\partial r}{\partial x} = -\frac{1}{r} \sin \theta \cos \theta \cos \theta \]

\[ \frac{2}{dx} = \sin \theta \cos \phi \frac{2}{d\phi} + \frac{1}{r} \cos \theta \sin \phi \frac{d\phi}{dx} - \frac{1}{r} \sin \phi \frac{\partial}{d\phi} \]

\[ -\sin \theta \frac{dy}{d\phi} = -\frac{x}{r^2} \frac{dy}{dx} = -\frac{1}{r} \sin \theta \sin \phi \cos \phi \]

\[ \frac{dy}{d\phi} = +\frac{1}{r} \cos \theta \sin \phi \frac{d\phi}{dx} = \frac{x}{x^2 + y^2} = \frac{1}{r} \sin \theta \sin \phi \]

\[ \frac{dy}{d\phi} = \sin \theta \cos \phi \frac{2}{d\phi} + \frac{1}{r} \sin \phi \frac{d\phi}{dx} = +\frac{1}{r} \cos \phi \frac{2}{d\phi} \]

\[ x \frac{d}{dy} - y \frac{d}{dx} = 0 \cdot \frac{d}{dy} + 0 \cdot \frac{d}{dx} + \frac{1}{r} \sin \left[ y \sin \phi \cos \phi + y \sin \phi \sin \phi \right] \]

\[ = \frac{2}{d\phi} \frac{d}{d\phi} = \frac{d}{d\phi} \quad \text{as it should be} \]
\[ \psi = \mathcal{R}(\chi) \mathcal{S}(\theta, \phi) \]

Take \( m = 1, l = 1, n = 1 \).

\[ P_{1,1}(x) = \frac{1}{2} \frac{d}{dx} (x^2 - 1) = x = c \cos \theta \]

\[ P_{1,1} = (1 - x^2)^{\frac{1}{2}} \left( \frac{d}{dx} \right) P_1(x) = (1 - x^2)^{\frac{1}{2}} x \sin \theta \]

\[ S = e^{i\phi} \sin \theta \]
\[ \mathcal{R}(\chi) = \epsilon \ e^{-\chi^2} \mathcal{L}_2^3(\chi) \]

\[ \mathcal{L}_2^3(x) = \left( \frac{d}{dx} \right)^3 \mathcal{L}_2(x) \]

\[ \mathcal{L}_2(x) = e^{x} \left( \frac{d}{dx} \right)^2 e^{-x} x^2 \]

\[ = (x-2)x^2 = (x^2-4)x^2 \]

\[ = 4x^2 - 8x + 2 \]

So \( \mathcal{L}_2^3 = 0 \).

In fact \( n \geq l+1 \) is required.

Try \( m = 1, l = 2, n = 4 \).

\[ P_{2,1}(x) = P_{21}(x) = (1 - x^2)^{\frac{1}{2}} \left( \frac{d}{dx} \right) P_2(x) \]

\[ = (1 - x^2)^{\frac{1}{2}} \frac{d}{dx} \frac{3x^2 - 1}{2} = (1 - x^2)^{\frac{1}{2}} \cdot 3x. \]

This is \( e^{i\phi} \sin \theta \cos \theta \).
What happens when spherically symmetrical field is not Coulomb? 

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{df}{dr} \right) - \frac{\ell(\ell+1)}{r^2} f + (E + \alpha^2 r^2) f = 0
\]

\[
r^2 f'' + 2rf' + (E_\text{r}^2 - \ell(\ell+1) + \alpha^2 r^2) f = 0.
\]

Try \( a = \frac{2}{3} \).

\[
r^2 f'' + 2rf' + (E_\text{r}^2 - \ell(\ell+1) + \alpha_\text{r}^2) f = 0.
\]

Let \( f = e^{\gamma r} u \), \( f' = e^{\gamma r} \gamma u - e^{\gamma r} u' \), \( f'' = e^{\gamma r} u'' - 2e^{\gamma r} u' + \gamma e^{\gamma r} u \).

\[
0 = r^2 u'' - 2r u' + \frac{\gamma^2}{r} u
\]

\[
+ 2r u' - \frac{2\gamma}{r} u
\]

\[
+ \alpha (E_\text{r}^2 - \ell(\ell+1) + \alpha_\text{r}^2) u
\]

Choose \( \gamma \) so that \( E + \gamma^2 = 0 \).

\[
0 = r^2 u'' + \gamma u' (2r - 2\gamma^2) + \alpha (\alpha - \ell(\ell+1) - 2\gamma)
\]

\[
\text{weight: } 0 \quad 0 \quad +1 \quad 0 \quad +1
\]

\[
\text{Index of } u = \gamma(p-1) + 2r - \ell(\ell+1) + \alpha = \frac{p(p-1) - \ell(\ell+1)}{\alpha - \gamma/2}.
\]

\[
\text{Use value } p = \frac{\ell + 1}{2}
\]

\[
0 = \gamma^{\frac{\ell+1}{2}} u + 2\gamma^{\frac{\ell+1}{2}} u' + \frac{\ell+1}{2} \gamma^{\frac{\ell+1}{2}} u
\]

\[
+ 2 \gamma^{\frac{\ell+1}{2}} u' + 2 \gamma^{\frac{\ell+1}{2}} u
\]

\[
- 2 \gamma^{\frac{\ell+1}{2}} u' - 2 \gamma^{\frac{\ell+1}{2}} u
\]

\[
+ \alpha (\alpha - \gamma/2 - \ell(\ell+1))
\]

\[
0 = \gamma + p - \ell(\ell+1) + \alpha
\]

\[
p = \frac{-1 \pm \sqrt{1 + 4(l^2 + \ell(\ell+1))}}{2}
\]

Let \( p \) denote the real will + \( \gamma \).
June-July 67: They raised question of calculating $\int e^{x^2} \, dx$.

Integrals for now: $\int e^{-x^2} \, dx$.

Euler's formula for $x$: $e^{\pi i} = -1$.

Aug. Theory of vibrating strings: Solution of $f(x-t)+F(x+t)$.

Them of revolution for large oscillations.


October: D.E. with current waves

Circuit theory using complex numbers,

just for induction etc.


Prove that planets describe ellipses.

Make on a vertical circle.

Feb. Make al. min in 2 dimensions

April: Calculate $\int e^{-x^2} \, dx$ by $\int e^{-x^2} \, dx$.

Surface for which $\frac{\partial F}{\partial x} = 0$.

Change of partials: $f(x,y) = f(x,z)$.

Power series by formula of differential equation found.

May.
Coefficients of Legendre Polynomials of \( n \)th, \( L_n(x) \).

\[
L_n(x) = \sum_{r=0}^{n} C_{nr} x^r, \quad C_{00} = 1
\]

\[
C_{nr} = -C_{n,r-1} \frac{(n-r+1)(n+r)}{r^2}, \quad \text{integer arithmetic}
\]

\[
C_{n+1,r} = (-1)^r \frac{(n+r)(n+2r+3)}{2r} C_{n,r}
\]

This is number of choices for \( n \) things, \( r \) things of which \( n-r \) are identical, also \( r \) and \( r \) sets of identical objects, hence always whole number.

Missing code snippet:

```
DIM C(60, 60)
FOR N = 0 TO 60
P. "M = ": INPUT M
DIM C#(M, M)
FOR N = 0 TO M: C#(N, 0) = 1: NEXT.
N = 1
REPEAT
FOR R = 1 TO N
C#(N, R) = -(N-R+1)(N+R) C#(N, R-1) / R / R.
NEXT
N = N + 1
UNTIL N > M
END.
```

Choose \( M \) large, and see what shape array is.

Too large. Then choose \( M \) below this.
\[
\sum_{r=1}^{n-1} \frac{1}{(n+\frac{1}{2}-r)\sqrt{r}} = \sum_{r=1}^{n-1} \frac{1}{\sqrt{r}} \left( \frac{1}{n+\frac{1}{2}} + \frac{1}{n+\frac{1}{2}-r} \right) = \frac{1}{n+\frac{1}{2}} \sum_{r=1}^{n-1} \frac{1}{\sqrt{r}} + \frac{1}{n+\frac{1}{2}-r} \sum_{r=1}^{n-1} \frac{1}{\sqrt{r}} = \frac{1}{n+\frac{1}{2}} \left( 1 + \frac{1}{2} + \ldots + \frac{1}{n-1} \right) \\
\frac{1}{2} + \frac{3}{4} + \ldots + \frac{1}{n} \leq \int_{1}^{n} \frac{1}{x} \, dx = \ln n.
\]

\[
\sum_{r=1}^{n} \frac{1}{(n+\frac{1}{2}-r)\sqrt{r}} = \sum_{r=1}^{n} \frac{1}{\sqrt{r}} \left( \frac{1}{n+\frac{1}{2}} + \frac{1}{n+\frac{1}{2}-r} \right) = \frac{1}{n+\frac{1}{2}} \sum_{r=1}^{n} \frac{1}{\sqrt{r}} + \frac{1}{n+\frac{1}{2}-r} \sum_{r=1}^{n} \frac{1}{\sqrt{r}} = \frac{1}{n+\frac{1}{2}} \left( 1 + \frac{1}{2} + \ldots + \frac{1}{n-1} \right) \\
\frac{1}{2} + \frac{3}{4} + \ldots + \frac{1}{n} \leq \int_{1}^{n} \frac{1}{x} \, dx = \ln n.
\]

The sum of the 'triangles' is less than or equal to \( \frac{1}{2} \) as \( n \) approaches infinity.

\[
\Psi(x+h) = \log x - \sum_{k=1}^{x} (-1)^{k} B_{1}(k) \frac{B_{1}(k)}{x} + \sum_{k=0}^{\infty} \frac{B_{1}(k)}{(x+3)^{k+2}} \\
\Psi(x) = \log x + \frac{B_{1}(k)}{x} + \sum_{k=0}^{\infty} \frac{B_{1}(k)}{(x+3)^{k+2}} \\
B_{1}(k) = \int_{0}^{1} \frac{B_{1}(x)}{(x+3)^{k+2}} \, dx
\]

\[
\int_{k}^{k+1} \frac{B_{1}(x)}{(x+3)^{k+2}} \, dx = \int_{0}^{1} \frac{B_{1}(x)}{(x+3)^{k+2}} \, dx = \int_{0}^{1} \frac{8 - \frac{1}{2} x^2}{(3+x-k)^2} \, dx \\
\frac{3}{2} = \left( \frac{3+x-k}{3+x-k} \right) \frac{1}{3+x-k} \frac{d}{dx} \left[ \log (3+x-k) \right] - \log (x-k) = \frac{1}{x+\frac{3}{2}-k} \left[ \frac{1}{3+x-k} \right]_{0}^{1} \left( x+\frac{3}{2}-k \right) \left[ \frac{1}{3+x-k} \right]_{0}^{1} \left( x+\frac{3}{2}-k \right)
\]
The Phase I Procedure (Illustrated by Problem C).

Unlike the transportation problem, or the case when all the slacks are positive, an initial feasible solution may not be available immediately in certain problems. Indeed, a feasible solution may not exist, i.e., it is impossible to satisfy the constraints with non-negative values of the activities.

Consider the following system:

Maximize \( x_1 \)

subject to \( x_1 + 3x_2 \geq 6 \)
\( 2x_1 + x_2 \geq 4 \)
\( x_1 + x_2 \leq 4 \)

and \( x_1, x_2 \geq 0 \).

When put into standard form, this becomes

Minimize \( z = -x_1 \) \hspace{1cm} (0)

subject to \( x_1 + 3x_2 - x_3 = 6 \) \hspace{1cm} (1)
\( 2x_1 + x_2 - x_4 = 4 \) \hspace{1cm} (2)
\( x_1 + x_2 + x_5 = 4 \) \hspace{1cm} (3)

and \( x_j \geq 0 \quad j = 1 \) to \( 5 \).

\( x_5 \), being a positive slack can go into the initial basis, but not so \( x_3 \) and \( x_4 \) which are negative slacks.

Graphical picture

\( z \) max = 3 at \( B(3, 1) \)

Introduction of artificial variables

Artificial variables \( y_1, y_2 \) are introduced into the rows that need them to give an initial basis.

A new objective function \( w = y_1 + y_2 + \ldots \) is introduced and this is minimized (as for Phase II with \( z \)). Whilst \( w \) is positive, Phase I lasts, until finally Phase I ends with \( w = 0 \) and each artificial variable has been eliminated from the basis (i.e., has become non-basic).
Unlike what happened to \( x_6 \) in Problem B, once an artificial is removed from the basis it cannot return. If \( w \) cannot be reduced to zero, the problem is infeasible.
Problem B.

Consider the following problem in 2 variables:

Maximize \[ z = 2x_1 + x_2, \] \hspace{1cm} (1)
\[ \begin{align*}
x_1 + 2x_2 & \leq 10 \\
x_1 + x_2 & \leq 6 \\
x_1 - x_2 & \leq 2 \\
x_1 - 2x_2 & \leq 1
\end{align*} \] \hspace{1cm} (2)

and \[ x_1, x_2 \geq 0 \] \hspace{1cm} (3)

It appears that here we have \( m > n \) (i.e. \( m = 4, n = 2 \)); but when we put in the four positive slacks \( x_3, x_4, x_5, x_6 \), we have \( m = 4, n = 6 \) with \( m < n \), as it should be.

Graphical Interpretation of equations (2).

If we plot \( x_1 \) against \( x_2 \) on a graph, we need only concern ourselves with the positive quadrant, as \( x_1 \geq 0, x_2 \geq 0 \).

Also the constraints (2) above form four straight line boundaries:

\[ z_{\text{max}} = 10 \] at \( D(4, 2) \)
Types of Solution

1. A feasible solution.

Any set of values of the $x_j$ which satisfy the constraints (2) and non-negative conditions (3).
(e.g. in Problem B, any point $(x_1, x_2)$ in or on polygon ABCDEF).

2. A basic solution.

A solution of equations (2) obtained by letting $n - m$ of the variables be zero, and solving for the remaining $m$ variables. The $m$ chosen variables constitute what is called the basis.

A basic feasible solution is defined as a basic solution of equations (2), which also satisfies conditions (3).

For example, in Problem B, when the slacks are added, equations (2) become:

\[
\begin{align*}
    x_1 + 2x_2 + x_3 &= 10 \\
    x_1 + x_2 + x_4 &= 6 \\
    x_1 - x_2 + x_5 &= 2 \\
    x_1 - 2x_2 + x_6 &= 1
\end{align*}
\]

Here we can set 2 out of the 6 variables zero and solve for the other 4 in $6C_2 = 15$ different ways, viz:—

\[
\begin{array}{cccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
0 & 0 & 0 & 0 & 0 & 10 & 6 & 2 & 1 & 2 & 14/3 & 11/2 & 4 & 13/3 & 3 \\
0 & 5 & 6 & -2 & -1 & 0 & 0 & 0 & 0 & 4 & 8/3 & 9/4 & 2 & 5/3 & 1 \\
10 & 0 & -2 & 14 & 11 & 0 & 4 & 8 & 9 & 0 & 0 & 2 & 7/3 & 5 & 0 \\
6 & 1 & 0 & 8 & 64 & -4 & 0 & 2 & 5 & 0 & -4/3 & -7/4 & 0 & 0 & 2 \\
2 & 7 & -4 & 0 & 24 & -8 & -4 & 0 & 1 & 4 & 0 & -5/4 & 0 & -2/3 & 0 \\
11 & 1 & -11 & -3 & 0 & -9 & -5 & -1 & 0 & 7 & 5/3 & 0 & 1 & 0 & 0 \\
A & F & B & E & D & C
\end{array}
\]

It is seen that nine out of the 15 basic solutions are non-feasible as they violate the non-negative conditions. (The 6 feasibles are marked A to F in table).

Also the 6 solutions which are feasible, correspond to the corners ABCDEF of the polygon (shaded on graph). This is a property of basic feasible solutions.

With 2 variables, we always get a convex polygon like ABCDEF;

With 3 variables, we always get a convex polygon, with basic solutions at corners;
orientations. Some kind of network topology.

ABCEGFDB

ABCE  E is in front of AE

BCFE  G = BE

CGBF  F = CA

and so on.
Topics.

Lockstep, Grades 5-8: W. M. Brown. Affects our school work of some of our students. Much of the work "ordinary" or "average." Today, more of the "advanced" work is required to lockstep. However, we must work carefully to code the work. Many students in which good variety of pace - some higher pupils fear to become complacent. Smiles - desired work. Need to go on to daily work with systematic textbook. Occasional conflict for gifted - scale altogether.

Works.

The escape from bad teaching:
Seven (years) & 50 - partly back to school. My model a good teacher.
I.2.88. \( Q(z) = \sum_{n=0}^{\infty} c_n z^n \), \( R(z) = \sum_{n=0}^{\infty} d_n z^n \), \( P(z) = \frac{a_0}{z} \). If \( |z| < \rho \), \( R(z) \to 1 \) for \( |z| > \rho \). If \( r > \rho \), we have a convergence ring of \( Q(z) \), Laurent series.

Then \( |c_n|/\rho^n \leq M \).

CHAPTER 3.

I.3.81. A collection of power series is called a monogenic system if (1) each power series is a continuation of any other, (2) every "remodeling" of any power series belongs to the system.

\( f(z) \) defined as follows. For \( z \in \rho \), we consider the power series of the system that have \( z = \rho \) inside circle of convergence. We define \( f(z) \).
P(3|a) and \( P_1(3|a) \)
and a region \( D \) lies inside both circles of convergence.
If \( P(3|a) = P_1(3|a) \)
for \( z = b_1, b_2, b_3, \ldots \) (point on \( D \))
where \( b_n \to b \) \( b \) in \( D \),
then the series are equal for every \( z \) in \( D \).

I.2.49. Laurent series
\[ G(z) = \sum_{\infty}^{\infty} c_n z^n \]
\[ P(z) = \sum_{0}^{\infty} c_n z^n \quad P_1(z) = \sum_{\infty}^{\infty} c_n z^n \]
\[ P(z) \text{ for } |z| < r \]
\[ P_1(z) \text{ for } |z| > r. \]

Convergence in \( z \) if \( r > r_1 \).
Suppose \( r_1 < r < r \)
\[ M = \max_{|z| = r} |G(z)| \text{ on the circle with } |z| = r \]
Then \[ |c_n| r^n \leq M \quad -\infty < n < +\infty. \]

[We have met this from Cauchy \( c_n = \frac{1}{2\pi i} \int_{|z| = r} \frac{f(z) dz}{z^{n+1}} \)]

Chapter 2 The concept of an analytic function.
§1. Monogenic system. A system of power series:
1. Any power series in it is a continuation of any other in \( D \).
2. Any remodelling of a power series in it belongs to the system.

