STRINGS

A\$ + B\$ gives A\$ followed by B\$.

110 STRING\$(B% − A%, "0") + A\$
makes a string as long as B\$ by putting the appropriate numbers of zeros (B% − A%) in front of A\$.
B% = LEN B\$

MID\$(A\$, L%, 1) gives string consisting of one symbol, the L%th in A\$.
(L% tells where to look, 1 says how many to take)

VAL MID\$(A\$, L%, 1) takes the numerical value of this, if it is a number, which it is.

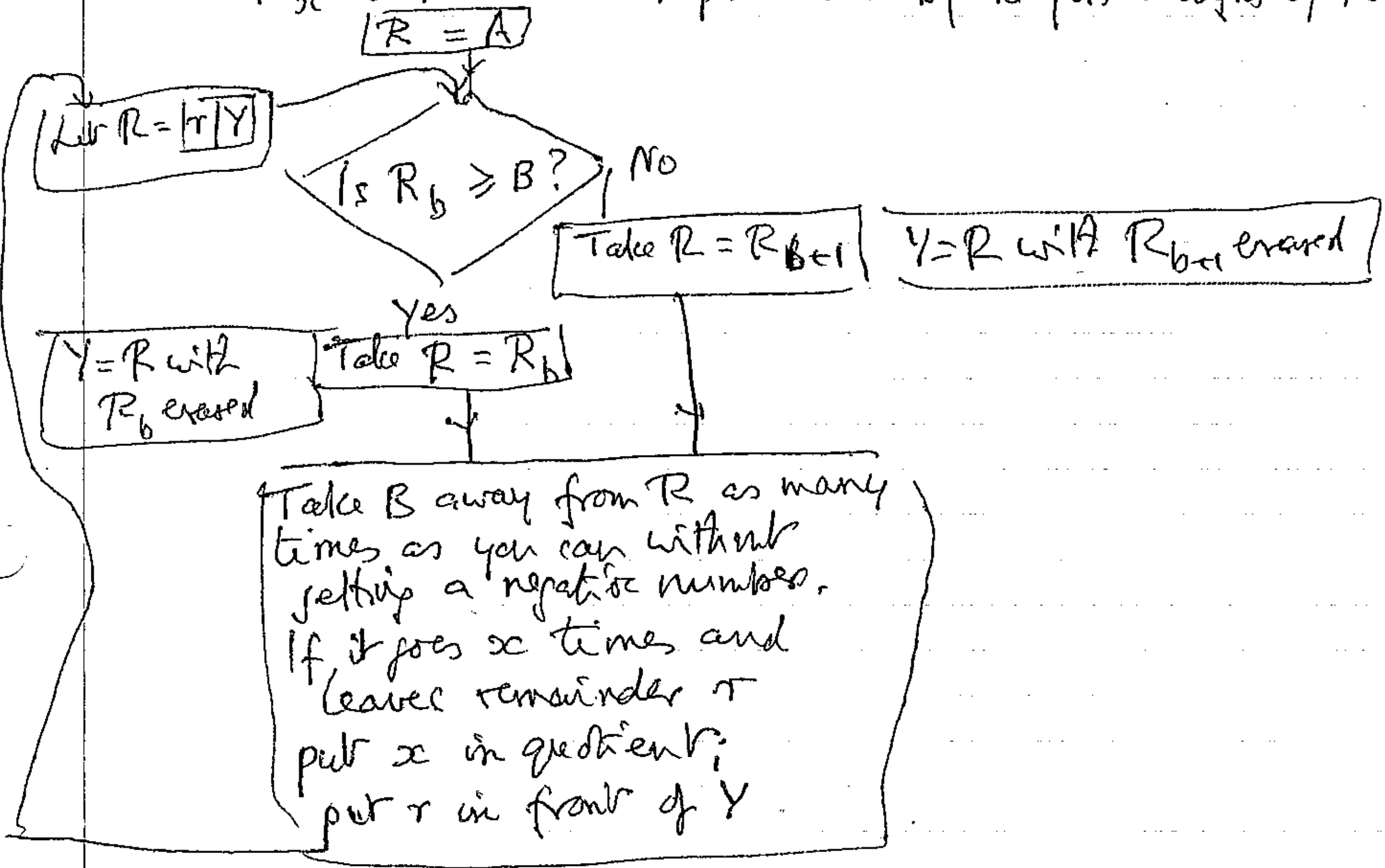
C% is the number carried. T% begins by being x + y + C%. The new C% is 1 if the sum ≥ 10.
Then T% → T% mod 10 the units digit in sum
160 R\$ → STR\$(T%) + R\$. It puts the T% just found in front of the numbers already found.

L% goes down to 1. Then if there is carry, 1 is put in front. 180, 190.

DEFADD. 70–200 DEFSUBTRACT 210–330
DEFFN strip. Takes initial zeros off. DEFFN multiply 410–570
~~DEFPROCdiv (a\$, b\$)~~ PROCDIV 820–940
PROCDIV rsdiv is subroutine in this 58–680
690–810 FNet. Purpose?
960–1090 √
1100–

The divisor is B with b digits.

R_x is the number represented by the first x digits of R



Osgood. Lehrbuch der Funktionentheorie
gives example of series that cannot be integrated term by term

$a_n(x) = S_n(x) - S_{n-1}(x)$.