§2. We have a monogenic system. We define a function possibly "many-valued" as follows. For a point \( z_0 \) we construct three power series in this system for which \( z_0 \) lies inside the circle of convergence. We associate with \( z_0 \) the value (s) given by these. We can define this: let \( P(3|z_0) \) belong to the system, and be \( z_0^{\infty} c_n (z-z_0)^n \). We take \( c_0 \) as a value of \( f(z_0) \).
\[ f(3) = \sum_0^\infty a_n r^n \]

\[ a_n = \frac{1}{2\pi i} \oint \frac{f(3) \, dg}{g^{n+1}} \]

Take \( O \) as circle of radius \( r \):

\[ M = \max |f(3)| \]

\[ |a_n| \leq \frac{1}{2\pi} \oint \frac{|f(3)| \, dg}{|g|^{n+1}} \]

\[ = \frac{1}{2\pi} \oint \frac{1 |f(3)| \, dg}{3^{n+1}} \]

\[ \leq \frac{1}{2\pi} \frac{M}{r^{n+1}} \]

\[ = \frac{M}{r^n} \]

\[ \therefore |a_n| r^n \leq M \]
\[ \frac{x^2}{x+4} \div \frac{x-4}{x^2-4x} = \frac{x^2}{x+4} \cdot \frac{x(x-4)}{x-4} = \frac{x^3}{x+4} \]

Given that \( x^3 \div 16 \) will have a remainder of 16.

For \( x = 5 \), \( 25 \div 16 \) will have a remainder of 9.

\[ x(\text{remainder}) = x^2 + x + n \]

Given that \( x \) is a prime number.

\[ \frac{x^2 + x}{x+1} = x - 1 \]

The remainder is 0.

\[ \text{remainder} = \frac{\text{remainder}}{16} = \frac{9}{16} \]

\[ (x-1)^2 \cdot (x+1) = x^3 - x + 1 \]

The remainder is 0.

\[ \text{remainder} = \frac{(x-1)^2}{16} = \frac{9}{16} \]

\[ (x-1)^2(x+1) = x^3 - x + 1 \]

\[ \text{remainder} = 0 \]

\[ \text{remainder} = \frac{(x-1)^2}{16} = \frac{9}{16} \]

\[ (x-1)^2(x+1) = x^3 - x + 1 \]

\[ \text{remainder} = 0 \]
\[ Z = 5, \quad \frac{1}{2}(Z-2)(Z-3) = \frac{1}{2} \cdot 3 = 3. \]

Take \( h_{ij} = Y_{i-1,j-1} \)

\[
\begin{bmatrix}
1 & 0.5 & 0.2 & 0.05 & 0.005 \\
0.5 & 1 & 0.5 & 0.25 & 0.025 \\
0.2 & 0.5 & 1 & 0.5 & 0.25 \\
0.05 & 0.25 & 0.5 & 1 & 0.5 \\
0.005 & 0.025 & 0.25 & 0.5 & 1
\end{bmatrix}
\]

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Polar coordinates

\[ \tan \psi = \frac{\frac{dr}{d\theta}}{r} = \frac{\frac{dr}{d\theta}}{\frac{d\theta}{d\phi}} \]

If \( r = a e^{b\phi} \)

\[ dr = bae^{b\phi} d\phi \]

\[ \tan \psi = \frac{1}{b} \]
\[ L = \frac{1}{\gamma} \sqrt{\frac{v^2}{\gamma^2 + v^2 \Theta^2}} \]

\[ \frac{d}{dt} \left[ \frac{v}{\gamma \sqrt{\frac{v^2}{\gamma^2 + v^2 \Theta^2}} \Theta} \right] = -\frac{1}{\gamma^2} \sqrt{\frac{v^2}{\gamma^2 + v^2 \Theta^2}} + \frac{1}{\gamma^2 \sqrt{\frac{v^2}{\gamma^2 + v^2 \Theta^2}}} \dot{\Theta} \]

\[ = -\frac{v^2}{\gamma^2 \sqrt{\frac{v^2}{\gamma^2 + v^2 \Theta^2}}} \frac{\Theta}{\gamma^2} + \frac{v^2}{\gamma^2 \sqrt{\frac{v^2}{\gamma^2 + v^2 \Theta^2}}} \dot{\Theta} \]

\[ = -\frac{v^2}{\gamma^2 \sqrt{\frac{v^2}{\gamma^2 + v^2 \Theta^2}}} \frac{\Theta}{\gamma^2} - \frac{v^2}{\gamma^2 \sqrt{\frac{v^2}{\gamma^2 + v^2 \Theta^2}}} \dot{\Theta} \]

\[ = \frac{v^2}{\gamma^2 \sqrt{\frac{v^2}{\gamma^2 + v^2 \Theta^2}}} \left( \frac{\Theta}{\gamma^2} - \dot{\Theta} \right) \]

If \( \gamma = b \), \( \frac{dv}{dt} = \frac{b}{b^2} = b \varepsilon \)

\[ \frac{1}{\gamma \sqrt{\frac{v^2}{\gamma^2 + v^2 \Theta^2}}} \Theta = -\frac{b^2 \varepsilon^2 \Theta}{\gamma^2 \sqrt{\frac{v^2}{\gamma^2 + v^2 \Theta^2}}} = -\frac{b^2 \Theta}{\gamma \sqrt{b^2 + 1}} \]

\[ \frac{\dot{\Theta}}{\gamma \sqrt{\frac{v^2}{\gamma^2 + v^2 \Theta^2}}} = \frac{b \varepsilon}{\gamma} \]

\[ \frac{d}{dt} \left( \frac{v}{\sqrt{b^2 + 1}} \right) = \frac{b \varepsilon^2 \dot{\Theta}}{\gamma^2 \sqrt{b^2 + 1}} - \frac{b \varepsilon \dot{\Theta}}{\gamma^2 \sqrt{b^2 + 1}} \]

\[ = \frac{b \varepsilon^2 \dot{\Theta}}{\gamma^2 \sqrt{b^2 + 1}} - \frac{b \varepsilon \dot{\Theta}}{\gamma^2 \sqrt{b^2 + 1}} \]

So PE is conserved.
\[ L F(x, \sigma) = \int \frac{1}{1 - \sigma x y} F(y, \sigma) \, dy \]

\[ L F(x, \sigma) = \frac{\frac{1}{2} F(a, \sigma) + \frac{1}{2} F(b, \sigma)}{1 - \sigma ax + 1 - \sigma bx} = \frac{A(\sigma)}{1 - \sigma ax} + \frac{B(\sigma)}{1 - \sigma bx} \]

\[ F(a, \sigma) = \frac{A(\sigma)}{1 - \sigma ax} + \frac{B(\sigma)}{1 - \sigma bx} \]

\[ F(b, \sigma) = \frac{A(\sigma)}{1 - \sigma ab} + \frac{B(\sigma)}{1 - \sigma bx} \]

\[ \Delta = \left[ \frac{1}{1 - \sigma a^2} - \frac{1}{1 - \sigma b^2} \right]^2 + \frac{4}{(1 - \sigma a^2)(1 - \sigma b^2)} \]

\[ \Delta = \left[ \frac{1}{1 - \sigma a^2} - \frac{1}{1 - \sigma b^2} \right]^2 + \frac{4}{(1 - \sigma ab)^2} \]

\[ \sigma(a^2 - b^2) - \sigma^2 ab(a^2 - b^2) = 2i \left[ 1 - \sigma(a^2 + b^2) + \sigma^2 a^2 b^2 \right] \]

\[ 0 = 2i + \sigma(b^2a^2 - 2ia^2 - 2ib^2) + \sigma^2 ab(a^2 - b^2 + 2iab) \]

\[ = 2i + \sigma(b^2 - 2a^2 - 2ib^2) + \sigma^2 ab(a + ib)^2 \]

\[ b^2(1 - 2i) - a^2(1 + 2i) = \left( \frac{4 - 3 + 4i}{8i \sigma^2 - \delta^2} \right) + 8i \sigma^2 + 16 \sigma \delta^2 \]

\[ \delta = 1 - 4 - 4i \]

\[ \sigma(\cdot, 1.9) = 2i \]

\[ \sigma(\cdot, 1.9) = \frac{2i}{(1 - \sigma)(1 - 1.8i \delta)} = 1 - \delta(1.9) \]

\[ 1.9 \delta - \delta^2 = 2i \left( 1 - 1.8i \delta + \delta(\sigma^2) \right) = 2i - 2.62i \delta + \delta(\sigma^2) \]

\[ \sigma^2(1.71 + 1.62i) - (1.9 + 3.62i) \delta + 2i \delta = 0 \]

\[ \sigma^2 = 1.9 + 3.62i \pm \sqrt{(1.9 + 3.62i)^2 - 8i(1.71 + 1.62i)} \]

\[ 2(1.71 + 1.62i) \]
\[ t = ct, \quad t' = ct' = \beta (ct - \frac{v}{c}x) = \beta (T - \frac{vx}{c^2}) \]
\[ \alpha' = \beta (\alpha - vt) = \beta (x - \frac{v}{c}t) \]
Take \( c = 1 \).
\[ x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \]
\[ t' = \frac{T - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ x'^2 = vt' \]

\[ \frac{x'\cos \theta}{1} = f \]
\[ x'\cos \theta = \text{g} \]
\[ \sqrt{\left( \frac{x}{2} \right) \text{m/s}} = \text{g} \]
\[
\left(\frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6} + \frac{x^6}{7}\right)^2
\]
\[
= \frac{x^2}{4} + \frac{x^3}{3} + x\left(\frac{x}{9} + \frac{1}{4}\right) + x^2\left(\frac{1}{5} + \frac{1}{6}\right) + x^3\left(\frac{1}{6} + \frac{2}{15}\right) + \ldots
\]
\[
x^7\left(\frac{1}{10} + \frac{1}{9} + \frac{1}{7}\right)
\]

\[
\begin{array}{ccccccccc}
& a & 5 & b & 3 & 12 \\
9 & c & 16 & 0 \\
& e & 2 & 9 & 4 & 7 \\
8 & 7 & 4 & 7 & 19 \\
6 & 8 & 4 & 20 & 7
\end{array}
\]
\[
a + b = 25
\]
\[
c + d = 25
\]
\[
f + g =
\]
played a decisive role. In analysis we are much concerned with questions about limits. For both real and complex numbers, a sequence of numbers \( z_n \) is tending to a limit \( L \) if

the distance of \( z_n \) from \( L \) is tending to zero. He decided that, if it was possible to find a satisfactory definition of distance between two mathematical objects (of any kind), it would be possible to find theorems about these objects analogous to the theorems about real and complex numbers.

The first question then is: what is a satisfactory definition of distance? He looked at the traditional proofs and found the only properties of distance used were the following very simple ones:

1. Distance is measured by a real number, which is never negative.
2. A distance is zero if, and only if, it is the distance between a point and itself.
3. The distance from \( A \) to \( B \) is the same as the distance from \( B \) to \( A \).
4. You cannot shorten your journey by breaking it. If you go from \( A \) to \( C \), and then from \( C \) to \( B \), the total distance cannot be less than the distance from \( A \) to \( B \). (It may of course be equal, if \( C \) lies on the direct route from \( A \) to \( B \).) This is known as the triangle axiom. It corresponds to Euclid’s remark, that the sum of the lengths of two sides of a triangle must exceed the length of the third side.

Frechet's investigation was extraordinarily fruitful. It was found possible to find a satisfactory definition for the distance between two matrices, two transformations, two functions, two operations that may involve differentiation and integration. At one blow, this opens the door to a whole series of results concerning the most varied situations.

It is often possible to find more than one definition of distance for given objects. For instance, on a chessboard we can define distance as the minimum number of moves a king needs to get from one square to another. We get a different definition if we consider a rook instead of a king. Options can equally well arise in more serious mathematical contexts.

The distance from \( A \) to \( B \) is the length of \( AB \). We now consider defining length. Of the many possible definitions we shall here consider only those that lead to our usual geometry, or to a geometry very similar to it. In all the spaces now to be listed, Pythagoras Theorem is true in some sense. All these spaces are vector spaces.

The symbol \( ||u|| \) will be used for the length of the vector \( u \). As \( v-u \) is the vector that goes from the point \( u \) to the point \( v \), \( ||v-u|| \) gives the distance of the point \( u \) from the point \( v \).

### LENGTH

1. Euclidean space of 2 dimensions.
   If \( u=(u_1, u_2) \), we define length by
   
   \[
   ||u||^2 = u_1^2 + u_2^2.
   
   With the help of Pythagoras Theorem, we can define
\[ f(x_1)^2 + f(x_2)^2 + \ldots + f(x_n)^2. \]

The distances between the points so obtained for various functions might give us a useful way of measuring the distances between the functions. However, the method is rather rough and ready. How big should \( n \) be? Why choose the mid-point of each interval?

Now a sum resembling that just written would appear if we were making an estimate of the value of \( \int_p^q f(x)^2 \, dx \).

This suggests that we might define the length of the function \( f \) by
\[ ||f||^2 = \int_p^q f(x)^2 \, dx. \]

This leads to something quite novel. We can define a dot product for this space, and thus find a meaning for one function being perpendicular to another.

The work follows the same pattern as before, but with integrals instead of sums.

When we were finding the condition for \( u \) to be perpendicular to \( v \), the length of the hypotenuse was
\[ ||(v-u)|| = \sqrt{||u||^2 + ||v||^2}. \]

If \( u \) corresponds to \( f(x) \) and \( v \) to \( g(x) \), \( v-u \) will correspond to \( g(x) - f(x) \). With the definition of length just found this condition will become
\[ c^2 = a^2 + b^2 \]
where
\[ a^2 = \int_p^q (g(x)-f(x))^2 \, dx, \quad b^2 = \int_p^q g(x)^2 \, dx. \]

We now need to multiply out the bracket that appears in \( c^2 \).

This gives
\[ c^2 = \int_p^q g(x)^2 - 2f(x)g(x) + f(x)^2 \, dx. \]

The integrals of \( f(x)^2 \) and \( g(x)^2 \) appear on both sides of the equation \( c^2 = a^2 + b^2 \), and cancel just as the squares did in the earlier examples. Again we divide by \(-2\) to arrive at the definition
\[ f.g = \int_p^q f(x)g(x) \, dx. \]

We shall say that the functions are perpendicular if this dot product is zero.

"Orthogonal" is a synonym for "perpendicular" and it is the custom to-day to speak of orthogonal functions rather than perpendicular ones. I do not know the reason for this. Perhaps the idea of functions being perpendicular is felt to be rather shocking, and the more learned word is used to lessen the shock.
The fact is extremely close. If we take \( u_n \) as the vector representing \( \sin nx \), equation (5) can be written \( u_m \cdot u_n = 0 \), which means that the sines are represented by mutually perpendicular vectors, while equation (6) says
\[ f \cdot u_3 = c_3 \cdot u_2 \cdot u_3, \]
which is of exactly the same form as equation (3), earlier.

Thus finding the coefficients in a Fourier series turns out to be just the problem of expressing a vector in a new system of perpendicular axes. This immediately suggests a thought. There are many ways to choose a set of perpendicular axes. There must be many other systems of perpendicular functions, from which we could derive series by exactly the same procedure. One such system is much simpler, and could even be used to compose problems in beginning calculus; all the functions in it are polynomials.

ORTHOGONAL POLYNOMIALS

With \([-1,1]\) as the basic interval, let \( P_0(x) = 1, \)
\[ P_1(x) = x, \quad P_2(x) = 3x^2 - 1, \]
\[ P_3(x) = 5x^3 - 3x, \quad P_4(x) = 35x^4 - 30x^2 + 3. \]
You can check that these are orthogonal. The work can be reduced by using the following observation: the dot product involves the vectors linearly. For example,
\[ f(aU + bV + cW) = a(f \cdot U) + b(f \cdot V) + c(f \cdot W). \]
This means that, if \( f \) is perpendicular to \( u \), to \( v \), and to \( w \), it is bound to be perpendicular to \( aU + bV + cW \), for any \( a, b, c \). This holds for any number of vectors in the bracket; if \( f \) is perpendicular to each of them, it is perpendicular to any linear mixture of them.

So, if we check that \( P_4(x) \), for example, is perpendicular to \( 1 \), to \( x \), to \( x^2 \) and to \( x^3 \), it is bound to be perpendicular to any linear combination of these, hence to any polynomial with degree less than 4, hence in particular to \( P_0(x) \), to \( P_1(x) \), to \( P_2(x) \) and to \( P_3(x) \). The same idea can be applied to \( P_n(x) \).

Further polynomials in this sequence can be obtained by taking \( P_n(x) = (d/dx)^n (x^2 - 1)^n \). That \( P_n(x) \) is perpendicular to \( x^m \) if \( m < n \) can be proved by integration by parts. Observe that \( (d/dx)^n (x^2 - 1)^n \) is zero for \( x = 1 \) and \( x = -1 \) if \( s \leq n \); do not expand any of the powers of \( x^2 - 1 \).

If multiplied by certain constants, these polynomials give the Legendre polynomials, which play an important part in electromagnetic theory and other branches of science. They can be used to build series in much the same way that sines are for Fourier series. Like Fourier series, they are capable of representing functions that have discontinuities.
and \((x, y, z) = 0\) means that \(\int \text{d}x + m\text{d}y + n\text{d}z = 0\) is independent of path. The surface \(\int \text{d}x + m\text{d}y + n\text{d}z = \text{constant}\) is a surface.

81. If there is no system of surfaces \(F\) rays, the problem of tracing is impossible.

If \(\text{ray comes to} x, y, z\) after crossing a surface at \(x_1, y_1, z_1\),

distance \(p = \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}\).

\[dx - dx = \text{d}p + p\text{d}e\text{cl}\]

Multiply, by \(p\) and add, \(\text{d}d + m\text{d}y + n\text{d}z = 0\) by \(1\).

\[\text{d}d + m\text{d}y + n\text{d}z = 0\text{ as before.}\]

Thus \(\text{d}E = p\text{d}p\) \(\text{can be put in the form}\) \(\text{d}p + p\text{d}e = 0\).

This enables us to construct mirror surfaces. Choose points on the normals to surface such that sum of distances to required focus is constant. (Difference of rays diverge from focus.)

The surface of constant action.

82. We have shown that a system of \(F\) rays can be brought to a point only if there are \(F\) surfaces. If rays diverge from a point and are reflected by any kind of mirror, these mirror surfaces \(F\) reflected rays. He shows that after any \(n\) of reflections, a surface cutting rays \(F\) makes total path constant.

83. The argument seems to be that variation at each point of reflection leaves the path in the same stationary, and variation in \(F\) surface leaves first length stationary, so total variation is zero.
$Sp + \mu Sp' = 0$ leads to $\int nd\sigma$ as formula for action.
Here again, rays are perpendicular to surfaces of equal action.

If ray has direction $(l, m, n)$ and $V = \theta_1 \theta_2 \theta_3$

$(l, m, n) = \text{grad} V$

Here $V = \xi m \Delta_s$, $m$ being the ratio of refractive indices for refracted and incident rays.

If $I = \int \mu ds$

$\lambda = \frac{1}{\mu} \frac{\partial I}{\partial x}, \ \ m = \frac{\partial I}{\partial y}, \ \ n = \frac{\partial I}{\partial z}$

(This agrees with Whittaker.)

$I$ is the characteristic for for curved rays, and will be denoted by $\nu$ in future.

$\nu^2 = \left(\frac{\partial I}{\partial x}\right)^2 + \left(\frac{\partial I}{\partial y}\right)^2 + \left(\frac{\partial I}{\partial z}\right)^2$

518. For repeated rays $\Sigma p = \text{constant}$ gives surface $I$ rays.

"If light be a material subsance its velocity in unphysical medium is proportional to the refractive power and is not altered by reflection" these surfaces will be called "surfaces of constant action."

Thus we have $\mu ds = \text{constant}, \nu$ being the velocity expected in particle $\mu$ of light.

\[\mu ds = \text{constant}, \nu \text{ being the velocity expected in particle $\mu$ of light.}\]

PROOF. $\frac{\partial x}{\partial z}, \frac{\partial y}{\partial z}, \frac{\partial z}{\partial z}$ are obtained by expressing $v$ as a homogeneous polynomial of degree 1 in $x, y, z$, while $\alpha, \beta, \gamma$ are $\xi, \eta, \zeta$ for the ray. The identity $x^2 + y^2 + z^2 = \alpha \beta \gamma$ is used to make the homogenous in rays described.
\[ \int_{-1}^{1} \int_{-1}^{1} x \left( \frac{\ln(1+x) - \ln(1-x)}{1-2xy} \right) \, dx \, dy = \frac{8}{3} + \frac{4}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} \cdot 3 + \ldots \]

This is obtained by using partial fractions and series expansions.

Thus,

\[ = 4 + \frac{8}{9} x^2 + \frac{4}{5} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot 3 + \ldots \]

\[ = 4 + \frac{2}{3} x^2 + \frac{2}{5} \cdot \frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot 3 + \ldots \]

So,

\[ \int_{-1}^{1} \int_{-1}^{1} \left( \frac{x}{1-2xy} \right) \, dx \, dy = \frac{8}{3} \cdot 4 \]

\[ = 2 \int_{-1}^{1} \int_{-1}^{1} \frac{x}{(1-2xy)(x-3)} \, dx \, dy = 8 \mu_2. \]
\[ \text{Proof Page 131:} \quad I \text{ think the result of this section can be obtained directly from \( \varepsilon(28) \). Condition is} \]
\[ S P_i dQ_i - S Q_i dP_i = \quad \text{(with tensor summation implied)} \]
\[ = S p_i dQ_i - S q_i dP_i. \]

\[ S P_i dQ_i - S Q_i dP_i = (\frac{\partial P_i}{\partial Q_i} + \frac{\partial Q_i}{\partial P_i}) (S dP_i + S dQ_i) - \]
\[ - \left( \frac{\partial P_i}{\partial Q_i} + \frac{\partial Q_i}{\partial P_i} \right) \left( \frac{\partial P_i}{\partial Q_i} + \frac{\partial Q_i}{\partial P_i} \right). \]

\[ S p_i dQ_i \text{ does not occur in } \varepsilon. \text{ Thus we must have } u = 0. \]

\[ \text{Consider } u = 1, \; v = 2. \]

\[ \frac{\partial P_i}{\partial Q_i} \; \frac{\partial Q_i}{\partial P_i} \; \text{ Take } n = 2. \]

\[ \left( \frac{\partial P_i}{\partial Q_i} + \frac{\partial Q_i}{\partial P_i} \right) \left( \frac{\partial P_i}{\partial Q_i} + \frac{\partial Q_i}{\partial P_i} \right). \]
We thus find:

\[ \frac{A_{n,n+1}}{A_{n,n}} = n \sum_{i=0}^{n-1} \frac{L(n, r+i)}{L(n, r)} = \frac{L(n+1, n-1)}{L(n, n)} + \]

Such ratios seem to offer the most convenient method of working.

Thus:

\[ \frac{L(n+1, n+2)}{L(n, n+1)} = \Delta \left( \frac{A_{n,n+1} - L(n+1, n-1)}{A_{n,n}} - \frac{L(n+1, n+1)}{L(n, n)} \right) \]

\[ = \Delta \left[ \frac{(n+1)(2n+1)}{(4n+3)^2} \right]^2 \]

after straightforward calculations.

To some extent, this sets the pattern for later work. Initially, when seeking the values of \( L(n, n+c) \), we find:

\[ \frac{A_{n-n+c}}{A_{n-n}} = \sum_{i=0}^{n-c} \frac{A_{n-i}}{n-i} + \text{terms already known.} \]

However:

\[ \frac{A_{n-n+c}}{A_{n-n}} \approx C(n+2)^2 \]

\[ A_{n-n+c}/A_{n-n} \approx (n/2)^{2c} \]

while it seems:

\[ L(n, n+c)/L(n, n) = O(n^c). \]

and many terms must cancel, so that this approach does not seem promising as a way of calculating \( L(n, n+c)/L(n, n) \) or proving its properties.
A MATHEMATICIAN'S APOLOGY. G.H. Hardy. (Published 1940.)
(Opening paragraph) "It is a melancholy experience for a professional mathematician to find himself writing about mathematics. The function of a mathematician is to do something, to prove new theorems, to add to mathematics, and not to talk about what he or other mathematicians have done. Statements, despise publicists, painters despise art-critics, and physiologists, physicists and mathematicians have usually similar feelings; there is no scorn more profound, or on the whole more justifiable, than that of the men who make for the men who explain. Exposition, criticism, appreciation, is work for second-rate minds." 

Probably it is a biological necessity for a research worker to feel like this. There would be serious trouble if one grafted the brain of an eagle into the body of an elephant. Hardy continued to work at the theory of numbers through several of the most disturbed decades in history. Compare Einstein's General Theory of Relativity, 1916; contrast Einstein's attitude. Both hated war. Not criticise either; different genes.

p 88-9. "It is plain now that my life, for what it is worth, is finished, and that nothing I can do can perceptibly increase or diminish its value. It is very difficult to be dispassionate, but I count it a "success"; I have had more reward and not less than was due to a man of my particular grade of ability. I have held a series of comfortable and "dignified" positions. I have had very little trouble with the duller routine of universities. I hate "teaching" and have had little teaching to do. I have done having been almost entirely supervision of research; I love lecturing, and have lectured a great deal to extremely able classes; and I have always had plenty of leisure for the researches which have been the one great permanent happiness of my life.

p.90 "I have never done anything "useful". No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world... Judged by all practical standards, the value of my mathematical life is nil."

p.56 (defining "useful") "Mathematics may, like poetry or music promote and sustain a lofty habit of mind", and so increase the happiness of mathematicians and even of other people; but to defend it on that ground would be merely to elaborate what I have said already. What we have to consider now is the 'crude' utility of mathematics."

p71. "I was not thinking only of pure mathematicians. I count Maxwell and Einstein, Eddington and Dirac, among "real" mathematicians. The great modern achievements of applied mathematics have been in relativity and quantum mechanics, and these subjects are, at present at any rate, almost as "useless" as the theory of numbers."

Written about five years before the atom bomb.
At meiosis, a double division occurs; the diagram suggests a single crossover to occur.

Germ cells:

Any one of these is equally likely to be contributed to offspring. In male, each becomes a sperm, but only one sperm gets into egg. In female, three are discarded. One forms egg.
\[ \begin{align*}
a_1 &= a^1 \\
a_2 &= a^2 \\
a_3 &= a^3 \\
a_4 &= -gt^2 a^1 + (-1 + g^2 t^2) a^4
\end{align*} \]

\[ (-1 + g^2 t^2) a_1 + g \cdot a_4 = a \left[ (-1 + g^2 t^2 - g^2 t^2) \right] = -a^1 \]

\[ g \cdot a_1 + a_4 = a \left[ (-g^2 t^2 - 1 + g^2 t^2) \right] = -a^4 \]

\[ g'' = 1 - g^2 t^2 \]

\[ g^{44} = -g^2 \]

\[ g^{44} = -1 \]

\[ g^{22} = 0 \]

\[ g^{33} = 0 \]

\[ dK_{ac} = 0. \]

If \( \sigma = \lambda \), \( g^{00} = 0 \) unless we have \( g^{14} \) or \( g^{41} \).

If \( \sigma \neq \lambda \), and \( \lambda = 2 \) or \( \lambda = 3 \), then \( \sum_{\mu \nu \lambda} \sigma \neq 0 \).

So \( \sum_{\mu \nu \sigma} \phi > 0 \) if \( \sigma = 2 \) or \( 3 \).

If \( \mu = \nu = 0 \) or \( \mu = 2 \) or \( 3 \), then \( \lambda = 2 \) or \( 3 \).

So only \( \sum_{\mu \nu} \phi > 0 \) when \( \lambda = 2 \) or \( 3 \). We may have \( \sigma = 0 \) or \( \lambda = 0 \).

We may then have \( \phi = 4 \).

\[ \begin{align*}
\{44,1\} &= g^{11} \left[ 14,1 \right] + g^{14} \left[ 14,4 \right] \\
\{14,4\} &= g^{41} \left[ 14,1 \right] + g^{44} \left[ 14,4 \right] \\
\{44,1\} &= g^{11} \left[ 14,1 \right] + g^{14} \left[ 14,4 \right] \\
\{44,4\} &= g^{41} \left[ 44,1 \right] + g^{44} \left[ 44,4 \right] \\
\{44,4\} &= g^{44} \left[ 44,4 \right] + g^{44} \left[ 44,4 \right] \\
\end{align*} \]
Equations of a geodesic.

In \((\mathbb{S}, \mathcal{F}, g, p, x)\) must contain two \(x^s\).

If \(x^1 \neq 4, \mu = 4, \nu = 4\).
If \(x^4, \mu, \nu\) close \(p, 1, 4\) with or less are \(4\).

\[
\frac{d^2 x}{ds^2} + \left\{ 4,4,4 \right\} \frac{dx}{ds} \frac{dx}{ds} = 0
\]

\[
i_2 \frac{d^2 x}{ds^2} - g(1 + g^{24}) (\frac{dx}{ds})^2 = 0.
\]

\[
\frac{d^2 x}{ds^2} = 0 \quad \frac{d^2 x}{ds^2} = 0.
\]

\[
\frac{d^2 x}{ds^2} + \left\{ 4,4,4 \right\} \frac{dx}{ds} \frac{dx}{ds} + \left\{ 4,4,4 \right\} \frac{dx}{ds} \frac{dx}{ds}
\]

\[
\left\{ 4,4,4 \right\} \frac{dx}{ds} \frac{dx}{ds} = 0
\]

\[
\frac{d^2 x}{ds^2} = - g^{24} (\frac{dx}{ds})^2
\]
\[
\begin{align*}
\lambda = 2, & \quad E_{22} \quad \lambda_0 V_{20} = m_{222} V_{20}, \\
E_{33} & \quad \lambda_0 V_{31} = m_{223} V_{20}, \\
E_{44} & \quad \lambda_0 V_{42} = m_{422} V_{20}, \\
\end{align*}
\]

\[
\begin{align*}
E_{12} & \quad m_{111} V_{11} + m_{222} V_{20} = 0, \\
E_{02} & \quad m_{011} V_{02} + m_{022} V_{20} = 0, \\
E_{13} & \quad m_{111} V_{12} = \lambda_0 V_{11} - m_{102} V_{02} - m_{122} V_{11} - m_{222} V_{20}, \\
E_{03} & \quad m_{010} V_{03} + m_{022} V_{02} = -m_{011} V_{02} - m_{012} V_{11} - m_{023} V_{20},
\end{align*}
\]

There are only 2 rows above \( V_2 \) with \( \delta V \) reached.

These

\begin{align*}
E_{(s)} & \quad \gamma = 0, 1, 2, 3, 4 \\
E_{(s)} & \quad s = 0, 1, 2, 3, 4
\end{align*}

If \( \gamma \geq 2 \) and \( \gamma > s \), \( E_{(s)} \) gives \( V = 0 \).

\[
\begin{align*}
E_{(s)} & \quad (MV)_{(s)} = \lambda V_{(s)} \quad \text{coeff of} \quad \lambda^s \\
\lambda & = \lambda_0 \sigma^2 + \ldots \quad V_{(s)} = V_{(\gamma = \gamma)} \sigma^{\gamma - 2} \\
\lambda V_{(s)} & \rightarrow \sigma^s \quad s < \gamma \quad \text{so} \quad 0 \quad \text{in} \quad \lambda V_{(s)} \\
(MV)_{(s)} & \leq m_{72} V_{(s)} \quad \text{at} \quad m_{72} \rightarrow \sigma_{\text{max}}(\kappa \xi) = \sigma^* \quad \text{coeff of} \quad \delta^s = 0.
\end{align*}
\]
The formula for $V_{n-2,2}$.

\[ V_{n-2,2} = V_{n-1,1} \]

\[ V_{n-1,1} = \frac{V_{n-2,2}}{m_{n-2,1} m_{n-1,1}} \]

\[ V_{n-2,2} = \frac{m_{n-1,1} m_{n-2,1}}{m_{n-2,2}} \]

Derivative under $V_{n-2,2}$ is

\[ [a_{n-2,n} b_{n-1,n-1} a_{n-2,n-2} b_{n-3,n-2}] \]

\[ \begin{vmatrix} a_{n-2,n} & a_{n-2,n-1} & b_{n-2,n-2} \\ a_{n-2,n-1} & a_{n-2,n-2} & b_{n-3,n-2} \end{vmatrix} \]

\[ = b_{n-2,n-2} a_{n-2,n} - b_{n-2,n-2} a_{n-2,n-1} \]

Determinant leads to $O(n^2)$, so no cancellations needed. First part is

\[ C_20 = \frac{b_{n+1,n+1}}{b_{n,n}} \]

\[ \frac{2n(2n-1)...2}{[(4n-1)(4n-3)...2]^{\frac{3}{2}}} \]

\[ \frac{(4n-1)(4n-3)...2}{[(4n-1)(4n-3)...2]^{\frac{3}{2}}} \]

\[ \frac{2n(2n-1)(2n-3)...2}{[(4n-1)(4n-3)...2]^{\frac{3}{2}}} \]

\[ \frac{2n(2n-1)(2n-3)...2}{[(4n-1)(4n-3)...2]^{\frac{3}{2}}} \]

\[ \frac{2n(2n-1)(2n-3)...2}{[(4n-1)(4n-3)...2]^{\frac{3}{2}}} \]

\[ \frac{2n(2n-1)(2n-3)...2}{[(4n-1)(4n-3)...2]^{\frac{3}{2}}} \]

\[ \frac{2n(2n-1)(2n-3)...2}{[(4n-1)(4n-3)...2]^{\frac{3}{2}}} \]

\[ \frac{2n(2n-1)(2n-3)...2}{[(4n-1)(4n-3)...2]^{\frac{3}{2}}} \]

\[ \frac{2n(2n-1)(2n-3)...2}{[(4n-1)(4n-3)...2]^{\frac{3}{2}}} \]

\[ \frac{2n(2n-1)(2n-3)...2}{[(4n-1)(4n-3)...2]^{\frac{3}{2}}} \]

\[ \frac{2n(2n-1)(2n-3)...2}{[(4n-1)(4n-3)...2]^{\frac{3}{2}}} \]

\[ \frac{2n(2n-1)(2n-3)...2}{[(4n-1)(4n-3)...2]^{\frac{3}{2}}} \]
\[ \mu_n = \frac{1}{(2n+1)(2n+3) \frac{\pi}{2}} \quad \text{moment function} \quad \text{(from Weyl's Hilbert)} \]

\[ \mu_{n+1} = \frac{1}{(2n+3)(2n+5) \frac{\pi}{2}} \quad \text{method of} \quad \frac{1}{(2n+3)(2n+5)} \frac{\pi}{2} \]

\[ = \left( \frac{2n+1}{2n+3} \right) \frac{\pi}{2} \]

\[ \psi(t) = \sum_{n=0}^{\infty} (2n+1) \pi \int L_n(u) du \]

If we use the Hermite differential equation, we can rewrite

\[ w(t) = \psi'(t) = \sum_{n=0}^{\infty} (2n+1) \pi L_n(t). \]

\[ |L_n(t)| \leq 1. \text{ We shall have uniqueness if} \]

\[ \sum_{n=0}^{\infty} \frac{(2n+1) \pi}{2n+3} \]

\[ L_0(t) = 1 \]
\[ L_1(t) = 2t \]
\[ L_2(t) = 6t^2 - 6t \]
\[ L_3(t) = 20t^3 - 30t^2 + 12t \]

\[ \mu_n = \frac{1}{(2n+1)(2n+3) \frac{\pi}{2}} \quad \text{to} \quad \frac{1}{2n+3} \]

\[ \lambda_0 = 1, \lambda_1 = \frac{\gamma}{2} \]
\[ \lambda_2 = \frac{7}{2 \pi} \quad \lambda_3 = \frac{149}{735} \]

\[ (2n+1) \pi \cdot \pi_0 = 1 \quad \pi_1 = \frac{15}{9} \quad \pi_2 = \frac{35}{25} \quad \pi_3 = \frac{149}{105} \]

\[ 1.667 \quad 1.4 \quad 1.419 \]
Divide the fraction curriculum into units and within any one unit (lasting at least two weeks and often three or more) limit the number of fractions the children are to work with and give each a very specific meaning.

Within each unit, the fractions that we use are made specific by giving the whole number "one" a fixed and narrow meaning. The "ones" that we use are described below.

a) Concrete "Ones" Fractions can be given specific meanings by attaching them to some concrete material. For example, in our first unit we call the orange Cuisenaire rod "one". The yellow rod is thus named "one half", the red rod "one fifth" and the white rod "one tenth".

\[
\begin{array}{c|c}
\text{orange} & 1 \\
\text{yellow} & \frac{1}{2} \\
\text{red} & \frac{1}{5} \\
\text{white} & \frac{1}{10}
\end{array}
\]

In this unit, which takes perhaps 3 weeks, these fractions and their multiples are the only ones to which the children are exposed. Furthermore, during this period, all symbolic work refers exclusively to the rods. Thus for example, the children are taught that the numeral \( \frac{3}{5} \) means three of the fifths, that is, three red rods. Even with this kind of narrow understanding they can discover that \( \frac{2}{5} > \frac{3}{10} \) by comparing a "train" of two reds with a train of three whites. Similarly, it is easy for them to see that \( \frac{5}{10} = \frac{1}{2} \). However, at least temporarily, the only meaning here is that a train of 5 white rods is just as long as a yellow.
\[
\frac{d}{dx} (x^2-1) y' - n(n+1) y = 0
\]

\[(x^2-1)y'' + 2xy' - n(n+1)y = 0.\]

Let \( y = \sum \alpha_s x^s \), from coeff of \( x^{s-2} \)

\[
\alpha_{s-2} \left[ (s-2)(s-3) + 2(s-2) - n(n+1) \right] = \alpha_s \cdot s(s-1)
\]

\[
\left[ \right] = (s-2)(s-1) - n(n+1)
\]

\[
= (s-2-n)(s-1+n)
\]

\[
\therefore \quad \alpha_s = \frac{\alpha_{s-2} \cdot (s-2-n)(s-1+n)}{s(s-1)}
\]

If \( \alpha_0 = 1 \)

\[
\alpha_2 = -\frac{n(n+1)}{2}
\]

\[
\alpha_4 = \frac{n(n+1)(n+2)(n+3)}{4!}
\]

\[
\alpha_s = \frac{\alpha_{s-2} \cdot (s-2-n)(s-1+n)}{s(s-1)}
\]

\[
\alpha_s = \frac{1}{1 - \frac{n}{s}} \left( 1 + \frac{n-1}{s} \right)
\]

\[
\approx \frac{1 + \frac{n+2}{5} \left( 1 + \frac{1-n}{s} \right)}{\left( 1 + \frac{1}{s} \right) \left( 1 + \frac{n+2}{5} \right)}
\]

\[
n = 4 \quad \alpha_{5,2} = \frac{(s-6)(s+3)}{5(s-1)} \alpha_{s-2}
\]

\[
s=2 \quad \alpha_2 = -\frac{4}{2} \alpha_0 \quad \alpha_4 = -\frac{2}{4!}
\]
8127. The linear covariant of a differential form.

\[ \Omega_a = \sum x_i \, dx_i \quad \Omega_b = \sum x_i \, \delta x_i \]

\[ S \Omega_a - d \Omega_b = \sum (\frac{\partial x_i}{\partial x_j} - \frac{\partial x_j}{\partial x_i}) \, dx_i \, \delta x_j \]

\[ \alpha_{ij} = \frac{\partial x_i}{\partial x_j} - \frac{\partial x_j}{\partial x_i} \quad \text{This is the type of expression found in co-ord.} \]

If \((y_1, \ldots, y_n)\) are variables, and \(b_{ij} = \frac{\partial y_j}{\partial y_i} \quad \text{we have} \)

\[ \sum \alpha_{ij} \, dx_i \, dx_j = \sum b_{ij} \, dy_i \, dy_j \]
Then \( x_1 = g_{11} x^{(1)} + g_{12} x^{(2)} = g_{1j} x^j \)
\[
x_2 = g_{21} x^{(1)} + g_{22} x^{(2)} = g_{2j} x^j
\]
So \( x_i = g_{ij} x^j \quad \text{(1)} \)

Now \( \sigma = x \cdot x = x^i x_i = x^i g_{ij} x^j = g_{ij} x^i x^j \quad \text{(2)} \)

Often \( g_{ij} \) is introduced in this way. \( \sigma \) is supposed to be the length of the vector \( x \), so given by
\[
\sigma = g_{ij} x^i x^j \quad \text{(2)}
\]

This is accidentally where \( g_{ij} = g_{ji} \).