$S_n(x) = nxe^{-nx^2}$.

He has given geometrical description of $S_n(x)$ earlier.

W. F. Osgood. Cambridge, Mass. 1928.

R. Courant Vorlesung über Differential- und Integralrechnung (Göttingen 1930)

VIII §3.

$f(x) = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \frac{x^2}{(1+x^2)^3} \cdots$

$x \neq 0$. $f(x) = \frac{x^2}{1-\frac{1}{1+x^2}} = \frac{x^2(1+x^2)}{x^2} = 1+x^2$

$x = 0$ $f(0) = 0$

 is given, with graph, VIII §4.

example 2

At any stage there is a remainder. If it is A7
123 456 789 987 654 321
This is recorded as
R(5) = 123 R(4) = 456 R(3) = 789
R(2) = 987 R(1) = 654 R(0) = 321.

The degree of R( ) is shown as N. Here N = 5.

It is sometimes necessary to use numbers larger than 1000, just as in ordinary long division we may need numbers bigger than 10. e.g. 2117 ÷ 23
2 lower degree than 23
21 same degree but less than 23.
211 . 9 × 23 + 4.

The divisor will always go into a number of higher degree. If the divisor is less than 1000 000 it will be sufficient to have a 9-figure number in a stage.

It is not satisfactory to carry out division by repeated subtraction. If at some stage we have

$$\begin{array}{r} 999 \\ \underline{1} \end{array}$$

this may come from 999 999 with 1 000 partial quotient 999, or from 999 999 / 1 999 with partial quotient 500.

Suppose 1000a + b is the first part of the dividend. However large the later digits, any number beginning 1000a + b + 1 will be larger than the dividend. Dividing by this will give us an underestimate of the partial quotient.

Now $$\frac{x}{1000a+b} - \frac{x}{1000a+b+1} = \frac{x}{(1000a+b)(1000a+b+1)}$$

As $x < 1000\,000$, this is certainly less than 1.
So it will be sufficient to divide by 1000a + b + 1, and then test whether the quotient may be increased by 1.

Let the number we are dividing by be 1000 B(1) + B(0) with degree M = 1, or B(0) with degree M = 0.

A9

At some stage we have to consider $X = 1000\,R(N) + R(N-1)$ which we have checked to be at least as large as $Y = 1000\,B(1) + R(0)$.

Let $Z = Y + 1$.

Let $C = \mathrm{INT}(X/Z)$

Continued fractions.

An expression such as $a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$

To express a number as a continued fractn.

Example, $52/23$. $\frac{52}{23} = 2\frac{6}{23}$ $1/\frac{6}{23} = \frac{23}{6}$

$\frac{23}{6} = 3\frac{5}{6}$ $1/\frac{5}{6} = \frac{6}{5}$

$\frac{6}{5} = 1\frac{1}{5}$

$\therefore \frac{52}{23} = 2 + \frac{1}{3+}\frac{1}{1+}\frac{1}{5}$.

More interesting, express $\sqrt{2}$ as a continued fraction.

$\sqrt{2} = 1 + (\sqrt{2}-1) = 1 + \frac{1}{\sqrt{2}+1}$

$\sqrt{2}+1 = 2 + (\sqrt{2}-1) = 2 + \frac{1}{\sqrt{2}+1}$

$\sqrt{2}+1 = 2 + (\sqrt{2}-1)$ From now on, we get 2 at each step.

$\sqrt{2} = 1 + \frac{1}{2+}\frac{1}{2+}\frac{1}{2+}$ ...... for ever.

It is of course necessary to check that the R.H.S. does converge to a definite limit.

We can break off, as 1, or $1+\frac{1}{2}$, or $1+\frac{1}{2+}\frac{1}{2}$ etc

$1+\frac{1}{2} = \frac{3}{2}$ $1 + \frac{1}{2\frac{1}{2}} = 1\frac{2}{5} = \frac{7}{5}$

Note $\left(\frac{3}{2}\right)^2 = \frac{9}{4} = 2\frac{1}{4}$ $\left(\frac{7}{5}\right)^2 = \frac{49}{25} = 2 - \frac{1}{25}$

The sequence continues $\frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \ldots$

$\left(\frac{17}{12}\right)^2 = \frac{289}{144} = 2\frac{1}{144}$; $\left(\frac{41}{29}\right)^2 = \frac{1681}{841} = 2 - \frac{1}{841}$

$\left(\frac{99}{70}\right)^2 = \frac{9801}{4900} = 2 + \frac{1}{4900}$

It seems that we get ever better approximations, alternately above and below.

Convergents.

If we break off $a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\frac{1}{a_4+}\cdots$ at various points we get

$$\frac{p_1}{q_1} = \frac{a_1}{1};\ \frac{p_2}{q_2} = \frac{a_1a_2+1}{a_2};\ \frac{p_3}{q_3} = \frac{a_1a_2a_3 + a_1 + a_3}{a_2a_3+1}$$

We observe $p_3 = a_3p_2 + p_1$

$q_3 = a_3q_2 + q_1$.