Equations (1) and (2) form \( \sigma \) in any number of dimensions.

The Polar Form
Consider any two vectors \( x \) and \( y \). Let \( |x| \leq 1 \) denote the length of \( x \).

\[
|x+y|^2 = g_{ij} (x^i + y^i) (x^j + y^j) = g_{ij} x^i x^j + g_{ij} y^i y^j + g_{ij} x^i y^j + g_{ij} y^i x^j
\]
\[
= |x|^2 + 2 g_{ij} x^i y^j + |y|^2
\]

as \( g_{ij} \) is symmetric.

\[
2 g_{ij} x^i y^j = (x+y)^2 - |x|^2 - |y|^2
\]

So \( g_{ij} x^i y^j \) is invariant, i.e. it is a scalar the same in every system.

From equation (1), \( x^i = g_{ij} y^j \)

So \( g_{ij} x^i y^j = x^i y^i \)

Earlier, we obtained equation by considering a particular case in 2 dimensions. Here we have shown that equation (1) leads to a definition of \( y^i \) that makes \( x^i y^j \) invariant, as \( x_i \) should be, and the proof holds in space of any number of dimensions where \( g_{ij} \) is defined by (2).
Continuation of p. 94.

We are concerned with \( \leq \frac{\Delta}{\Delta + \Delta + \Delta} \leq \frac{\Delta}{\Delta + \Delta} \).

\[ \Delta \leq \frac{\Delta + \Delta + \Delta}{\Delta + \Delta} \leq \frac{\Delta + \Delta}{\Delta} \]

\[ \Delta \leq \frac{\Delta + \Delta + \Delta}{\Delta + \Delta} \leq \frac{\Delta + \Delta}{\Delta} \]

\[ \Delta \leq \frac{\Delta + \Delta + \Delta}{\Delta + \Delta} \leq \frac{\Delta + \Delta}{\Delta} \]

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\[ \Delta \leq \frac{\Delta + \Delta + \Delta}{\Delta + \Delta} \leq \frac{\Delta + \Delta}{\Delta} \]

\[ \Delta \leq \frac{\Delta + \Delta + \Delta}{\Delta + \Delta} \leq \frac{\Delta + \Delta}{\Delta} \]

\[ \Delta \leq \frac{\Delta + \Delta + \Delta}{\Delta + \Delta} \leq \frac{\Delta + \Delta}{\Delta} \]

\[ \Delta \leq \frac{\Delta + \Delta + \Delta}{\Delta + \Delta} \leq \frac{\Delta + \Delta}{\Delta} \]

\[ \Delta \leq \frac{\Delta + \Delta + \Delta}{\Delta + \Delta} \leq \frac{\Delta + \Delta}{\Delta} \]
Can we prove \( \lambda_{m+1} / \lambda_m > 0 \) by using simple overestimates for the subdifferential terms?

\[
\lambda_{m+1}/\lambda_m = \frac{(2n+1)^2(2n+2)}{(4m+1)(4m+2)(4m+3)(4m+4)}
\]

Note: denominator =

\[
(16n^2 + 24n + 5)(16n^2 + 24n + 9)
\]

\[
= \frac{16n^4 + 48n^3 + 52n^2 + 24n + 5}{256n^4 + 768n^3 + 800n^2 + 386n + 45}
\]

\[
= \frac{1 + \frac{2n^3 + 3n + 1.25}{16}}{256n^4 + 768n^3 + 800n^2 + 386n + 45}
\]

Numerically:

\[
\approx \frac{(4n^2 + 6n + 2)^2}{(4n^2 + 6n + 2)^2}
\]

so probably close to \( \frac{1}{16} \).

So

\[
\frac{\lambda_{m+1}/\lambda_m}{\lambda_{m+1}} < \frac{1}{16} + \frac{1}{100n^2} \quad \text{or} \quad \frac{1}{16} + \frac{1}{128n^2} \quad \text{if preferred.}
\]

\[
\frac{\lambda_{m+1}/\lambda_m}{\lambda_{m+1}} < \frac{1}{16} + \frac{1}{(100(m+1))^2} \quad \text{if preferred.}
\]

\[
\frac{\lambda_{m+1}/\lambda_m}{\lambda_{m+1}} < \frac{1}{256} + \frac{1}{200n^2 + 40n - 4}
\]

\[
\frac{\lambda_{m+1}/\lambda_m}{\lambda_{m+1}} < \frac{1}{256} + \frac{1}{1000n^2} \quad \text{if preferred.}
\]

\[
h \geq 1 \quad \frac{1}{16} + \frac{1}{128n^2} < \frac{1}{6} + \frac{1}{128} = \frac{9}{128} < 14.
\]

\[
\frac{\lambda_{m+1}/\lambda_m}{\lambda_{m+1}} < \frac{1}{14} \quad \frac{\lambda_{m+1}/\lambda_m}{\lambda_{m+1}} < \frac{1}{96}
\]

We may deal with these before taking \( \Delta \). (case needed.)

\[
\phi(n) = \frac{(n+1)(2n+1)}{4n+3}
\]
\[
\lambda_{n+1, n+1} = \frac{1}{16} + \frac{2n^2 + 3n + 19/16}{256n^4 + 768n^3 + 800n^2 + 336n + 43}
\]

\[
\frac{1}{16} \leq \frac{\lambda_{n+1, n+1}}{\lambda_{nn}} \leq \frac{1}{16} + \frac{1}{128n^2}
\]

\[
\frac{1}{16} \leq \frac{\Delta \lambda_{n+1, n+1}}{\lambda_{nn}} \leq \left[ \frac{1}{16} + \frac{1}{128(n+1)^2} \right] - \frac{1}{16} = \frac{1}{128n(n+1)^2}
\]

\[
\frac{1}{16} \leq \frac{\lambda_{n, n+1}}{\lambda_{nn}} \leq \frac{1}{16} \cdot \frac{1}{(28n(n-1))^2} \leq \frac{1}{16} \cdot \frac{1}{256}
\]

\[
\frac{1}{16} \leq \frac{\lambda_{n, n+1}}{\lambda_{nn}} \leq \left( \frac{1}{16} \cdot \frac{1}{(28n(n-1))^2} \right) = \frac{1}{256} \frac{1024n^2 - 256}{256n^2} = \frac{1}{256} + \frac{1}{512n^2}
\]

\[
\frac{1}{16} \leq \frac{\Delta \lambda_{n, n+1}}{\lambda_{nn}} \leq \frac{1}{512n^2}
\]

In difference \( \Delta F(n) = F(n+1) - F(n) \) we find overestimates of \( F(n+1) \) and underestimates of \( F(n) \).

\[
\frac{1}{16} \leq \frac{\lambda_{n+1, n+1}}{\lambda_{nn}} \leq \frac{1}{16} \cdot \frac{1}{256n^2}
\]

\[
\frac{1}{16} \leq \frac{\Delta \lambda_{n+1, n+1}}{\lambda_{nn}} \leq \frac{1}{512n^2}
\]

\[
\frac{1}{16} \leq \frac{\Delta \left[ \frac{\lambda_{n+1, n+2}}{\lambda_{n+1, n+1}} + \frac{\lambda_{n, n+1}}{\lambda_{nn}} \right]}{\lambda_{nn}} \leq \frac{1}{256(n-1)^2}
\]
In general relativity, the object is to make laws in such a way that they have the same form in any (analytic) system of coordinates. The central theme of tensor analysis is seen here—to compare statements as they appear in different coordinate systems.

On the surface of the earth, a point can be specified by \( \theta \), the colatitude and \( \phi \), the longitude. Distances are calculated using the equation
\[
d s^2 = d\theta^2 + \sin^2 \theta \, d\phi^2.
\]
Relativity uses coordinates of this type. A point may be specified by \((x_1, \ldots, x_n)\), and distances are given by a quadratic expression
\[
d s^2 = \sum_{i,j} g_{ij} \, dx^i \, dx^j.
\]
Summation repeatedly occurs in tensor theory, and here is a convention—sum over any index number that occurs twice. Thus the equation above would be written
\[
d s^2 = \sum_{i,j} g_{ij} \, dx^i \, dx^j.
\]

Tensors can also be used in connection with different coordinate systems in a plane or other linear surface. The systems have the same origin. Distance will now be given by
\[
t^2 = \sum_{i,j} g_{ij} \, x^i \, x^j
\]
for the length of a vector \( t \).

In different coordinate systems a point will have different specifications. We suppose that people in different systems are aware of the correspondence: it is known how to transform the number specifying a point in system A so as to obtain the numbers used in system B. The transformation is assumed linear. If \((x^i)\) in A specifies the same point as \((x'^i)\) in B we have
\[
x^i = t^i_j \, x'^j
\]
(summation convention).

When we say that something is a vector, we imply that the coordinates are known in every system and that these are connected by the transformation that apply to points, (3).
Vector and Tensor Theory.

This is essentially concerned with geometrical and physical situations, as recorded in different coordinate systems.

In a plane, for instance, in one system a displacement may be specified by \(x^1, x^2\), in another by \(\xi^1, \xi^2\). (\(1\) and \(2\) are labels, not powers.) The systems are supposed to be related by linear equations:

\[
\begin{align*}
x^1 &= a_{11} \xi^1 + a_{12} \xi^2, \\
x^2 &= a_{21} \xi^1 + a_{22} \xi^2.
\end{align*}
\]

These express algebraically that the vector called \((x^1, x^2)\) in the first system is geometrically the same as that called \((\xi^1, \xi^2)\) in the other.

In general, if \(i, j, \ldots, n\) are \(n\) dimensions \(x^1, \ldots, x^n\) in geometrically identified with \(\xi^1, \ldots, \xi^n\),

\[
\begin{align*}
x^i &= \sum_{j=1}^{n} a_{ij} \xi^j, \\
i &= 1, \ldots, n.
\end{align*}
\]

Tensor convention. When a letter such as \(j\) occurs twice, once above and once below, it is understood that we sum 1 to \(n\), as in the work around such work. Thus we write \(x^i = t^i_j \xi^j\).

A number is called a scalar if it is the same in all systems, \(x^i\). Electric potential \(f\) is such a quantity

\[
\frac{df}{dx^i} = \frac{\partial f}{\partial \xi^j} a_{ij} \xi^j.
\]

So

\[
\frac{\partial f}{\partial \xi^j} = \frac{df}{dx^i} a_{ij} \xi^j.
\]

If we write

\[
\sigma_i = \frac{df}{dx^i}, \quad \sigma_i = \frac{df}{\delta^i}
\]

we have

\[
\sigma_j = \sigma_i t^i_j.
\]

Note that (1) and (10) are not the same transformation.

\[
\begin{align*}
\sigma_j &= \sum_{i=1}^{n} \sigma_i t^i_j, \\
\sigma_j &= \sigma_i t^i_j.
\end{align*}
\]
Covariant and Contravariant.

The distinction between $v_i$ and $v^i$ is often referred to as that between covariant and contravariant vectors. I find this terminology confusing. There are two different kinds of vectors: I would rather speak of the covariant and contravariant methods of specifying a vector.

(i) \[ \begin{array}{c}
\text{O} \\
\begin{array}{c}
\text{x}' \\
\text{x}^2
\end{array}
\end{array} \]

(ii) \[ \begin{array}{c}
\text{O} \\
\begin{array}{c}
\text{x}_1 \\
\text{F}
\end{array}
\end{array} \]

Consider a plane with axes taken at an angle $\alpha$.

A displacement $OP$ can be specified by $(x', x^2)$ as in (i). A force $(F, F')$ should be specified so that $x_1 x' + x_2 x^2$ represents the work done by the force in the displacement. To achieve this, we must take $x_1, x_2$ as the projections of $OF$ on the axes, as in (ii).

$(x_1, x_2)$ is called covariant, $(x', x^2)$ contravariant.

The difference lies in the form of the specification. Nothing stops us specifying OP by its projections on the axes, or of $(x', x^2)$ components parallel to the axes. In fact there is a definite procedure for obtaining the representation $(x_1, x_2)$ from $(x', x^2)$.

The work done by a force in a displacement is a scalar product, and appears as $x_1 x^2$.

If we wanted to find the length of $OP$ we would need to take the scalar product of $OP$ with itself. $OP^2 = x_1 x^2$.

$x_i = x' + x^2 \cos \alpha$, $x_2 = x' \sin \alpha + x^2$, so $x_1 x^2 = x' (x' + x^2 \cos \alpha) + x^2 (x' \sin \alpha + x^2) = (2x')^2 + 2x' x^2 \cos \alpha + (x^2)^2$. 
If we write  
\[ x_1 = g_{11}x^1 + g_{12}x^2 \]
\[ x_2 = g_{21}x^1 + g_{22}x^2 \]
this will agree with our result above if 
\[ g_{11} = 1 \quad g_{12} = 0 \quad g_{21} = 0 \quad g_{22} = 1 \]
and we have 
\[ O^p_j = g_{ij}(x^j)^2 + g_{21}x^1x^2 + g_{22}x^2x^2 \]
These equations may be written concisely
\[ x_i = g_{ij}x^j, \quad O^p_j = g_{ij}x^ix^j. \]

\( g_{ij} \) can be generalized to apply to any case in which a length is defined as the square root of a quadratic form.

If, in the general case, we have another coordinate system \( (t_i, \tau^j) \), with \( x^i \rightarrow t_i, \tau^j \), we expect \( O^p_j = g_{ij}t_i \tau^j \sqrt{\tau^k \tau^k} \), and in fact we find this is so, for
\[ O^p_j = g_{ij}x^ix^j = g_{ij} (t_i \tau^j)(t_i \tau^j) \]
\[ = g_{ij} t_i t_i \tau^j \tau^j \]
and
\[ \tau^{\mu} = g_{ij} t_i t_j \tau^j \tau^j \]

Thus, we have a procedure for going from one system to another when dealing with something like \( g_{ij} \) with two subscripts.
\[
A_{ij} = \sum_{k=0}^{\infty} \frac{\lambda_{ij}^{k+1}}{\lambda_{ii}^{k+1}} + \sum_{k=0}^{\infty} \frac{\lambda_{ij}^{k+1}}{\lambda_{ii}^{k}} + \sum_{k=0}^{\infty} \frac{\lambda_{ij}^{k+1}}{\lambda_{ij}^{k+1}}
\]

Take \(\Delta = \frac{A_{ij}}{\lambda_{ii}}\)

\[
\Delta = \left[ \frac{\lambda_{ij}^{k+1}}{\lambda_{ii}^{k+1}} + \frac{\lambda_{ij}^{k+1}}{\lambda_{ij}^{k}} + \frac{\lambda_{ij}^{k+1}}{\lambda_{ij}^{k+1}} \right]
\]

\[
\Delta (a_y b_y) = a_y + b_y - a_y b_y
\]

\[
= b_y (a_{y+1} - a_y) + a_y (b_{y+1} - b_y)
\]

\[
= b_y a_y \Delta a_y + a_y \Delta b_y
\]

\[
\phi(x) = \frac{\lambda_{ij}^{k+1}}{\lambda_{ii}^{k+1}}
\]

\[
\Delta = \frac{\lambda_{ij}^{k+1}}{\lambda_{ii}^{k+1}} + \frac{\lambda_{ij}^{k+1}}{\lambda_{ij}^{k+1}} + \phi(x) \Delta = \frac{\lambda_{ij}^{k+1}}{\lambda_{ii}^{k+1}} + \phi(x) \Delta = \frac{\lambda_{ij}^{k+1}}{\lambda_{ii}^{k+1}} + \phi(x) \Delta
\]

\[
\Delta = \frac{\lambda_{ij}^{k+1}}{\lambda_{ii}^{k+1}} + \frac{\lambda_{ij}^{k+1}}{\lambda_{ij}^{k+1}} + \phi(x) \Delta
\]

\[
\Delta = \frac{\lambda_{ij}^{k+1}}{\lambda_{ii}^{k+1}} + \frac{\lambda_{ij}^{k+1}}{\lambda_{ij}^{k+1}} + \phi(x) \Delta
\]

So \(\lambda_{ij}^{k+1} = \frac{\lambda_{ij}^{k+1}}{\lambda_{ii}^{k+1}} + \phi(x) \Delta
\]

\[
\lambda_{ij}^{k+1} = \frac{(x+y)^2}{x+y} \frac{(x+y)^2}{x+y} \frac{(x+y)^2}{x+y}
\]

\[
\lambda_{ij}^{k+1} = \frac{(2x+1)(2x+2)}{(2x+1)(2x+2)(2x+3)} \frac{(2x+1)(2x+2)}{(2x+1)(2x+2)(2x+3)} \frac{(2x+1)(2x+2)}{(2x+1)(2x+2)(2x+3)}
\]

\[
\lambda_{ij}^{k+1} = \frac{(x+y)(x+y)}{(x+y)(x+y)(x+y)} \frac{(x+y)(x+y)}{(x+y)(x+y)(x+y)} \frac{(x+y)(x+y)}{(x+y)(x+y)(x+y)}
\]

\[
\lambda_{ij}^{k+1} = \frac{(x+y)(x+y)(x+y)}{(x+y)(x+y)(x+y)(x+y)} \frac{(x+y)(x+y)(x+y)}{(x+y)(x+y)(x+y)(x+y)} \frac{(x+y)(x+y)(x+y)}{(x+y)(x+y)(x+y)(x+y)}
\]
\[ \Delta q_{n+1} = \frac{(2n+1)^2(2n+2)^2}{(4n+1)(4n+2)+(4n+5)} \Delta q_{n-1} \]