This suggests that perhaps

$$p_n = a_np_{n-1} + p_{n-2} \qquad (Ia)$$
$$q_n = a_nq_{n-1} + q_{n-2} \qquad (Ib)$$

Proof by induction This holds for $n = 3$. If it holds as far as $n = N$. p_{N+1}/q_{N+1} is formed by going as far as a_{N+1}. This means that we replace

$$a_1 + 1/a_2 + \cdots + 1/a_N$$

by

$$a_1 + \frac{1}{a_2} \qquad + \frac{1}{/a_N + \frac{1}{a_{N+1}}}$$

So we get p_{N+1}/q_{N+1} by changing a_N to $a_N + \frac{1}{a_{N+1}}$ in the equations for p_N/q_N.

i.e.

$$\frac{p_{N+1}}{q_{N+1}} = \frac{(a_N + \frac{1}{a_{N+1}})p_{N-1} + p_{N-2}}{(a_N + \frac{1}{a_{N+1}})q_{N-1} + q_{N-2}}$$

$$= \frac{a_Na_{N+1}p_{N-1} + p_{N-1} + a_{N+1}p_{N-2}}{a_Na_{N+1}q_{N-1} + q_{N-1} + a_{N+1}q_{N-2}}$$

$$= \frac{a_{N+1}(a_Np_{N-1} + p_{N-2}) + p_{N-1}}{a_{N+1}(a_Nq_{N-1} + q_{N-2}) + q_{N-1}}$$

$$= \frac{a_{N+1}p_N + p_{N-1}}{a_{N+1}q_N + q_{N-1}}$$

Thus, if the formula is correct up to $n = N$ it is correct up to $n = N+1$.

The convergents (i.e p_n/q_n) to $\sqrt{2}$ are

$$\frac{1}{1} \quad \frac{3}{2} \quad \frac{7}{5} \quad \frac{17}{12} \quad \frac{41}{29} \quad \frac{99}{70} \quad \ldots$$

$$-\frac{1}{1}+\frac{3}{2} = +\frac{1}{2} \ : -\frac{3}{2}+\frac{7}{5} = \frac{-1}{10} ; -\frac{7}{5}+\frac{17}{12} = \frac{+1}{60} :$$

$$-\frac{17}{12}+\frac{41}{29} = \frac{-1}{12\times 29} \qquad -\frac{41}{29}+\frac{99}{70} = \frac{+1}{29\times 70}$$

These results suggest (1) the numerator is always ± 1, (2) the denominator is the product of the denominators. i.e it is suggested $\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^n}{q_n q_{n-1}}$. (IIa)

~~Proof by induction Suppose this true as far as~~

This means $\quad p_n q_{n-1} - q_n p_{n-1} = (-1)^n$. (IIb)

Proof by Induction If this is true as far as $n = N$ the next expression is $p_{N+1} q_N - q_{N+1} p_N$

$$= (a_{N+1} p_N + p_{N-1}) q_N - (a_{N+1} q_N + q_{N-1}) p_N$$

$$= p_{N-1} q_N - p_N q_{N-1} = -(-1)^N = (-1)^{N+1}.$$

Corollary. The convergents, as found by the rule, are in their lowest terms (III)

For if p_n and q_n had a common factor, it would be a factor of $(-1)^n$.

The equation $\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^n}{q_n q_{n-1}}$ shows that, if n is even, $\frac{p_n}{q_n} > \frac{p_{n-1}}{q_{n-1}}$; if n is odd (IV)

$\frac{p_n}{q_n} < \frac{p_{n-1}}{q_{n-1}}$.

Corollary. If p, q are two integers with no common factor, we can find integers p', q' :- $pq' - p'q = 1$.

Proof. Express p/q as a continued fraction let p'/q' be the convergent before the final p/q.

It follows from Ia and Ib that $p_n > p_{n-1}$ and $q_n > q_{n-1}$ (V)

(VI) $p_n q_{n-2} - p_{n-2} q_n = (-1)^{n-1} a_n.$

Proof From I

$p_n q_{n-2} - p_{n-2} q_n = (a_n p_{n-1} + p_{n-2}) q_{n-2} - p_{n-2}(a_n q_{n-1} + q_{n-2})$

$= a_n (p_{n-1} q_{n-2} - p_{n-2} q_{n-1}) = a_n (-1)^{n-1}.$

(VII) It follows from (VI) that

$$\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = \frac{(-1)^{n-1} a_n}{q_n q_{n-2}}.$$

(VIII) Hence the even convergents continually decrease, while the odd convergents continually increase.

(IX) Hence even p_n/q_n decrease to a limit, the odd p_n/q_n increase to a limit.

(X) These limits are the same, for

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^n}{q_n q_{n-1}}$$ and RHS $\to 0$ by (V).

A complex variable $w = f(z)$, $w = u + iv$, $z = x + iy$ is called analytic in a region if at each point of the region the derived function $f'(z)$ exists, not depending on the direction of $dx : dy$.

For a simple treatment it is helpful to make certain assumptions about the continuity of the partial derivatives (these assumptions can be reduced).

We require then $du + i\,dv = f'(x+iy)(dx + i\,dy)$

Let $f'(x+iy) = p + iq$.

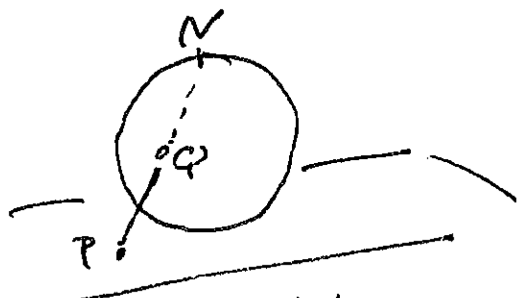
Then
$$du + i\,dv = (p+iq)(dx + i\,dy)$$
$$= p\,dx - q\,dy + i(q\,dx + p\,dy)$$
$$\therefore \quad du = p\,dx - q\,dy, \quad dv = q\,dx + p\,dy$$
$$\therefore \quad \frac{\partial u}{\partial x} = p \quad \frac{\partial u}{\partial y} = -q \quad \frac{\partial v}{\partial x} = q \quad \frac{\partial v}{\partial y} = p$$

So $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$: $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$ Cauchy-Riemann.

These are the equations satisfied by the potential and stream function for an incompressible flow in 2 dimensions with a velocity potential. Current in a copper sheet may be taken as an example of this.

The equations fail at point where current enters or leaves. These correspond to singularities.

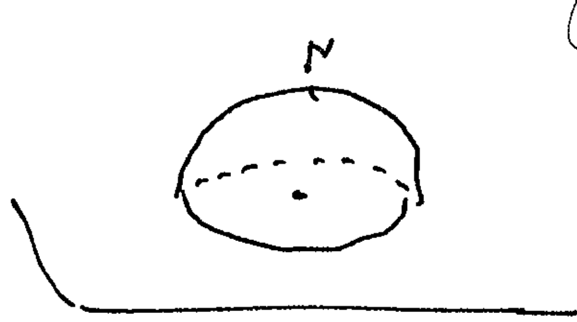
Stereographic projection.



If we suppose a sphere placed with its centre at or above the origin, and point P of the complex plane projected to the correspond[ing] point Q, it is found that we obtain u, v for an incompressible flow on the sphere. This removes the special status of ∞, which now is seen at the point N.

Circles in the plane project to circles on the sphere.

The equations for stereographic projection



Let $z=0$ be taken as the complex plane, $x^2+y^2+z^2=1$ as the sphere, $(1,0,0)$ as N.

$a+ib$ is represented by the point $(a,b,0)=P$. We need to find where the line NP meets the sphere.

The vector $\vec{NP}$ is $(a,b,-1)$. The point $N+t\,NP$ has coordinates $(at, bt, 1-t)$. It moves from N to P, in a straight line, as t goes from 0 to 1.

It is on the sphere when $a^2t^2+b^2t^2+(1-t)^2=1$

i.e. $t^2(a^2+b^2+1)-2t=0$, so $t=\dfrac{2}{a^2+b^2+1}$

The point is then $\dfrac{2a}{a^2+b^2+1}, \dfrac{2b}{a^2+b^2+1}, \dfrac{a^2+b^2-1}{a^2+b^2+1}$

Inverse points as seen on the sphere.

$$\frac{1}{a+ib}=\frac{a-ib}{a^2+b^2}=\alpha+i\beta \text{ say}$$

Thus $\alpha=\dfrac{a}{a^2+b^2}$ $\quad\beta=-\dfrac{b}{a^2+b^2}$

The point corresponding to $\alpha+i\beta$ on the sphere is

$$\frac{2\alpha}{\alpha^2+\beta^2+1}, \frac{2\beta}{\alpha^2+\beta^2+1}, \frac{\alpha^2+\beta^2-1}{\alpha^2+\beta^2+1}$$

$$\alpha^2+\beta^2=\left(\frac{a}{a^2+b^2}\right)^2+\left(\frac{-b}{a^2+b^2}\right)^2=\frac{1}{a^2+b^2}$$

Thus $\dfrac{2\alpha}{\alpha^2+\beta^2+1}=\dfrac{\frac{2a}{a^2+b^2}}{\frac{1}{a^2+b^2}+1}=\dfrac{2a}{a^2+b^2+1}$ same as before.

$\dfrac{2\beta}{\alpha^2+\beta^2+1}=\dfrac{-2b/(a^2+b^2)}{\frac{1}{a^2+b^2}+1}=\dfrac{-2b}{a^2+b^2+1}$, minus previous value.

$\dfrac{\alpha^2+\beta^2-1}{\alpha^2+\beta^2+1}=\dfrac{\frac{1}{a^2+b^2}-1}{\frac{1}{a^2+b^2}+1}=\dfrac{1-a^2-b^2}{a^2+b^2+1}=-$ previous value.

Thus, if $a+ib$ maps to (x,y,z) on sphere (C10) z

$\alpha + i\beta$ maps to $(x, -y, -z)$.

These are connected by a rotation of 180° about OX.

Circle of convergence.

If $f(z)$ is analytic in a region containing the origin, and $z=k$ is the nearest singularity, it is possible to expand $f(z)$ in an absolutely and uniformly convergent series $f(z) = \sum_0^\infty a_n z^n$ provided $|z| < k$, i.e. within a ~~cent~~ circle, centre O, with nearest singularity on circumference.

For example $f(z) = \frac{1}{1-z}$ has only one singularity $z=1$.

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots \qquad (\mathrm{I})$$

convergent for $|z| < 1$, divergent for $|z| > 1$.