\text{New A:}
- [1] 1 2

\text{New B:}
- [1] 2 2

* (2n+1)/(2n+2) in C

C to A

C to B

* (2n+1)(2n+2)^2 in C

\text{New B:}
- [F] 1 2 2 16 160 1 2 8

* C to S numeraco in S

\text{New A:}
- [1] 1 4

\text{New B:}
- [1] 3 4

* C to A

C to B

* C to A

C to B

* C to A

\text{New B:}
- [F] 9 -18 -82 256 256

* \text{denom} A in C

\text{New S to A}

C to B

\text{1}
Checking results used in study of $\lambda_{ij}$ from characteristic eqn

$$\frac{\Delta_{nm}}{\Delta_{nn}} = \frac{(n+1)^2 (2n+1)^2 - n^2 (2n-1)^2}{(4n+3)^2 - (4n-1)^2}$$

$n=0$

$$\frac{\Delta_{12}}{\Delta_{00}} = \frac{1}{9}$$

$n=1$

$$\frac{\Delta_{12}}{\Delta_{22}} = \frac{36}{49} = \frac{1}{9} = 1.623 \times 10^{-2}$$

$n=2$

$$\frac{\Delta_{12}}{\Delta_{22}} = \frac{1.124 \times 10^{-2}}{253}$$

From computed results $\lambda_{11} = 1.623 \times 10^{-2}$

$\lambda_{22} = 1.124 \times 10^{-2}$

The most significant term in $\frac{\Delta_{nm}}{\Delta_{nn}}$ is

$$z = \frac{\lambda_{0,n+1}}{\lambda_{n,n}} \cdot \frac{\lambda_{1,1}}{\lambda_{2,2}}$$

$$\Delta_{3} = \frac{\lambda_{0,n+1}}{\lambda_{n,n}}$$

$$\phi(1) \cdot \phi(1)$$

$$\phi(1) = \frac{1}{9} \cdot 0.623 \times 10^{-2} = 0.692 \times 10^{-2}$$

$$\phi(2) = \frac{1}{9} \cdot 0.623 \times 10^{-2} = 0.692 \times 10^{-2}$$

$$\phi(1) \cdot \phi(1) = 0.692 \times 10^{-2}$$

$$\phi(2) \cdot \phi(2) = 0.692 \times 10^{-2}$$

So $\Delta = \frac{\lambda_{0,n+1}}{\lambda_{n,n}}$ is confirmed.
\[ \int x(x-3)^5 \, dx \]

Let \( s = x-3 \), \( ds = dx \)

\[ x-3 = s \]
\[ x = s+3 \]

\[ \int = \int (s+3)^5 \, ds \]
\[ = \int s^5 + 3s^4 \, ds \]
\[ = \frac{s^6}{6} + \frac{3s^5}{5} \]
\[ = \frac{(x-3)^6}{6} + \frac{(x-3)^5}{5} \]

\[ \left[ \frac{(x-3)^6}{6} + \frac{(x-3)^5}{5} \right] \]

\[ \sum \frac{2x-6 + 7}{14} = \frac{2x+1}{14} \]

\[ \frac{6s^7 + 21s^6}{42} \]
\[ \frac{s^6(6s+21)}{42} \]
\[ \frac{s^6(2s+7)}{42} \]

\[ 2x-6 + 7 = 2x+1 \]
If \( \ell = 0 \), we are simply using the genus \( f(x, \sigma) \).

What about \( \ell = 1, \ k = 0, \ j = -1 \)?

\[ Q_2(x, \sigma) = 1 + 3\left(\frac{x^2}{3} + \frac{x^2}{15} + \frac{1}{28}\right) + 9\left(\frac{x^6}{7} + \frac{x^6}{35} + \frac{1}{63}\right) + \ldots \]

\[ Q_2(0, \sigma) = 1 + \frac{\sigma^2}{\delta} + \frac{\sigma^2}{25} + \frac{\sigma^2}{67} + \ldots \]

\[ \frac{Q_2(x, \sigma)}{Q_2(0, \sigma)} = 1 + \frac{\sigma x^2}{3} + \frac{\sigma x^2}{5} + \frac{\sigma^3 x^6}{7} + \ldots \]

The underlined terms represent an increase on \( 0 \) in \( f(x, \sigma) \)

\[ f(x, \sigma) = 1 + \frac{\sigma x^2}{3} + \frac{\sigma x^2}{5} + \frac{\sigma^3 x^6}{7} + \ldots \]

\[ \frac{Q_2(x, \sigma)}{Q_2(0, \sigma)} - f(x, \sigma) = \frac{\sigma^2 x^2}{(15 - 28)} + \frac{\sigma^2 x^2}{(15 - 28)} + \ldots \]

\[ Q_2(x, \sigma) - Q_2(0, \sigma) = f(x, \sigma) \]

\[ = 1 + \sigma \left(\frac{x^2}{3} + \frac{1}{15}\right) + \sigma^2 \left(\frac{x^6}{7} + \frac{x^6}{35} + \frac{1}{63}\right) - \]

\[ - \left[1 + \frac{\sigma^2}{3} + \frac{\sigma^2}{25} + \frac{\sigma^2}{67}\right] \left[1 + \frac{\sigma x^2}{3} + \frac{\sigma x^2}{5} + \frac{\sigma^3 x^6}{7} + \ldots \right] \]

\( \sigma^2 x^2 \) comes from \( \sigma^2 x^2 \) in \( Q_2(x, \sigma) \), but \( \sigma^2 x^2 \) from \( \frac{\sigma}{15}, \frac{\sigma^2}{28} \)

The could be a term \( \sigma^2 x^2 \) in \( f \), but it is not at the same
Mr. R. Mirchandani,
Penguin Books Limited,
27 Wrights Lane
W8 5TZ.

Dear Mr. Mirchandani,

Thank you for your letter of September 5th.
You say something about the possibility of a new preface. At present of course it does not have a preface but goes straight to Chapter 1, which was designed as an opening, and I should like it to stay that way. You may have in mind the fact that it was written nearly 50 years ago. I think you might deal with this in the publisher's blurb, something along the following lines; this book first appeared in 1943 in the Forces Book Club. It was designed to help those in the forces who needed to become engineers in a hurry. However the fact that it has stayed in print until the present indicates that it has been useful to readers other than those for whom it was originally planned. Since 1943 there have been changes in money, weights and measures. References to the old system are only incidental, and it does not seem necessary to change these.

I think you have my biography up to 1976, when I retired from the University of Toronto and came, with my wife, to live in Cambridge (England). Since then I have written a book showing in detail some practical applications of modern mathematics - a thing the "Modern Mathematics" campaign in U.S.A. notably failed to do. I have also written two booklets with problems and topics to keep the quickest pupils busy when they have finished the regular stint of exercises. Since 1977 I have been meeting a small group of interested secondary school pupils on Saturday mornings.

I hope you find this useful,
With best wishes,
Yours sincerely,
\[ \frac{\lambda_{n+1, x+1}}{\lambda_{n, x}} = \left( \frac{1}{2} \frac{x+1}{2} \right)^2 \] 

where \( \nu = 4 \).

\[ \frac{\lambda_{n, x}}{(x+1)(x+2)^2(x+5)} \]

\[ = \frac{1}{16} \frac{(x+2)^2(x+4)^2}{(x+5)^2(x+5)} \]

Partial fractions: \( x^2 - 1 \) gives \( \frac{9}{16} \) \( x = -5 \) gives \( -\frac{9}{16} \)

\[ \frac{9/16}{x+1} - \frac{9/16}{x+5} = \frac{9/4}{(2x+5)(x+5)} \]

\[ (x+2)^2(x+4)^2 - \frac{9}{4} (x+5)^2 = (2x+5)^2 - (\frac{3x-9}{2}) \]

4 times this: \( (2x^2 + 12x + 16)^2 - (3x + 9)^2 \)

\[ = (2x^2 + 15x + 25)(2x^2 + 9x + 7) \]

\[ = (x+5)(2x+5)(x+1)(2x+7) \]

So

\[ \frac{(x+2)^2(x+4)^2}{(x+1)(x+5)^2} \]

\[ = 4 - \frac{1}{(x+3)^2} \]

\[ \frac{9/16}{x+1} - \frac{9/16}{x+5} = \frac{9/4}{(2x+5)(x+5)} \]

\[ = \frac{1}{16} \left( \frac{9/16 - 9/16 - \frac{9/4}{(x+3)^2}}{4x+1} - \frac{9/4}{(4x+3)^2} \right) \]

A detailed analysis of (2) may be unnecessary, as this part is of low order.

\[ \Delta \frac{\lambda_{n+1, x+1}}{\lambda_{n, x}} = \frac{1}{16} \left\{ \frac{9/16}{4x+5} - \frac{9/16}{4x+9} - \frac{9/4}{(4x+7)^2} \right\} \]

\[ = \frac{1}{16} \left\{ \frac{-32}{(4x+1)(4x+5)(4x+9)} + \frac{32n + 60}{(4x+7)^2(4x+3)^2} \right\} \]

\[ = \frac{2x + 2.5}{(4x+3)^2(4x+7)^2} - \frac{2}{(4x+7)^2(4x+5)(4x+9)} \]

\[ = O \left( \frac{1}{x^2} \right) \]

\( \phi(x+1/x+1) \) and does not make any contribution to the integral part.
Cloud on p.49.

For $A_{5}$, $X_{0}^{i+2}=1, X_{0}^{i+2}=2, X_{0}^{i+3}=1$.

The squares of these are $1$-valued in $A^{\text{even}}/M$.

We require $P_{n}^{(2)/P_{n}}, P_{n}^{(3)/P_{n}}, P_{n}^{(4)/P_{n}}$.

p.65b. Partition list of elements gives $\lambda$ coeff.

\[
\frac{3}{2} (a^{2}+b^{2}+c^{2}+d^{2}+e^{2}) - 12a - 9b - 6c - 3d.
\]

I think we get correct results by taking $a=0, b=0$ etc.

[3] $c=3, x=0$. \(\frac{3}{2}(3^{2}) = 27/2 = 13\frac{1}{2}\) agrees with p.65b.

(1) $c=2, d=1, x=0$. \(\frac{3}{2}(4+1) - 3 = 15 - 3 = 4\frac{1}{2}\) agrees.

(11) $c=1, d=1, c=1$. \(\frac{3}{2}(1+1) - 3 - 6 = 4\frac{1}{2} - 9 = -4\frac{1}{2}\) agrees.

\[
1 + \frac{13\frac{1}{2}}{n} + 4(1 + \frac{4\frac{1}{2}}{n}) + (1 - \frac{4\frac{1}{2}}{n})
\]

\[
= 6 + \frac{27}{n} = 6(1 + \frac{4\frac{1}{2}}{n}) \text{ on agreement in p.69.}
\]
P. 92 suggests that it may be necessary to consider second terms for asymptotic behaviour. Consider the process with estimates of accuracy.

\[ \frac{X_{i,j+1} - X_{i,j}}{\lambda_{i,j}} = \frac{(n+1)^2(2n+1)}{(4n+3)^2} \]

\[ = \frac{n^2}{4} + \frac{3}{8} n + \frac{7}{64} - \left[ \frac{1}{8(n+3)} \right]^2 \]

**Argument of p. 87.**

\[ \lambda_{i,j+1} \lambda_{i,j} \approx \Delta := \frac{\lambda_{i,j+1}}{\lambda_{i,j}} \approx \Delta(n) = \frac{n}{2} + \frac{5}{8} + O\left(\frac{1}{n}\right) \]

\[ \Delta_{i,j} := \left( \frac{n}{2} + \frac{5}{8} + O\left(\frac{1}{n}\right) \right) \left( \frac{n^2}{4} + \frac{3}{8} n + \frac{7}{64} + O\left(\frac{1}{n}\right) \right) \]

\[ = \frac{n^3}{8} + \frac{33}{32} n^2 + \frac{35}{128} n + O\left(\frac{1}{n}\right) \]

\[ \Rightarrow 0 \leq i < j \leq n \Rightarrow \lambda_{i,j+1} \lambda_{i,j} \approx \frac{n^2}{32} + \frac{5}{96} n^2 + \frac{1}{256} n + O\left(\frac{1}{n}\right) \]

\[ a(4n^3 + 6n^2 + 4n + 1) \]

\[ + b(7n^2 + 3n + 1) \]

\[ + c(2n + 1) \]

\[ + d \]

\[ 4a = \frac{1}{3}, \quad a = \frac{1}{8} \]

\[ 2b = \frac{4}{32} \quad 3b = \frac{15}{2} \]

\[ 2c = \frac{32}{3} - \frac{5}{32} \quad c = \frac{1}{128} \]

\[ d = \frac{32}{152} \quad 5d = \frac{1}{32} \]
$z_n = 4 \ (x_1, x_2, x_3, x_4)$ 4-vectors. $\Sigma x_i^2$ simpler scalar

Quadratic form $\Sigma 2.1: x_i x_j$. Invariant, its 4-tensor in $\Delta(x_i, x_j)$

called $\partial \ - \ $Tensor. Invariant, its 4-coupling in $\Delta(x_i, x_j)$

an antisymmetric bilinear term $\Sigma \delta_{ik}(x_i x_j - x_j x_i)$

is called a 6-Tensor. The two simpler scalars are $\Sigma \delta_{ik}$ and $\Lambda = \delta_{12} \delta_{23} + \delta_{13} \delta_{21} + \delta_{14} \delta_{32}$

Note: Terminology has changed since spread of general relativity theory.
Change of axes.

A matrix is involved whenever we change axes. The point \((x,y)\) in our original axes is associated with the vector \((x(1), y(0))\). This equation brings out the fact that \((1, 0)\) and \((0, 1)\) are the unit vectors along the axes. When we change axes, other vectors are made to play these roles, so we interpret the point \((X,Y)\) as corresponding to the vector \((X(a) Y(c)\) \((b) + (d)\). Here of course the co-ordinates \((a,b)\) and \((c,d)\) refer to our original graph paper. In the new system they will be \((1,0)\) and \((0,1)\). Accordingly the connection between the old and new co-ordinates is given by the equation \((x) = X(a) Y(c) i.e. x=Ax+cY\) \((y) = (b) + (d)\), \(y=bX+dY\), which is equivalent to the matrix equation \((x) = (a \ c)(X)\) \((y) = (b \ d)(Y)\).

If the co-ordinates in the old system are the components of the vector \(v\), and in the new system of the vector \(V\), these will be related by some matrix \(S\), so that \(v=SV\). We shall use \(V=S^{-1}v\) when we need to express the new co-ordinates in terms of the old ones.

It is important to notice that a matrix expressing a transformation and a matrix defining a conic behave differently under change of axes.

For the transformation \(v \rightarrow v^*\) we have \(v^* = Mv\).

When the axes are changed by the matrix \(S\) we have \(v=SV\) and \(v^* = SV^*\), so \(SV^* = MSV\), which means \(V^* = S^{-1}MSV\), and the matrix for the transformation in the new axes is \(S^{-1}MS\).

On the other hand, when the matrix \(M\) defines the conic \(v^TMv = c\), \(v=SV\) and \(v^T = V^TS^T\). Accordingly the conic \(v^TMv = c\) appears in the form \(V^TS^TMSV = c\), and the matrix appearing here is \(S^{-1}MS\), which is not, as a rule, the same as \(S^{-1}MS\), which we had for the transformation.

Later we shall meet an important exception in which the matrices still agree in the new system.
If we write \( Q \) for the matrix \( S^T S \), we want \( U^T Q V \) to be the same as \( U^T V \), whatever \( U \) and \( V \) are. Let \( U \) and \( V \) be the column vectors with elements \( u_1 \) and \( v_1 \), where \( i \) goes from 1 to \( n \). Then \( U^T V = \langle u_i, v_j \rangle \), the sum being from 1 to \( n \), while \( U^T Q V = \langle u_i, Q_{ij}, v_j \rangle \), \( i \) and \( j \) being summed from 1 to \( n \). Here the coefficient of \( u_i, v_j \) is \( Q_{ij} \). In \( U^T V \) the coefficient of \( u_i, v_j \) is \( 1 \) when \( i = j \) and 0 when \( i \) and \( j \) are unequal. This means that \( Q \) must equal the identity matrix, that is, that \( S^T S = I \).

We have shown that the transformation \( S \) preserves scalar products if, and only if, \( S^T S = I \). This means that \( S^{-1} = S^T \); the inverse and the transpose must be the same. This means that, when we are dealing with an \( S \) of this type, we need not bother whether a matrix \( M \) is being used to define a transformation or a conic. Both will behave in the same way when they undergo \( S \).

A transformation, \( S \), that satisfies this condition is said to be orthogonal. A matrix that specifies a change of position by rotation or reflection is clearly orthogonal; we know that lengths and angles are not changing. The same applies if we are changing axes by rotating or reflecting them.

Usually finding the inverse of a matrix is liable to be awkward. In the case of an orthogonal matrix this difficulty disappears. We simply transpose it.

The determinant of an orthogonal transformation.

A matrix and its transpose have the same determinant. Also the determinant of the product of two matrices is the product of their determinants. If we consider the equation \( S^T S = I \), and take determinants, we see that the square of the determinant of \( S \) must be 1. Accordingly, the determinant itself of \( S \) must be \( +1 \) or \( -1 \), and thus orthogonal transformations fall into two classes. Those with determinant \( +1 \) turn out to be rotations; they can be achieved by a continuous motion. Those with determinant \( -1 \) cannot. We can imagine a right-hand glove being transformed into a left-hand glove, but there is no way in which we can actually turn the one to become the other.
A problem I was working on involved the principal
integral \[ \int_{0}^{1} \frac{1}{k-b} \, db \]
which occurs near \( k = 1 \).

\[ \begin{align*}
N = 220 & \quad \text{N} = 220 \\
\text{C/N} & \quad \text{C/N} \\
(0.142857143) & = 9.98577425 \\
(0.285714286) & = 9.21034037 \\
(0.428571429) & = 9.48559992 \\
(0.571428571) & = 10.7295861 \\
(0.714285714) & = 13.8629436 \\
(0.857142857) & = 24.0794561 \\
(1.142857143) & = -18.3258146 \\
(1.28571429) & = -7.98507696 \\
(1.42857143) & = -4.6209812 \\
(1.571428571) & = -2.989185 \\
(1.714285714) & = -2.04330249 \\
(1.857142857) & = -1.4394305 \\
(2.142857143) & = -1.04907127 \\
(2.28571429) & = -0.719205181 \\
(2.42857143) & = -0.495874558 \\
(2.571428571) & = -0.325037859 \\
(2.714285714) & = -0.191564574 \\
(2.714285714) & = -0.0854888241
\end{align*} \]
The V. M. C. Poles
of Maths Clubs.

I regard maths clubs as the decisive instrument for the improvement of mathematical education.

Teacher Training

Teaching is a very complicated process. You have three things simultaneously in mind — the mathematician itself, the difficulties children pupils will have in understanding it, and keeping order. When I was training teachers, I tried to separate these. First we would get clear about the mathematician. Then the students would take one or two children and try to help them with their difficulties. After that, they would run a club for 9–10 year olds, who voluntarily stayed after school for mathematical stimulation. Children of that age still show the enthusiasm that ought to be the norm in a mathematician class.