If we want a series valid for $|z| > 1$, we put

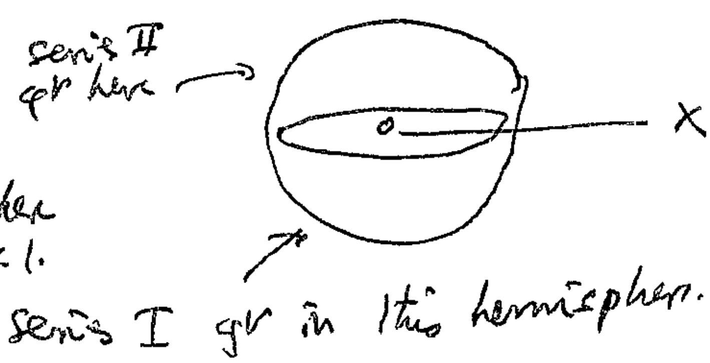
$$f(z) = \frac{1}{1-z} = \frac{-1}{z-1} = \frac{-1/z}{1-1/z}$$

$$= -\frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \dots\right]$$

$$= -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} \dots\dots \qquad (\mathrm{II})$$

If $\zeta = \frac{1}{z}$ $\quad f(z) = -\zeta - \zeta^2 - \zeta^3 \dots$

$\zeta = 0$ corresponds to $z = \infty$. We call the above series the series about $z = \infty$. It converges for $|\zeta| < 1$.



z is changed to ζ by a rotation of 180° about OX.

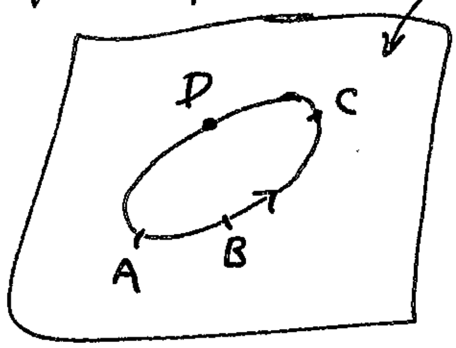
Thus the upper hemisphere counts as the circle $|\zeta| < 1$.

It corresponds to all the points outside $|z| = 1$

Thus we may think of the points outside a circle as being inside a circle, centre ∞.

An important property of integrals

f(z) analytic in this region



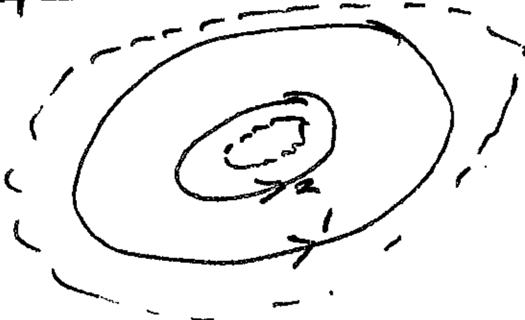
$$\oint f(z)\,dz = 0$$

for any closed path lying entirely in the region

$$0 = \oint = \int_{ABCDA} = \int_{ABC} + \int_{CDA} = \int_{ABC} - \int_{ADC}$$

$$\therefore \int_{ABC} f(z)\,dz = \int_{ADC} f(z)\,dz$$

The value of the integral is not altered if the path is deformed, the end points staying the same, it is essential that the path does not pass over any singularity.



If $f(z)$ is analytic in the region between the two dotted curves, then

$$\int_{C_1} = \int_{C_2}.$$

Proof. Consider $\int$

If $f(z)$ ~~resembles~~ $\sim \frac{k}{z}$ near $z=0$, for C, a circle around origin,

$$\int_C f(z)\,dz \sim \int_C \frac{k}{z}\,dz$$

$$= \int_0^{2\pi} \frac{k\, i r e^{i\theta}\, d\theta}{r e^{i\theta}} = k \cdot 2\pi i$$

Moment functions

$$f(z) = \int^{b} \frac{w(t)\,dt}{z-t} \quad \text{where } w(t) \text{ is real}, \geq 0$$

is called a moment function.

$$f(z) = \int_a^b \frac{w(t)\,dt}{z(1-\frac{t}{z})} = \int_a^b \frac{w(t)\,dt}{z}\left(1+\frac{t}{z}+\frac{t^2}{z^2}+\dots\right)$$

$$= \int_a^b w(t)\,dt\left(\frac{1}{z}+\frac{t}{z}+\frac{t^2}{z^2}+\dots\right)$$

Coefficient of $\frac{1}{z}$ is $\int_a^b w(t)\,dt$, the total mass

of $\frac{1}{z^2}$ is $\int_a^b t\,w(t)\,dt$ the moment about origin

of $\frac{t}{z^3}$ is $\int_a^b t^2 w(t)\,dt$ moment of inertia about origin

$\int_a^b t^n w(t)\,dt$ is known as the nth moment.

Analytic nature of $f(z)$

(1) If z is any point not on the interval $[a,b]$, $f(z)$ is analytic at z.

Proof $$f'(z) = \frac{d}{dz}\int_a^b \frac{w(t)\,dt}{z-t}$$

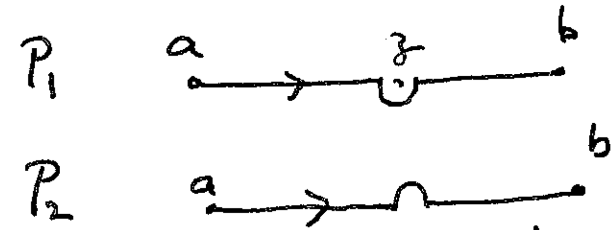
$$= \int_a^b w(t)\,dt \cdot \frac{\partial}{\partial z}(z-t)$$

$$= \int_a^b \frac{-w(t)}{(z-t)^2}\,dt.$$

As z is not on $[a,b]$, $|z-t|$, the distance of z from point t, has a minimum greater than zero, so $\frac{1}{(z-t)^2}$ will be finite for $0 \leq t \leq 1$, and the integral will be satisfactory.