In Canada, I used to ask future teachers to pride something they regarded as a deficit in the educational system and run a small club in an attempt to remedy it. Some felt that high students were held back, and became bored and delinquent. They offered more advanced work to them. Others were troubled by the problem of the physically active child, who dislikes sitting at a desk. They started a club for six restless boys, selected by the principal of an elementary school. The idea was to have athletic competitions, with elaborate scoring, records of averages, etc., and see how much mathematics they could learn in a critical three hours noticing it.

Improvement of Education

A club then is not simply a luxury or an entertainment, but represents a situation in which you strive to provide something that is lacking in existing schools, or is not done as well as you think it should be.

Mathematics, like any other subject, flourishes only when taught by those who both love it and are able to communicate their enthusiasm to their classes. There have always been some such teachers, who have preserved the tradition of mathematics as a living subject. But their numbers have been limited; many teachers in their childhood were not lit by the magic of dead
Let \( C_i(\sigma) = \sum_{j=0}^{\sigma} W_{ij} \sigma^j \)

Then \( F(x, \sigma) = \sum x^i \sum W_{ij} \sigma^j \)

\[ \sum \sigma^j \{ \sum x^i W_{ij} \} \]

Coef of \( \sigma^j \) in \( \sigma \) of degree \( j+2 \).

\[ \sum x^i W_{ij} \sigma^j \] in degree \( j+2 \).

\[ W_{ij} = 0 \quad \text{if} \quad i > j+2 \]
\[ i < j \quad \text{if} \quad i < j+2 \]

Example. For \( q = 2 \),

\[
\begin{align*}
W_{00} & \quad w_{01} & \quad W_{02} & \quad w_{03} \\
W_{10} & \quad w_{11} & \quad W_{12} & \quad w_{13} \\
W_{20} & \quad w_{21} & \quad W_{22} & \quad w_{23} \\
& \vdots & \quad \vdots & \quad \vdots \\
& \quad W_{41} & \quad w_{51} & \quad W_{52} & \quad w_{53}
\end{align*}
\]

Here \( C_5(\sigma) = W_{53} \sigma^2 + \ldots \),

so has factor \( \sigma^2 \), which is \( i - q \) power of \( \sigma \).

Let \( d_i(\sigma) = \sigma^{q} C_i(\sigma) \).

Then \( d_i(\sigma) \) has factor \( \sigma^i \). As \( \sigma^2 \) does not depend on \( i \), \( d_i(\sigma) \) is still an eigenvector of \( \sigma \).

\[
\lambda d = A^2 t d \quad \text{where} \quad A = A \quad \text{and} \quad \lambda = \lambda \quad \text{and} \quad t = t
\]

Let \( b_i = A^{-1} d_i \).

\[
\begin{align*}
\lambda d_i(\sigma) & \quad \text{factor} \quad \sigma^i \\
(b_i - \sigma_i) & \quad \text{factor} \quad \sigma^i \\
b_i & = \sigma_i \quad \sum b_i \sigma^i \\
b_i(\sigma) & = \sigma_i \quad \text{factor} \quad \sigma^i \\
& \quad \text{where} \quad \sigma_i \sigma^i = \sigma^i
\end{align*}
\]

Let \( b_i = \sigma_i \quad \sum b_i \sigma^i \).

\[
\begin{align*}
Tb & = M b_i \quad \sum T_{ij} b_j = M b_j \quad \text{where} \quad M = \| \sigma_i \sigma^i \|
\end{align*}
\]

The community matrix \( T \) has

\[
\begin{align*}
T_{ii} & = -(1+\sigma^2) (i(i+1)) \\
T_{i+1,i} & = \sigma (i+1)^2 \\
T_{i+1,i+1} & = \sigma (i+1)^2 \\
T_{i+1,i+1} & = \sigma (i+1)^2 
\end{align*}
\]

\( T_{i, i-1} = \sigma i^2 \)
\[ X(0) = F(x_0) = \int_{-\infty}^{x_0} f(x) \, dx \]

**Theorem:**

Let \( f(x) \) be a continuous function. Then, for any \( a \) and \( b \) in the domain of \( f(x) \), the definite integral of \( f(x) \) from \( a \) to \( b \) is defined as

\[ \int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x \]

where \( x_i \) are points in the interval \([a, b]\) and \( \Delta x \) is the width of each subinterval.
\( q = 2 \)

\( \ell(2,0) \) give \( M_0 b_{20} = -2.3 b_{20} \) \( \therefore M_0 = -6 \)

\( \ell(0,2) \) give \( M_0 = b_{11} \) \( \therefore b_{11} = -6 \)

\( \ell(1,1) \) \[ M_0 b_{11} + M_1 b_{10} = b_{01} - 2[b_{11} + b_{10}] + 4b_{20} \]
\( b_{10} = 0 \) since \( 1 + 0 \leq 2 \). Also \( b_{01} \)
- \( 6b_{11} = -2b_{11} + 4b_{20} \)
- \( 6b_{11} = 4b_{20} \) \( b_{20} = -b_{11} = +6 \)

\( \ell(3,0) \)

\[ M_0 b_{30} = 9b_{20} - 12[b_{30} + b_{20}] + b_{12} \]
\( \therefore 6b_{30} = 9b_{20} \) \( b_{30} = \frac{9 \times 6}{6} = 9 \)

\( \ell(0,3) \): \( M_1 = b_{12} \)

\( \ell(2,1) \) \( M_0 b_{21} + M_1 b_{20} = 4b_{11} - 6[b_{21} + b_{20}] + 9b_{20} \)

\( \ell(9,1) \)

\[ M_0 b_{91} + M_1 b_{90} = 8^2 b_{8-1,1} - 2(9 \times 1)\sum b_{91} + b_{90} \]
\( + (9 \times 1)^2 \times b_{91,0} \)

By \( \ell(0,8) \) stage we know \( M_0, b_{90}, b_{91,0} \) \( b_{9-3,1}, b_{9+1,0} \)

\( q = 3 \)

\[ 6M_1 = -24 + 36 + 8 \]
\[ = 24 \]
\[ M_1 = \frac{24}{6} = \frac{7}{2} \]

\[ M_1 = \frac{3}{2}(q^2 + (q + 1) = \frac{3}{2}(4 \times 2 + 1) = \frac{7}{2} \]

\( \therefore \text{valid} \)
Field of a bar magnet
+ m at (x, y), - m at (-x, y)

Force in BP is $\frac{m}{r^2}$.

Horizontal component in
$\frac{m}{r^2} \cos \beta = \frac{m(x-a)}{r^2}$

Horizontal component of force in PA is $-\frac{m}{r^2} \cos \beta = -\frac{m(x+a)}{r^2}$

Total horizontal component is $\frac{m(x-a)}{r^2} - \frac{m(x+a)}{r^2}$

If $y=0$, $r = x-a$, $r_2 = x-a$, $r_3 = x+a$

$H_x = \frac{m(x-a) - m(x+a)}{(x-a)^2} = \frac{m \cdot 2a}{(x-a)^2}$

If $x > a$,

$H_x = \frac{m}{(x-a)^2} - \frac{m}{(x+a)^2} = \frac{m \cdot 2ax}{(x-a)^2}$
\[ I = \int \int \int \frac{y^3}{(x-a)(y-b)} \frac{1}{1-\alpha y^2} \, dx \, dy \, dz \, dx \]

\[ \int (x-a) \, dx = \ln(1+y) - \ln(1-y) \]

\[ \int \frac{1}{y-b} \, dy = \ln(1+y) - \ln(1-y) \]

\[ \int \left[ \ln(1+y) - \ln(1-y) \right] \frac{y^3}{1-\alpha y^2} \, dy \, dz \]

If \( y = 1 \), then change \( y \) to \( 1-\alpha y^2 \), and the integral is unchanged.

\[ \int_0^1 \int_0^1 \sum_{j=0}^\infty \left( \begin{array}{c} 3 \cr j \end{array} \right) \frac{y^j}{1-\alpha y^2} \, dy \, dz \]

\[ \Delta w \quad u = y^3, \quad v = 3, \quad \frac{\partial (u,v)}{\partial (y,z)} = \left| \begin{array}{cc} 3 & 0 \\ 0 & 1 \end{array} \right| = 3 \]

\[ I = 2 \int_0^1 \int_0^1 \left[ \ln(1+y) - \ln(1-y) \right] \frac{y^3}{1-\alpha y^2} \, dy \, dz \]

\[ = 2 \int_0^1 \int_{u=-1}^{u=1} \frac{[\ln(1+y) - \ln(1-y)] [\ln(u+y) - \ln(1-u)]}{1-\alpha u} \, du \, dv \]

\[ = 2 \int_{v=0}^{v=0} \int_{u=-v}^{u=v} \frac{[\ln(1+y) - \ln(1-y)] [\ln(u+y) - \ln(1-u)]}{1-\alpha u} \, du \, dv \]

Later, perform:

\[ = 2 \int_0^1 \int_{u=1}^{u=1} \frac{\ln(1+y) - \ln(1-y)}{1-\alpha u} \frac{[\ln(u+y) - \ln(1-u)]}{1-\alpha u} \, du \, dv \]

\[ = 2 \int_{v=0}^{v=0} \int_{u=1}^{u=1} \frac{\ln(1+y) - \ln(1-y)}{1-\alpha u} \frac{[\ln(u+y) - \ln(1-u)]}{1-\alpha u} \, du \, dv \]

(4 changes of variable)

While

\[ = 4 \int_{u=0}^{u=0} \int_{v=u}^{v=u} \frac{\ln(1+y) - \ln(1-y)}{1-\alpha u} \frac{[\ln(u+y) - \ln(1-u)]}{1-\alpha u} \, du \, dv \]
Suppose $\alpha \in (-1, 1, \infty)$

$s(z) = \frac{1 + \alpha}{\alpha}$
\[ \mu = \log \left( 1 - \frac{2}{\sqrt{3}} \right) \]

Redick method
With some clever diagonal. Three then occur in the other cases as well. Those we鹅

(\text{when } n = 0, \text{ the } \text{case } n = 0). Their \text{possible } \text{cases} \text{ are }

\begin{align*}
\pi &= \pi, \quad \text{if } n = 0, \\
\pi &= \pi + 1, \quad \text{if } n = 1, \\
\pi &= \pi + 2, \quad \text{if } n = 2.
\end{align*}

For all the \text{other cases}, \pi = \pi \text{ is the only possibility, as all the numbers can be \text{arranged} to 2 such numbers. Actually:}

\pi = \pi, \quad \text{if } n = 0.

The \text{only possibilities are:}

\pi = \pi, \quad \text{if } n = 0.

\pi = \pi + 1, \quad \text{if } n = 1.

\pi = \pi + 2, \quad \text{if } n = 2.

\text{If } \pi = \pi + 1, \text{ we much increase by } \pi \text{ when}

\pi = \pi - 1, \text{ but when}

\pi = \pi + 2, \text{ and}

\pi = \pi - 2.

\text{For all the numbers from } i \text{ to } n \text{ we have}\n
\pi = \pi, \quad \text{if } n = 0.

\pi = \pi + 1, \quad \text{if } n = 1.

\pi = \pi + 2, \quad \text{if } n = 2.

\text{Therefore, the only possibilities are:}

\pi = \pi, \quad \text{if } n = 0.

\pi = \pi + 1, \quad \text{if } n = 1.

\pi = \pi + 2, \quad \text{if } n = 2.
Differentiate lead to a coefficient of \( \frac{1}{2^n} \) that is

\[
\sum_{i=0}^{k} \frac{a_i}{2^n} \leq (i - s).
\]

\[
\sum_{i=1}^{k} \frac{a_i}{2^n} = \frac{a_0(1 + a_0)}{2} - sa_0
\]

\[
= \frac{1}{2} a_s^2 - (\frac{s - 1}{2})a_s
\]

Sum, see Volk. Correct for \( \frac{1}{2} \geq a_s^2 - \frac{k}{6}(s - \frac{1}{2})a_s \)

\[
\begin{align*}
\text{Together we have} & \quad \frac{1}{2} \cdot \frac{k}{6} a_0^2 + \frac{k}{6}(s + \frac{1}{2})a_0 \\
& \quad \frac{1}{4} \cdot \frac{k}{6} a_s^2 + \frac{k}{6}(\frac{k - 5}{2})a_s \\
& \quad \frac{k}{6} (-\frac{k}{2})a_s
\end{align*}
\]

\[
= \frac{3}{6} \frac{k}{6} a_0^2 + \frac{k}{6}(\frac{s}{2} - \frac{3}{2} s)a_s
\]

Square identity this.

\[
\begin{align*}
\sum_{s=0}^{k} & \quad \frac{3}{6} \frac{k}{6} a_s^2 + \frac{k}{6}(1 - 3s)a_s
\end{align*}
\]

Correction.

\[
\sum_{s=0}^{k} \frac{k}{6} (\frac{s}{2} - \frac{s}{2}s) = - \frac{k}{6} a_s.
\]

Together we finally have the coefficient of \( \frac{1}{n} \)

\[
\frac{3}{2} \leq a_{s}^2 - 3 \frac{k}{6} sa_s
\]
First part, \( \sum_{n=0}^{k} \prod_{i=1}^{n} (2i-2s-1+2i) \)

Taking out factors 2 and \( k \), we are led to

\[
\sum_{i=1}^{k} \left( \frac{i}{2} - \frac{s}{2} - \frac{k}{2} - \frac{1}{4} \right)
\]

\[
\left( \frac{i}{2} - \frac{s}{2} - \frac{k}{2} - \frac{1}{4} \right)
\]

\[
\left( \frac{1}{2} - \frac{s}{2} - \frac{k}{2} + \frac{1}{2} \right)
\]

\[
\frac{1}{2} a_s \left( \frac{1+a_s}{2} \right) + a_s \left( \frac{k-s}{2} + \frac{1}{4} \right)
\]

\[
\frac{1}{2} a_s + a_s \left( \frac{k-s}{2} \right)
\]

Sum over \( s \).

Checked against p. 656.

Second part:

\[
\prod_{i=1}^{k} (4n-2s-2i+1) \prod_{i=s+1}^{k} (4n+2a_s+2a_i-2s-2i+1)
\]

Checked against p. 615.

We have

\[
\sum_{i=s+1}^{k} \left( \frac{i}{2} - \frac{s}{2} - \frac{k}{2} + \frac{1}{4} \right)
\]

\[
\sum_{i=s+1}^{k} \left( \frac{i}{2} - \frac{s}{2} + \frac{a_s + a_i}{2} \right)
\]

\[
\sum_{i=s+1}^{k} \left( \frac{1}{2} - \frac{s}{2} + \frac{a_s + a_i}{2} \right)
\]

\[
\sum_{i=s+1}^{k} \left( \frac{1}{2} - \frac{s}{2} + \frac{a_s + a_i}{2} \right)
\]

Note: \( s \geq k \) gives no sum, so take sum for \( s = 0 \) to \( k-1 \).

Also:

\[
\sum_{i=s+1}^{k} \left( \frac{a_s + a_i}{2} \right)
\]

Consider as example:

\[
\sum_{s=0}^{k} \frac{a_s + a_i}{2}
\]

In groups:

\[
\begin{align*}
\sum_{s=0}^{a_0 + a_1 + a_2 + a_4} s = 0 \\
\sum_{s=1}^{a_1 + a_3 + a_4} s = 1 \\
\sum_{s=2}^{a_2 + a_3 + a_4} s = 2 \\
\sum_{s=3}^{a_3 + a_4} s = 3
\end{align*}
\]

\[
4(a_0 + a_1 + a_2 + a_3 + a_4)
\]

So sum above is:

\[
\frac{1}{2} \frac{k(k+1)}{2} a_s
\]
Approximation to $n!$

$$n! = \int_{0}^{\infty} x^n e^{-x} \, dx$$

$$\ln(x^n e^{-x}) = n \ln x - x$$

$$\frac{d}{dx}(x^n e^{-x}) = \frac{n}{x} - 1 = \frac{n-x}{x}$$

This is true for $x = n$: maximum of $x = n$.

For $x = y + n$:

$$n! = \int_{0}^{\infty} (y+n)^n e^{-x-y} \, dy$$

$$-n \int_{0}^{\infty} (1+\frac{y}{n})^n e^{-y} \, dy$$

$$= e^{\cdot n} \int_{0}^{\infty} (1+\frac{y}{n})^n e^{-y} \, dy$$

$$(1 + \frac{y}{n})^n = e^{\cdot n \ln(1 + \frac{y}{n})}$$

$$= e^{\cdot n \left[ \frac{y}{n} - \frac{y^2}{2n^2} + \frac{y^3}{3n^3} \right]}$$

$$= e^{\cdot y - \frac{y^2}{2n} + \frac{y^3}{3n^2}} \quad \text{for} \quad |y| < n$$

$$\therefore n! \approx n^ne^{-n} \frac{1}{e}$$

$$\int_{0}^{k} x^n e^{-x} \, dx < \int_{0}^{k} x^n \, dx = \frac{(x^{n+1})_k}{n+1} = \frac{k^{n+1}}{n+1}$$

$$H \quad k = n(1-\alpha)$$

$$k^{n+1} = [n^{n+1}(1-\alpha)^{n+1}$$

$$\therefore \alpha = \frac{1}{n+1}$$
\[
\frac{a \cdot e^{al} - 1}{e^{al} - 1} \leq \sum_{n=0}^{\infty} P_n(t) a^n
\]

1. If \( t = 0 \), \( \lim_{n \to \infty} \frac{a^{n+1} - 1}{a^n - 1} = 0 \).
2. \( P_n(0) = 0 \) for all \( n \).

1. If \( t \to 1 \), \( \lim_{n \to \infty} \frac{a^{n+1} - 1}{a^n - 1} = a \).
2. \( P_n(1) = 0 \) except for \( n = 1 \).
3. \( P_n(1) = 0 \) for \( n \geq 2 \).

Consider the power of \( \frac{e^{al} - 1}{e^{al} - 1} \):

\[
\frac{a \cdot e^{al} - 1}{e^{al} - 1} = a\frac{e^{al} - 1}{e^{al} - 1} = a\left(1 - \frac{1}{2a} - \frac{1}{2a^2} - \frac{1}{2a^3} - \cdots\right)
\]

\[
= a(1 - \frac{1}{2a} - \frac{1}{2a^2})
\]

\[
= at + a^2(\frac{1}{2} - \frac{1}{2} t)
\]

1. \( P_0(t) = 0 \), \( P_1(t) = a \), \( P_2(t) = \frac{1}{2} a^2 - \frac{1}{2} t \).
2. \( P_3(t) = t - \frac{1}{2} \).
1/z = (1/2)z^{-2} + (z+1)^{-2} + (z+2)^{-2} + \ldots \ldots \ldots \ldots
\hspace{1cm} -B_2z^{-3} - B_4z^{-5} + \ldots \ldots \ldots -B_{2r}z^{-2r-1} + R
\hspace{1cm} (7)

where R stands for a remainder term involving an infinite sum. The top line here, if we ignore the first term, is equal to L''(z). Solving for L''(z) we find

L''(z) = 1/z - 1/(2z^2) + B_2z^{-3} + B_4z^{-5} + \ldots B_{2r}z^{-2r-1} - R \hspace{1cm} (8)

Integrating gives

L'(z) = \ln(z) + 1/(2z) - B_2/(2z^2) - B_4/(4z^4) \ldots - B_{2r}/(z^{2r+1}) - R^* \hspace{1cm} (9)

where R^* is a remainder found by integrating R.