There will be a discontinuity across the cut (a,b). If z is real, $0 < z < t$, we shall get different values for $f(z)$ depending on whether we approach z from below or above.

If z is above the real axis, displacing the path to P_1 will not cause the integrand $\dfrac{w(t)}{z-t}$ to pass over any singularity, since $z-t \neq 0$ in this region.



Thus $f_+(z) = \displaystyle\int_{P_1} \frac{w(t)\,dt}{z-t}\,dt$, where $f_+(z)$ is the value of $f(z)$ found by continuation from above.

Similarly $f_-(z) = \displaystyle\int_{P_2} \frac{w(t)\,dt}{z-t}$

Now the difference of these is integral around 

$$\text{So } f_+(z) - f_-(z) = \int_C \frac{w(t)\,dt}{z-t} = \int_C \frac{-w(t)\,dt}{t-z}$$

$$f_+(z) - f_-(z) = -2\pi i\, w(z)$$

the radius of the circle having tended to 0.

<u>Note</u> The above proof uses $w(t)$ being analytic. The theorem holds true in many cases where this is not so, for instance if $w(t)$ has jumps such as

This theorem is used if we know $f(z)$ and want to find $w(t)$ to give it.

C14

In the course of my work I came across the function $\varphi(a) = \int_0^1 \frac{-\log v \, dv}{a+v}$

$-\ln v > 0$ for $0 < v < 1$.

Suppose $a > 0$. $\int_0^1 \frac{-\ln v \, dv}{a+v} < \frac{1}{a}\int_0^1 -\ln v \, dv$

$= -\frac{1}{a}\left[v \ln v - v \right]_0^1$

$v \ln v \to 0$ as $v \to 0$, so this $= \frac{1}{a}$.

Thus $\varphi(a)$ is finite when $a > 0$.

If $a = 0$ $\int_0^1 \frac{-\ln v \, dv}{v} = -\frac{1}{2}(\ln v)^2 \Big]_0^1 = +\infty$.

$\varphi(a) \to +\infty$ as $a \to 0$.

If $\underline{a}$ increases, clearly $\frac{1}{a+v}$ decreases.

Hence $\varphi(a)$ decreases steadily as $\underline{a}$ goes from 0 to $+\infty$.

$$\frac{1}{a+v} = \frac{1}{a(1+\frac{v}{a})} = \frac{1}{a}\left(1 - \frac{v}{a} + \frac{v^2}{a^2} - \frac{v^3}{a^3} \cdots\right)$$

$$\varphi(a) = \int_0^1 -\ln v . \sum_0^\infty \frac{(-1)^n v^n}{a^{n+1}}$$

$$\int_0^1 -v^n \ln v \, dv = \left[\frac{-v^{n+1}\ln v}{n+1}\right]_0^1 + \int_0^1 \frac{v^{n+1}}{n+1}\frac{1}{v}dv$$

$[\ ]_0^1 = 0$ $\quad \int_0^1 \frac{v^{n+1}}{n+1}\cdot\frac{1}{v}dv = \int_0^1 \frac{v^n \, dv}{n+1}$

$$= \frac{v^{n+1}}{(n+1)^2}\Big]_0^1 = \frac{1}{(n+1)^2}$$

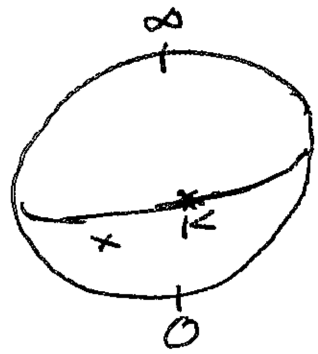
$$\text{Thus } \varphi(a) = \sum_0^\infty \frac{(-1)^n}{(n+1)^2 a^{n+1}} \qquad (1)$$

This series converges for $|a| > 1$, diverges if $|a| < 1$.

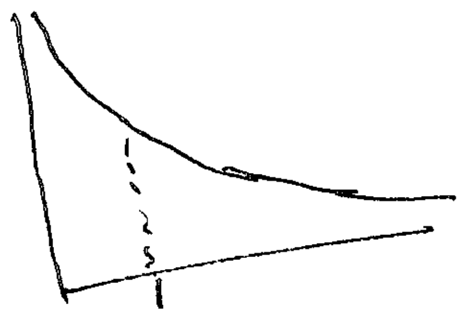
It diverges everywhere inside the circle $|a| = 1$ and converges everywhere outside it. This indicates that there must be a singularity somewhere on $|a| = 1$.

For if k is the singularity furthest from the origin, the series about ∞ converges for $|z| > |k|$.

If $|k|$ were less than 1, the region of convergence of series (1) would have some part inside $|a| = 1$.



The graph of $\varphi(a)$ for $a > 0$ is



There is no indication of a singularity at $a = 1$. The singularity must be elsewhere.