Integrating again, we have

\ln(z!) = C + (z+0.5)\ln(z) - z + B_2/(1.2z) + B_4/(3.4z^3) + \ldots \hspace{1cm} (10)

This equation is to be understood in the sense that, if we take a finite number of terms, this will differ from the left-hand side by less than the magnitude of the first term that is neglected.

It can be shown that C = (1/2) \ln(2\pi). The terms at the beginning give the usual Stirling approximation.
If $OP$ is a displacement and $OF$ represents a force, the work done by the force in the displacement is

$$x^{(1)}F_1 + x^{(2)}F_2,$$

which we shall write $x^iF_i$.

Sums continually occur in tensor theory, and the tensor convention stipulates that we sum over any index that occurs twice, as "$i$" does here. 

The raised index $i$ in $x^i$ indicates that we are specifying $x$ in the manner (i), known as contravariant specification. The lowered index $i$ in $F_i$ indicates that $F$ is being represented in the manner (ii), covariant specification.

Books often refer to "covariant vectors" and "contravariant vectors". This language I think is likely to confuse learners. There are two different forms of representation, not two different kinds of vectors - it is hard to see how there could be.

Note that $x^iF_i$ is the scalar product, $x \cdot F$ in vector notation.

We are perfectly free to represent $x$ in the covariant manner. If $A$ is the angle between the axes, by dropping perpendiculars we find

$$x_1 = x^{(1)} + x^{(2)} \cos A,$$

$$x_2 = x^{(1)} \cos A + x^{(2)}.$$
Change of Variables.
Thus \( x^i y^j \) is invariant. What does this tell us about the way \( y^i \) is transformed when axes are changed?

Suppose we bring in new variables, \( x^i \), defined by

\[
x^i = t^i_j x^j \quad \ldots \quad (3)
\]

Then \( x^i y^j = t^i_j x^j y^j \). In the new system this should be \( x^j y^j \) and we can make this so for every \( x^j \) only by taking

\[
y^i = t^i_j y^j \quad \ldots \quad (4)
\]

The transformations (3) and (4) differ. (3) gives \( x \) in terms of \( x \), (4) gives \( y \) in terms of \( y \). Thus we have different rules of transformation for vectors shown in covariant form and in contravariant form.

Note that when perpendicular axes are being used, the two modes of representation coincide, as is evident when the angle \( \alpha \) in Figure 1 is 90°.

Now \( g_{ij} \) too must be recognized as a tensor, for it tells us how to calculate the length of a vector, a quantity on which all systems are agreed. How does it transform? As before we introduce new co-ordinates \( x^i \) by \( x^i = t^i_j x^j \).

Then \( g_{ij} x^i x^j = g_{ij} t^i_p t^j_q x^p x^q \), which ought to be \( G_{pq} x^p x^q \).

Accordingly

\[
G_{pq} = t^i_p t^j_q g_{ij} \quad \ldots \quad (5)
\]

gives the transformation for \( g_{ij} \). Note that the transformation is the same as the one we would use if \( g_{ij} \) was the product of two vectors \( x^i y^j \). This property is sometimes used to define a tensor, by saying that a tensor transforms like the product of vectors. I have never found this approach particularly helpful.

A tensor with two subscripts does very much the same job in this symbolism as a matrix in ordinary vector-matrix notation. If we write equation (2) as \( OP = x^i g_{ij} x^j \) so that summation is over adjacent pairs of symbols, it is seen that, in matrix notation, we would have an expression of the form \( x^T M x \) with a matrix \( M \). This is the usual way to express a quadratic of two variables in matrix symbolism.

It is also possible to define a tensor corresponding to the matrix, \( M \), for a mapping \( y = M x \).
the velocity of a particle lies in the interval \( u, u+du \). The chance that the velocity of a particle has components within \( u, u+du \); \( v, v+dv \); \( w, w+dw \) is then

\[
 f(u)f(v)f(w)\ du\ dv\ dw. \quad \text{If we have a small region, of volume } \sqrt{V} \text{ in the } (u,v,w) \text{ space, at or around the position } (u,v,w), \text{ the chance of the velocity lying in this region is } f(u)f(v)f(w)\sqrt{V}. \quad \text{The reason for bringing in } \sqrt{V} \text{ is that this quantity, being an element of volume, does not depend on the choice of axes of co-ordinates. Since no direction is singled out for special privileges in a gas, this probability should not depend in any way on the direction of the velocity, but only on its magnitude. Thus}
\]

\[
f(u)f(v)f(w) \text{ should be a function of } u^2+v^2+w^2 \text{ alone. The same will be true of its logarithm. Accordingly we shall have}
\]

\[
 \ln f(u) + \ln f(v) + \ln f(w) = \phi(u^2+v^2+w^2)
\]

for some function \( \phi \). If we differentiate partially with respect to \( u \) we obtain

\[
 \frac{f'(u)}{f(u)} = 2u\phi'(u^2+v^2+w^2)
\]

Similarly we have

\[
 \frac{f'(v)}{f(v)} = 2v\phi'(u^2+v^2+w^2)
\]

and

\[
 \frac{f'(w)}{f(w)} = 2w\phi'(u^2+v^2+w^2)
\]

it will be seen that

\[
 \frac{f'(u)}{uf(u)} = \frac{f'(v)}{vf(v)} = \frac{f'(w)}{wf(w)}
\]

This can only be so if \( f'(u)/[uf(u)] \) is a constant, \( C \). Then

\[
 f'(u)/f(u) = Cu, \quad \text{and } ln f(u) = (C/2)u^2 + \text{constant.}
\]

Hence \( f(u) = A \exp[(C/2)u^2] \) for some constant \( A \). Clearly \( C \) must be negative; otherwise the velocity would be overwhelmingly likely to be infinite. If \( C/2 = -a \) we have

\[
 f(u) = Ae^{-au^2}
\]

To obtain \( A \) in terms of \( a \), we use the fact that the probability of the particle having some velocity is 1. Accordingly

\[
 1 = \int_{-\infty}^{\infty} A \exp(-au^2) \ du
\]

A standard result is

\[
 \int_{-\infty}^{\infty} \exp(-au^2) \ du = \sqrt{\pi/a} 
\]

Hence

\[
 A = \sqrt{\pi}a^{-1/2}, \quad \text{and}
\]

\[
 f(u) = \sqrt{\pi}a^{-1/2}e^{-au^2}
\]
The kinetic energy of a molecule is \((\frac{m}{2})(u^2 + v^2 + w^2)\). As \(u^2\), \(v^2\), and \(w^2\) all have the same value, it follows that the average kinetic energy of a molecule is \((\frac{3}{2})kT\). As a mole contains \(N\) molecules, the kinetic energy for a mole is \((\frac{3}{2})NkT\). As \(Nk=R\), it follows that the kinetic energy per mole of ideal gas is given by \(E=(\frac{3}{2})RT\).

The results here are in accordance with a general principle known as the equipartition of energy, that the average energy of a particle has \((\frac{1}{2})kT\) for each degree of freedom.

**Line Integrals. Exact Differentials.**

The role of line integrals can be seen by considering the calculation of the work done on a mass, which moves along a specified path in a field of force. The work done by a force \((X, Y)\) when the mass moves through \((dx, dy)\) is \(Xdx+Ydy\). \(X\) and \(Y\) we suppose to be known functions of \(x, y\). A convenient way to specify the path followed is to take \(x\) and \(y\) as functions of a parameter \(t\); \(x=x(t), y=y(t)\). Then \(x\) and \(y\) will be definite functions of \(t\). The work done will then be

\[
\int X(t) \, x'(t) + Y(t) \, y'(t) \, dt,
\]

taken between \(t\) and \(t_1\) corresponding to the beginning and the end of the path. The integral will normally be written as \(\int Xdx+Ydy\), with an indication of the path to be followed.

For instance, if we wanted the path to go round the triangle with corners \((0,0), (1,1), (2,0)\) we might take
- from \(t=0\) to \(t=1\) \(x=t, y=t\)
- from \(t=1\) to \(t=2\) \(x=t, y=1-t\)
- from \(t=2\) to \(t=4\) \(x=4-t, y=0\).

This rather elementary example is given, so that there is no sense of vagueness or mystery in the concept of a line integral. Curved paths of course require only the introduction of higher powers of \(t\).

If the field of force should happen to be one that has a potential, the work done in going from \(A\) to \(B\) will equal the change in the value of the potential from \(A\) to \(B\), whatever path may be taken, so the value of the integral will not depend on the path. The integral taken around a closed curve will give zero. In this case we say that \(Xdx+Ydy\) is an exact differential.

When the value of the integral depends on the path taken, \(Xdx+Ydy\) is said not to be an exact differential, and the integral around a closed path may fail to be zero. This happens, for example, if \(X=y, Y=0\), as can be verified by calculating the integral for the path specified above. It is obvious that the value of this integral must depend.
equal.

The First Law allowed us to define a new function of state, the energy, $E$. The Second Law leads to the definition of a new function of state, the entropy, $S$, though this is far from obvious. Clausius, who introduced the concept of entropy in 1854, is regarded as the founder of physical chemistry. (Pledge, p.145.)

There are two lines of thought that suggest the existence of entropy. I find it surprising that Clausius was able to reach this new concept, if he did it from the first of these, which runs as follows. Let $\Delta Q_1$ be the amount of heat taken in the part AB of the Carnot cycle, and $\Delta Q_2$ the amount taken in in CD (this of course is negative). We have

$$\Delta Q_1 = RT_1 (\ln V_2 - \ln V_1)$$
$$\Delta Q_2 = -RT_2 (\ln V_2 - \ln V_1)$$

It follows that

$$\frac{\Delta Q_1}{T_1} - \frac{\Delta Q_2}{T_2} = 0.$$  

Now the left-hand side of this last equation is what we would get if we found the line integral around ABCD of $dQ/T$ (if for once we allow ourselves to write $dQ$ for the infinitesimal amount of heat taken in). This means that we are dealing with an exact differential; there is a function of state, $S$, such that, in an infinitesimal change, $\Delta Q = TdS$. $S$ is called the entropy.

This result has been found only for a Carnot cycle. There is then a lengthy argument to show that any reversible cycle can be approximated by a combination of Carnot cycles.

The second approach demonstrates the existence of such a function of state for an ideal gas, and thus suggests that such a function may exist in other situations. For an ideal gas,

$$\Delta Q = C \,dT + (RT/V) \,dV$$

which is clearly not an exact differential. However, if we divide by $T$ we get

$$\frac{\Delta Q}{T} = CDT/T + RDV/V$$

which clearly is an exact differential, so entitled to be called $dS$ with

$$S = C \ln T + R \ln V + \text{constant}.$$
equal.

The First Law allowed us to define a new function of state, the energy, $E$. The Second Law leads to the definition of a new function of state, the entropy, $S$, though this is far from obvious. Clausius, who introduced the concept of entropy in 1854, is regarded as the founder of physical chemistry. (Pledge, p.145.)

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$$\Delta Q_1 = -RT_1 (\ln V_2 - \ln V_1)$$
$$\Delta Q_2 = -RT_2 (\ln V_2 - \ln V_1)$$

It follows that $\Delta Q_1/T_1 - \Delta Q_2/T_2 = 0$.

Now the left-hand side of this last equation is what we would get if we found the line integral around ABCD of $dQ/T$ (if for once we allow ourselves to write $dQ$ for the infinitesimal amount of heat taken in). This means that we are dealing with an exact differential; there is a function of state, $S$, such that, in an infinitesimal change, $\Delta Q = TdS$. $S$ is called the entropy.

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$$\Delta Q = C dT + (RT/V) dV$$

which is clearly not an exact differential. However, if we divide by $T$ we get

$$\Delta Q/T = C dT/T + RdV/V$$

which clearly is an exact differential, so entitled to be called $dS$ with

$$S = C \ln T + R \ln V + \text{constant}.$$
It is interesting that we can reach the main point of Carnot's argument without any detailed calculations about the behaviour of gases or other substances. His theorem is this: **no engine can ever be more efficient than a reversible engine.**

Suppose the reversible engine works between the temperatures $T$ and $t$. No engine, working between these temperatures, can produce more work for a given quantity of heat. For suppose the reversible engine can take in $H$ units of heat at temperature $T$ and use it to produce $W$ units of work at temperature $t$. Being reversible, it could take in $W'$ units of work at temperature $t$ and use this to give out $H$ units of work at temperature $T$. Now suppose some other engine could do better and change $H$ units of heat at temperature $T$ to $W'$ units of work at temperature $t$, with $W' > W$. Carnot argued as follows:-

First, let the improved engine change $H$ units of heat at $T$ to $W'$ units of work at $t$. Use $W'$ to do some useful job. This leaves $W$ units, which the reversible engine can change to $H$ units of heat at $T$. Repeat this process. Each time we get $W'-W$ units of work, and end back where we started. We have perpetual motion which Carnot believed was impossible.

Here we have an essential theorem of thermodynamics, freed of technical details.

**The Carnot cycle.**

The mathematical implications of this theorem are found by considering a particular series of operations, known as the Carnot cycle, involving the expansion and contraction of gas, with heating and cooling. We suppose the gas to obey the equations for an ideal gas. This might seem to bring in an element of unreality. However the behaviour of actual gases at sufficiently low pressures approximates very closely to that of the hypothetical perfect gas, so that in principle it would be possible to carry out an actual experiment as close as one may wish to a Carnot cycle.

All the changes in the volume of the gas are to be carried out extremely slowly. Some of them are to take place at constant temperature, such as might be achieved by having the gas in a metal container in contact with a large mass of water. These are known as isothermal. Others are to be done in such a way that no heat can enter or leave, the container being surrounded by insulation; these are called adiabatic.

As the Carnot cycle involves both these processes, we need to calculate the work, heat and pressure changes involved in each of them, before we can consider the Carnot cycle itself.

**Adiabatic expansion.**

Theory suggests and experiment verifies that the internal energy of a gas depends only on the temperature. For an ideal gas we assume $E=CT$, where $E$ is the energy and $T$ the absolute temperature.
from this decrease of energy.

Work in an isothermal process.

This is a process in which the temperature does not change so T is constant. As \( pV = RT \) for a mole of gas, \( p = RT/V \) and \( p\,dv = (RT)\,dv/V \). Integrating this leads to logarithms and we find

\[
W = RT \ln(V_2/V_1) \tag{6}
\]

The energy of the gas, being a function of temperature, does not change, so, in an isothermal expansion, this work can only have come from that amount of heat being absorbed.

If the gas is compressed, \( V_2 < V_1 \), and the work done by the gas is negative and that amount of heat comes out of the system.

Graphing the Carnot cycle.

In the Carnot cycle there are two isothermal processes and two adiabatic processes. In textbooks the cycle is usually shown graphically with co-ordinates \( p \) and \( v \). This has certain advantages. However it seems interesting to take \( T \) and \( v \) as the variables. In the usual graph, the curves for the two types of process do not look very different. If we take \( T \) as one of the variables, it immediately becomes obvious which processes are isothermal, for in them \( T \) is constant and the graph is a level straight line. The equation for an adiabatic process was given in equation (3) as \( pv^g = \text{constant} \). As \( pV = RT \), \( p = RT/V \) and the equation becomes \( T^p v^{g-1} = \text{constant} \). As \( g = (C+R)/C \), \( g-1 = R/C \).

Thus we have \( T^p V^{R/C} = \text{constant} \), and so

\[
\ln T + (R/C)\ln V = \text{constant}.
\]

Accordingly if we take \( x = \ln V \) and \( y = \ln T \), adiabatic changes will be shown by a descending straight line with gradient \(-R/C\).

The nature of the cycle can be read off from the diagram, Figure 1.

Figure 1. A --- B

\[\begin{array}{c}
\downarrow \\
\downarrow \\
C --- D
\end{array}\]

If we begin the cycle with the isothermal change, at temperature \( T_1 \), from A to B, the lines AC and BD represent adiabatics through A and B respectively. C and D are points on these at temperature \( T_2 \). In the cycle we go from A to B and then B to D; in both of these the volume increases. We then go from D to C and from C to A; in both of these the volume decreases.

We now consider the total work done and the heat changes in the cycle.

As we saw in equation (5), the work done in BD is \( C(T_1-T_2) \), as we go from temperature \( T_1 \) to \( T_2 \). In CA we do exactly the opposite; we go from \( T_2 \) to \( T_1 \), so the total work is zero.
walls are perfectly reflecting or not, since the radiation is in equilibrium with the material of the container. It does not matter whether the radiation is reflected or whether some of it is absorbed, raising the temperature of the container, and later radiated. He shows that the conservation of energy would be violated if this were not so (p.200).

Black Body.

Above we referred to a black body as one that absorbed all radiation falling onto it. There is no natural object that has this property. However, if a cavity in a body has a very small opening, so that light entering through this opening is reflected and partially absorbed many times in the interior, very little light eventually finds its way out through the opening, and this is found to give an effectively black body. This arrangement can also be used in the opposite direction, the material being heated and radiation allowed to pass out through the narrow opening. This has certain theoretical advantages which will not be discussed here. Such a source of radiation is used in the first and third stages of the cycle described in the next section, which follows the argument as presented by Richtmeier in his "Introduction to Modern Physics".

The Stefan-Boltzmann Law.

In 1879 Stefan suggested, on the basis of experimental observations, a law equivalent to saying that \( u \), the intensity of radiation inside a container, is proportional to the 4th power of the temperature. In 1884, Boltzmann showed that this result could be deduced by considering an "ether engine" on the analogy of the Carnot cycle.

As in the Carnot cycle, there are two stages in which the temperature is held constant and two in which heat is not allowed to enter or leave.

The constant temperature stages differ markedly from those in the Carnot cycle, for it can be shown that \( u \), the radiation density, is a function of temperature \( T \) alone. As the pressure is \( u/3 \), it follows that pressure is constant in an isothermal change, so such changes appear as horizontal lines on a \( p,V \) diagram.

The container of the radiation is a cylinder, inside which a piston can move. The piston has unit area.

Stage 1. We start at A, with temperature \( T \), pressure \( p \), and volume \( V \). A black body is held at temperature \( T \), and radiation coming out of its small opening passes into a small opening in the cylinder. The
Equating the two expressions for the efficiency we find

\[ \frac{dT}{T} = \frac{du}{4u} \]

Integrating this gives

\[ 4 \ln(T) = \ln(u) + \text{constant} \]

so

\[ u = a T^4 \]  \hfill (19)

Note. The fact that \( du \) and \( dT \) have been chosen to represent decreases does not invalidate the argument. If you have any doubts on this score, it is quite simple to carry through the argument with \( du, dp \) and \( dT \) regarded as increases with negative values.

**Question 1.** We have seen that isothermal curves have the equation \( p = \text{constant} \) on the \( p,V \) diagram. What equation do adiabatics satisfy?