<u>Series valid for $0 < a < 1$.</u>

$$\varphi(a) = \int_0^1 \frac{-\ln v \, dv}{a + v} \quad \text{let } v = au$$

$$\varphi(a) = \int_0^{\frac{1}{a}} \frac{-\ln(au) \cdot a \, du}{a + au} = \int_0^{\frac{1}{a}} \frac{-\ln a - \ln u}{1 + u} \, du$$

$$\int_0^{\frac{1}{a}} \frac{du}{1+u} = \Big[\ln(1+u)\Big]_0^{\frac{1}{a}} = \ln\left(1 + \frac{1}{a}\right)$$

Thus part of $\varphi(a)$ is $\underline{\underline{-\ln a \ln\left(1 + \frac{1}{a}\right)}}$ (α)

$$\int_0^{1/a} \frac{-\ln u \, du}{1+u} = \int_0^1 \frac{-\ln u \, du}{1+u} + \int_1^{1/a} \frac{-\ln u \, du}{1+u}.$$

$$\int_0^1 \frac{-\ln u \, du}{1+u} = \int_0^1 -\ln u \cdot \sum_0^\infty (-u)^n \, du = \sum_0^\infty \frac{(-1)^n}{(n+1)^2} \quad (\beta)$$

9

016

In $\int_1^{1/a} -\frac{\ln u\,du}{1+u}$ put $u = \frac{1}{w}$.

$$\int = \int_{w=1}^{a} \frac{\ln w}{1+\frac{1}{w}}\left(\frac{-dw}{w^2}\right) = \int_a^1 \frac{\ln w\,dw}{w(1+w)}$$

$$= \int_a^1 \ln w\left[\frac{1}{w} - \frac{1}{1+w}\right]dw$$

$$\cancel{\int_a^1 \frac{\ln w\,dw}{1+w}} = \int_a^1 \frac{\ln w\,dw}{w} = \left[\tfrac{1}{2}(\ln w)^2\right]_a^1 = -\tfrac{1}{2}(\ln a)^2 \quad (\gamma)$$

Finally $$\int_a^1 \frac{-\ln w\,dw}{1+w} = \int_a^1 -\ln w \sum_0^\infty (-w)^n\,dw$$

$$= \sum_0^\infty (-1)^{n+1}\int_a^1 w^n \ln w\,dw$$

$$= \sum_0^\infty (-1)^{n+1}\left[\frac{w^{n+1}}{n+1}\ln w - \frac{w^{n+1}}{(n+1)^2}\right]_a^1$$

$$= \sum_0^\infty (-1)^{n+1}\left\{-\frac{a^{n+1}\ln a}{n+1} - \frac{1-a^{n+1}}{(n+1)^2}\right\}$$

$$= \ln a \sum_0^\infty \frac{(-1)^n a^{n+1}}{n+1} + \sum_0^\infty \frac{(-1)^n(1-a^{n+1})}{(n+1)^2}$$

$$= \ln a \ln(1+a) + \sum_0^\infty \frac{(-1)^n}{(n+1)^2} + \sum_0^\infty \frac{(-a)^{n+1}}{(n+1)^2} \quad (\delta)$$

In (α) we have $-\ln a \ln(1+a) + (\ln a)^2$.
After cancelling we find

$$\varphi(a) = \tfrac{1}{2}(\ln a)^2 + 2\sum \frac{(-1)^n}{(n+1)^2} + \sum \frac{(-a)^{n+1}}{(n+1)^2}$$

For $|a|>1$ $\quad \varphi(a) = \sum_0^\infty \frac{(-1)^n}{(n+1)^2 a^{n+1}}$

How does this series behave if we place $\underline{a}$ on the unit circle?

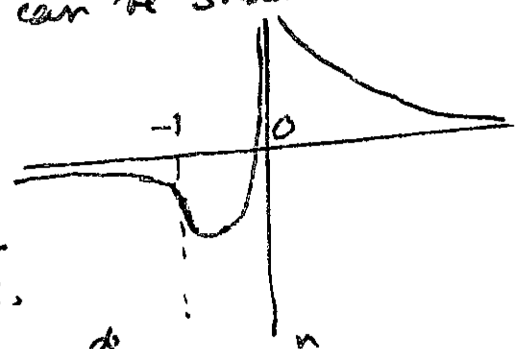
There is a theorem that, if $\sum c_n$ is absolutely convergent (i.e. $\sum |c_n|$ convergent) then $\sum c_n$ is convergent.

The absolute series corresponding to $\varphi(a)$ is
$$\sum_0^\infty \frac{1}{(n+1)^2 |a|^{n+1}}.$$

If $\underline{a}$ is on unit circle, $|a|=1$, and we have $\sum_0^\infty \frac{1}{(n+1)^2}$, which is convergent. It is in fact $\pi^2/6$. So $\varphi(a)$ is convergent everywhere on $|a|=1$. How then does it manage to have a singularity on $|a|=1$?

The clue was given by the graph for $a<0$.

The graph of $\varphi(a)$ can be shown to be,



and the tangent at $a=-1$ is vertical.