**Question 2.** Verify that the approximation treating ABCD as a parallelogram does not lead to an error.

**Planck's Version.**

It is instructive to compare and contrast the proof of this result given in Planck's book, pp. 200-201. He imagines a black body with variable volume, capable of doing work. This might be a cylinder with a piston. There is a small aperture at the end remote from the piston, through which radiation is emitted, so the cylinder constitutes a black body. Let \( U \) represent the energy of the radiation inside the cylinder, so that \( U \) replaces \( E \) in the universal equation (16). We have

\[ dU = T dS - pdV. \]  \hfill (20)

\( U = uV. \) All the quantities \( U, p, V, S, T \) are functions of state, so black body radiation has a definite temperature, namely the temperature of the enclosing wall. We wish to work with \( T \) and \( V \) as our variables. Now \( u \) is a function of \( T \) alone, so we may write

\[ du = d(Vu) = Vdu + udV = V(du/dT)dt + udV. \]

Now, from equation 20, \( T dS = du + pdV = du + (1/3)udV \) since \( p = u/3. \) Hence we have

\[ T dS = V(du/dT)dt + (4/3)udV. \]  \hfill Dividing by \( T, we obtain \]

\[ \frac{dS}{T} = \frac{V du}{dt} \quad \text{and} \quad \frac{dS}{V} = \frac{4u}{3T} \]

Applying \( d/dT \) to the first expression gives the same result as applying \( d/\partial T \) to the second, provided certain continuity conditions are satisfied. We find

\[ \frac{1}{T} \frac{du}{dT} = 4 \frac{du}{3T} - \frac{4}{3} \frac{u^2}{T} \]

On simplifying, this gives \( 4dU/T = du/u \) and integrating leads to \( u = aT^4 \) as before.
Tenets of a naturalistic mathematics

I shall now simply state the conclusions of my long inquiry into the possibility of approaching mathematics naturalistically. It sounds dreadfully pompous to say this, but since I have spent the better part of forty years reaching these conclusions, and have anyway already published many fragments of my thinking (which have puzzled and dismayed some readers), it would be foolish of me not to publish the overall result: the key which may enable those same bemused readers to see how it all makes sense.

The first tenet is important, because it may serve to curb any tendency to the kind of triumphalism which affected foundationalism, and led to its collapse, in the 1960s. We begin with the truism that mathematics is valued in society because of its predictive powers. This is reflected within mathematics by the relative importance of the idea of a sequence, and especially of the idea of a sequence which continues indefinitely in accordance with some rule. But there is a limit to the 'finding of rules' for sequences which continue indefinitely, because we can conceptualise a 'perverse random' sequence which will defeat any rule we try to use to predict it beforehand. Thus:

TENET 1 Indefinable sequences of events can be clearly and distinctly visualised, and they can, a fortiori, occur in experience. Such sequences lie outside the proper area of mathematics. Mathematics is fundamentally concerned with well-defined patterns, structures and objects. But indefinable sequences defeat every attempt at definition, indefinitely. They can never be incorporated into mathematics for this reason. Consequently there is a limit, which we can clearly and distinctly visualise, to the applicability of mathematics.

The second tenet is concerned with the possibility of applying mathematics to itself. This possibility, of applying mathematics to itself, is the great advance of Twentieth Century mathematics. It is used in the century's most significant result, Godel's Theorem; it was used by Turing in his conceptualisation of a Universal Machine; and it was celebrated by Hofstadter in his book Godel, Escher, Bach. (1979). But here, too, for all the new power opened-up by self-application, we eventually run into a limit. When one tries to formulate a statement which will assert its own falsity, one ends-up with a paradox. This is caused by the fact that one's provisional assessment of the statement's truth oscillates between 'true' and 'false'.

TENET 2 Once we legitimise the application of mathematics to itself, we open-up the possibility of oscillations of inconsistent partial meaning. This is a new, fundamental, species of contradiction, which may be called 'dynamic contradiction'. The paradoxes of set theory exhibit dynamic
If we plot $\frac{e^x}{T^5}$ against $\frac{1}{T}$ we get the same curve

$$\chi = \frac{e^x}{T^5} \quad \chi = \frac{1}{T^5} = \frac{1}{T^5} f(x)$$

$\chi$ the energy density inside the nucleus in proportion to $e_x$.

We have

$$\chi = \frac{\rho}{\pi^2}$$

Planck, for convenience of calculation, supposed every $e_x$ to have finite packets $\varepsilon$ and found

$$\chi = \frac{\delta \pi}{\chi} = \frac{\varepsilon}{\pi^2}$$

If we take $\varepsilon = \frac{1}{\lambda}$ we get agreement. As frequency $\nu$ is proportional to $\frac{1}{\lambda}$, we have results--an oscillator of frequency $\nu$ can have energies only multiples of $\frac{\varepsilon}{\pi^2}$. ($\hbar = 6.56 \times 10^{-27}$ erg sec.)

In Bohr model of atom, it is required that $\frac{\hbar}{p_0}$ be a multiple of $\frac{\varepsilon}{\pi^2}$. There are certain discrete orbits allowed by this. In such an orbit, radiation does not occur. Radiation involves a jump from one state to another.

* Please note that the writing at the top of this page suggests to me that it was written after one of his several mini-strokes which he had during his last six to eight years. Hopefully the content is valid.  June 19, 2009
\[ ds^2 = 2T \left( g_{11} \, dt^2 + g_{ik} \, dx^i \, dx^k \right) \quad (3) \]

We can now express angle between lines, divergence and curl, grad \( g \), div \( g \), \( \Delta = \text{div grad} \), and use them as simply as in Euclidean 3-D, with slightly more complicated analytic expressions from now on, all geometric statements refer to this space.

Important to distinguish between covariant and contravariant. Differently no greater than for oblique Cartesian axes.

d\( q \) are primitives for contravariant vector. The coefficients of \( d\theta \) in \( 2T \) give the fundamental covariant tensor \( 2T \) is the contravariant form corresponding to \( 2T \) for the moment, as is well known, are contravariant vectors corresponding to \( d\theta \), so the impulse is the covariant form of the velocity.

The LHS of (1) is the contravariant fundamental form with variables \( \tilde{\omega}/g_{ik} \). The \( \tilde{\omega}/g_{ik} \) are components of grad \( W \), which by its nature is covariant. Thus

\[ \left( \text{grad } W \right)^2 = 2(E-V). \]

\[ |\text{grad } W| \leq \sqrt{2(E-V)}. \]

The expression of \( \sqrt{2(E-V)} \) as the moment has the significance that covariant components can only be put into a contravariant form to give a sensible, i.e. invariant result.
Schrodinger.

Consider particle $T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ potential $V(x, y, z)$

Define $ds^2 = \frac{1}{2} m (d\dot{x}^2 + d\dot{y}^2 + d\dot{z}^2)$

\[
\frac{\partial W}{\partial t} + T \left( \frac{\partial W}{\partial \dot{q}} \right) + V(q) = 0,
\]

\[
dW = -E dR + S(q) = -E + T \left( \frac{\partial W}{\partial \dot{q}} \right) + S(q) = 0.
\]

\[
T \left( \frac{\partial W}{\partial \dot{q}} \right) = E - V
\]

\[
2T \left( \frac{\partial W}{\partial \dot{q}} \right) = 2(E - V)
\]

Grad $W$ has role of charge of $W$ component.

Distance in $x$ direction is $\sqrt{\frac{m}{2}} dx$.

Rate of change of $W$ is $\sqrt{\frac{2}{m}} \frac{\partial W}{\partial x}$

\[
(\text{grad } W)^2 = \frac{2}{m} \left( \frac{\partial W}{\partial x} \right)^2 = \frac{2}{m} \left( \frac{\partial W}{\partial y} \right)^2 + \frac{2}{m} \left( \frac{\partial W}{\partial z} \right)^2
\]

\[
T = \frac{p_x^2 + p_y^2 + p_z^2}{2m}
\]

\[
\frac{1}{m} 2\pi \left( \frac{\partial W}{\partial q} \right)^2 + \frac{2}{m} \left( \frac{\partial W}{\partial \dot{q}} \right)^2 - \frac{1}{2}(E - V)^2 = 2(E - V)
\]

\[
q_{ij} = \frac{m}{2} \delta_{ij}
\]

\[
dt = \frac{m}{2} \delta_{ij} (dx^i dx^j)
\]

\[
\text{Contravariant } g_{ij} dx^i = \frac{m}{2} dx^i
\]

\[
\frac{\partial W}{\partial q_k} =
\]

Back of W10
Schrödinger's principle says $V$ proportional to $\mu$.
Here $u$ and $v$ are constants, so no deduction possible.

When $V=0$, $E-V = T = \frac{1}{2} mv^2$

$$\frac{E}{\sqrt{2}(E-V)} = \frac{E}{\sqrt{mv^2}} \cdot \frac{E}{\sqrt{mv^2}}$$

in accord with stated result.
The constitution can clearly be replaced by
Huygen's elementary waves, with \( \mu \) proportional to \( \frac{1}{r} \)
from (6). The \( q \)-space is inhomogeneous but isotropic
(since phase velocity is all directions).

\[ W \text{ plays the role of phase. Fermat's principle} \]
\[ 0 = \int p \, dc = \int \frac{dc}{u} = \int \frac{\sqrt{2(E-V)}}{E} \, dv = \int 2 \, dv. \quad (7) \]

so Maupertuis' least Action principle. The "rays,

i.e. the orthogonal trajectories of the wave-surface,
are the paths of the system for energy \( E \), in
agreement with \( p_c = \frac{\partial W}{\partial q_c} \). \quad (8)

The concept of rays belongs to geometrical optics,
that is the only analogy we have established so far. The
wave fronts are only loosely related so far to mechanics,
for the representative point of the mechanical system
is no way moves with the velocity \( u \), \( du \) on the
contrary, the velocity is proportional to \( u \). From (8)
\[ \frac{du}{dt} = \sqrt{2T} = \sqrt{2(E-V)}. \]

This discrepancy is illuminating. From (6), the velocity is large where
\( \nabla W \) is large, i.e. where the surface is examined
close together, i.e. where \( u \) is small.

Secondly, \( W = \int |du| \). This naturally changes
during the motion (by \( (T-V)du \) and \( dW \)) so it is impossible
for the representative point to stay on the same \( W \) surface.
Note on Action:

\[ S = \int 2T \, dt = \int E \cdot dp \, dq \]

Consider mass moving under no force, with energy \( E \).

If \( x \) is distance from \((x_0, y_0, z_0)\) to \((x', y', z')\),

as momentum \( mV = m \sqrt{\frac{2E}{m}} = \sqrt{2mE} \)

so \( \int mV \, dx = \sqrt{2mE} \cdot x = S \).

\[
\frac{\partial S}{\partial E} = 1 \sqrt{2m} \cdot \frac{1}{2} E^{-\frac{1}{2}} = \gamma \sqrt{\frac{m}{2E}}
\]

\[
\bar{E} = \frac{1}{2} mV^2 \frac{m}{2E} = \frac{1}{2} V^2
\]

\[ \therefore \frac{\partial S}{\partial E} = \frac{\gamma}{V} = \alpha \frac{V}{V}
\]

as stated by Hamilton (Prop. 50 = 0).
It may also happen that
\[ a > c, \ b < c, \ a+b < 2c. \]

The factors \((2n+2b+1)\) - \((2n+2c)\)
are as in previous section. However, there are now more
factors in denominator \(\mathcal{N}\) than in numerator \(\mathcal{D}\). We have

\[ f(a,b,c) = \frac{(2n+2b+1)\cdot\cdots\cdot(2n+2c)}{(2n+2b)\cdot\cdots\cdot(2n+2c)} \]

(\(a-b\) factors)\(\frac{(2c-a-b)\text{ factors}}{}\).\(\frac{(2c-a-b)\text{ factors}}{}\).

Thus we seem to have independent effect when \(a > c\).

(a) If \(b > c\) then \((2n+2b+1)\cdot\cdots\cdot(2n+2b)\) in denominator.
If \(b < c\) then \((2n+2b+1)\cdot\cdots\cdot(2n+2c)\) in numerator.

(b) If \(a+b > 2c\) then \((4n+2c-1)\cdot\cdots\cdot(4n+2a+2b-1)\) in numerator.
If \(a+b < 2c\) then \((4n+2a+2b+1)\cdot\cdots\cdot(4n+2c-1)\) in denominator.

If \(b = c\), then \(x_1\) in (a) \(\times 2\) in (b).

Easy to arrange by process.

(a) \(\frac{(2c-a-b)\text{ factors}}{2.4\cdots(2a-2b)\text{ factors}}\) if \(b < a\).
An, n+q, is the coefficient of the \((n+q)^{th}\) power of \(s\) in the sum of products of the form

\[
\prod_{r=0}^{\infty} L \left[ i(r) \right].
\]

Now, \(L \left[ i(r) \right] = \sum_{j(r)=i(r)} L \left[ i(r), j(r) \right] s^{j(r)}\).

Accordingly, \(A_{n, n+q} = \sum_{\text{all products}} \prod_{r=0}^{\infty} L \left[ i(r) \right] L \left[ j(r) \right]\)

that satisfy the following conditions:

1. \(\sum_{j(r)} = 0+1+...+n+q,\)
2. The \(i(r)\) are distinct.
3. For each \(r,\) \(i(r) \leq j(r)\).

The task is to evaluate these sums.

\(P(n) = A_{n, n} -\)

At first, the equations are simple. If \(q=1,\)

we can form the sum for \(A_{n, n+1}\) by forming the product \(L(0,0) L(1,1) \cdots L(n,n)\)

by changing \(L(r,r)\) to \(L(r+1,r+1)\) and summing. We can form the sum for \(A_{n, n+q}\) by the following procedure:

1. In the product \(L(q,0) L(1,1) \cdots L(n,n)\)
   replace \(L(r,r)\) by \(L(r+q,r+q)\). Do this for each \(r\) and sum for \(r = 0\) to \(n\).
2. Replace \(L(n,n)\) by \(L(n+1,n+1)\) and add to the sum already found.
So we have \( \frac{1}{2} \int \ln(u+3) - \ln(u-3) \, du = |u| > 13 \)

\( \frac{1}{3} \ln(3-u) - \ln(3-u) \) \( \| \leq 13 \)

If \( |u| > 13 \), \( u+3 \) and \( u-3 \) have the same sign, so
\[ \frac{u+3}{u-3} > 0 \quad \implies \quad \frac{u+3}{u-3} = \frac{|u+3|}{|u-3|} \quad \text{at } \theta = \theta(u+3)-\theta(u-3). \]

Same argument applies in \( \text{and } \theta < \theta \).

\[ \text{We have} \quad \frac{1}{2} \frac{\ln(3+u) - \ln(3-u)}{u} \text{ for all } \theta. \]

\[ \therefore \quad I = \int_{u=-1}^{1} \int_{\theta=-1}^{1} \frac{u}{1-u} \left[ \ln(3+u) - \ln(3-u) \right] \frac{1}{3} \frac{1}{u} \, du \, d\theta. \]
Lagrangian equations

\[ \frac{\partial L}{\partial q_i} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \]

\[ \mu \to \sqrt{E-V} \]

\[ \text{and } E, V \text{ must be specified} \]
We have

\[
\frac{\text{ant. n} + \text{ant. brn. n} \quad \text{ant. n} \quad \text{ant. brn. n}}{\text{ant. n} \quad \text{ant. brn. n} \quad \text{ant. n} \quad \text{ant. brn. n}}
\]

\[
= \frac{\text{ant. n} \quad \text{ant. brn. n}}{\text{ant. n} \quad \text{ant. brn. n}}
\]

\[
+ \frac{\text{ant. n} \quad \text{ant. brn. n}}{\text{ant. n} \quad \text{ant. brn. n}}
\]

\[
+ \frac{\text{ant. n} \quad \text{ant. brn. n}}{\text{ant. n} \quad \text{ant. brn. n}}
\]

\[
\text{(ant. n} \quad \text{ant. n} \quad \text{ant. n} \quad \text{ant. n})
\]

\[
\text{(ant. n} \quad \text{ant. n} \quad \text{ant. n} \quad \text{ant. n})
\]

\[
\text{ant. brn. n} \quad \text{ant. brn. n} \quad \text{ant. brn. n}
\]
DEF PROC

FOR I = 1 TO Z
  FOR I = 1 TO Z-1
    FOR K = I+1 TO Z
      IF K < I+1 THEN PROC C.GEN (func)
        NEXT K
    NEXT I
  NEXT I

DEF PROC C.GEN

C(R, Z+I-1) = H(I, I-1, K)
C(R, Z+I) = H(I, I, K)
C(R, Z+I+1) = H(I+1, I, K)
C(R, Z+K-1) = -H(I, I, K-1)
C(R, K) = -H(C, K-1)
C(R, Z+K) = -H(I, I, K+1)
PROC. I have run into a lor of trouble
with this and think a sledgehammer melted may
be quickest.

\[ S a_{ij} h_{ik} = 0 \quad \text{omitted for } i = 3 \]

\[ a_{ij} \text{ is damped. This lema...} \]

\[ 0 = a_{ij} h_{ik} + a_{ik} h_{ij} = a_{i,i} h_{ii} \]

\[ - h_{ii} a_{k,k} = h_{ik} a_{k} - h_{ik} a_{k} - h_{ik} a_{k} + k \]

\[ a_{ij} = x_{i} \quad i = 1, 2, 3 \quad a_{i,i} = x_{i}^{2} + i = a_{i,i} \]

So

\[ 0 = h_{i-1,k} x_{j} x_{i-1} + h_{i,k} x_{j} + h_{i-1,k} x_{j}^{2} - \]

\[ - h_{i,k} x_{j} - h_{i,k} x_{j} - h_{i,k} x_{j} + k \]

We suppose \( k \geq i \), \( i = 1, \ldots, 3 \), \( k = i + 1 \ldots 3 \).

If \( k = i \), a repeated element can occur only

Then \( 0 = h_{i,k} x_{j} x_{i-1} + h_{i,k} x_{j} + h_{i,k} x_{j} + k \)

\[ - h_{i,k} x_{j} - h_{i,k} x_{j} - h_{i,k} x_{j} + k \]

\[ - h_{i-1,k} + h_{i,k} \quad \text{for } i = 2 \text{ or } \frac{1}{2} \text{ or } \frac{1}{4} \text{, which check.} \]

Here for \( I = 1 \) to \( Z-1 \), PROCP.

\[ C(R, Z+I-1) = H(I-1, I+1) \]

\[ C(R, I) = H(I, I+1) \]

\[ C(R, Z+I) = H(I+1, I+1) - H(I, I) \]

\[ C(R, I+1) = -H(I, I+1) \]

\[ C(R, Z+I+1) = -1 + H(I, I+2) \]

Call this \( \text{PROCAB. (AB for N, A+1)} \)

If \( Z = 5 \) \( I = 4 \) \( Z+I+1 = 10 \) outside range.

\[ R_{i, k+1} a_{k+1,i} k \]

\( k = 5 \) \( \text{outside range} \)

\( \text{PROCAB01E} \)
Balmer 1885. The q lines, the known in the spectrum of hydrogen.

Fitted

Wave length \( \lambda = \frac{1}{m^2 - 4} \)

\( m = 3, 4, 5, \ldots \)
Examples on Lagrange
in later parts
DP1, 2 Double pendulum
PS1 Pendula on sphere