In fact $\varphi(a) = \sum_0^\infty \frac{(-1)^n}{(n+1)^2 a^{n+1}}$

$$\varphi'(a) = \sum_0^\infty \frac{(-1)^n}{(n+1)^2}\left[-(n+1)a^{-n-2}\right]$$
$$= \sum_0^\infty \frac{(-1)^{n+1}}{(n+1)a^{n+2}}$$

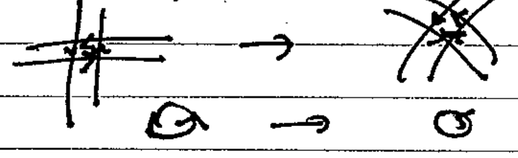
$$\log\left(1+\frac{1}{a}\right) = \frac{1}{a} - \frac{1}{2a^2} + \frac{1}{3a^3} - \frac{1}{4a^4} \cdots$$
$$= \sum_0^\infty \frac{(-1)^n}{(n+1)a^{n+1}}$$

$\therefore \varphi'(a) = \frac{-1}{a}\log\left(1+\frac{1}{a}\right)$ which is $-\infty$ for $a=-1$.

We are concerned with functions, $\mathbb{C} \to \mathbb{C}$, for which $f'(z)$ exists, i.e for $w = f(z)$, $\frac{dw}{dz} = f'(z)$

Consider a particular point z, and variations dz from it. Then $dw = f'(z)\,dz$. Here $f'(z)$ is some particular complex number. Multiplication by it produces a rotation and change of scale, i.e it leaves angles and ratios unchanged. Such a transformation is called <u>conformal</u>. As $dw = f'(z)\,dz$ holds only for differentials, this means that the geometry is preserved only in the limit as we considered neighbourhoods of z and w whose size $\to 0$. This in particular means that the angle between the tangents to two curves, at a place where they cross, is not altered.

Note that multiplication produces a <u>turning</u>: it cannot produce a <u>turning over</u>. Thus, if $z = x + iy$, $w = x - iy$ the function $z \to w$ cannot be analytic.

Always  direction of rotation preserved

Transformation, $w = \frac{az+b}{cz+d}$, maps conformally 1-1 the entire plane z to the entire plane w. It preserves the essential behaviour of functions at each point, and can give a very useful way of putting some situation in an easily understood form.

We suppose, of course, $ad - bc \neq 0$.

Such a transformation can be the combined effect of simpler transformations. For example

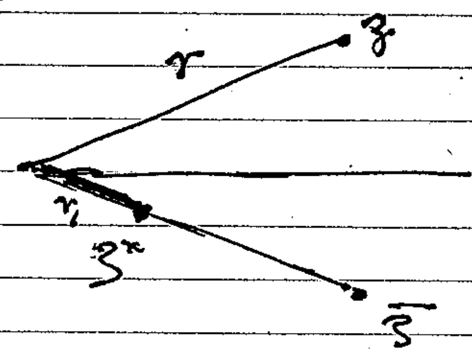
$$\frac{12z+25}{3z+6} = 4 + \frac{1}{3z+6} = 4 + \frac{1}{3(z+2)}$$

Thus $z^* = \dfrac{12z+25}{3z+6}$ can be achieved by the following sequence of steps.

$$z_1 = z + 2 \qquad (1)$$
$$z_2 = 3z_1 \qquad (2)$$
$$z_3 = 1/z_2 \qquad (3)$$
$$z^* = z_3 + 4. \qquad (4)$$

Such a decomposition is always possible.
(1) and (4) are simply translations.
(2) is a change of scale.
(3) is a new kind of transformation.

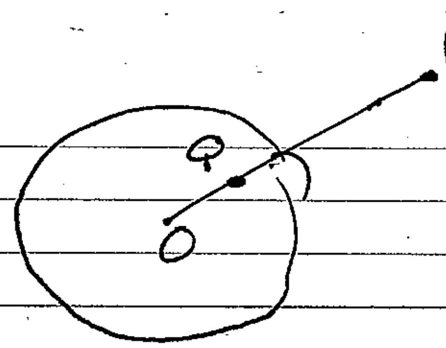
If $z^* = \dfrac{1}{z}$ and z is (r, θ) then z^* is $(\frac{1}{r}, -\theta)$



If $z = x+iy$, $\bar{z} = x - iy$, reflection in the real axis. $\bar{z}$ is at distance r from the origin, z^* at distance $\frac{1}{r}$.

$\bar{z} \to z^*$ is known as <u>inversion in the unit circle.</u>

Given any circle, radius R, centre O, inversion in it sends P to Q where OPQ is straight and $OP.OQ = R^2$.

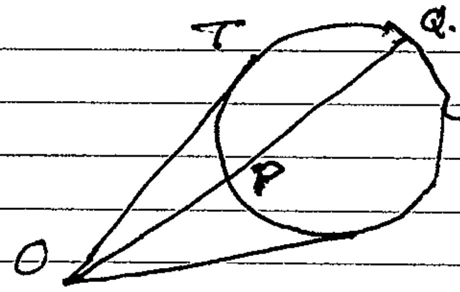


This operation has much in common with reflection. If $P \to Q$, then $Q \to P$. If the radius R becomes very large, it could be mistaken for reflection.

It reverses sense of rotation.

$z^* = 1/z$ is analytic: it is equivalent to 2 operations, each of which reverses sense of rotation. It could not be equivalent to a single such operation.

<u>Theorem</u> If we have a circle C, and if OT is a tangent to it, inversion of C in the circle, centre O, radius OT, makes $C \to C$.



<u>Proof</u> $OP.OQ = OT^2$ is a known theorem.

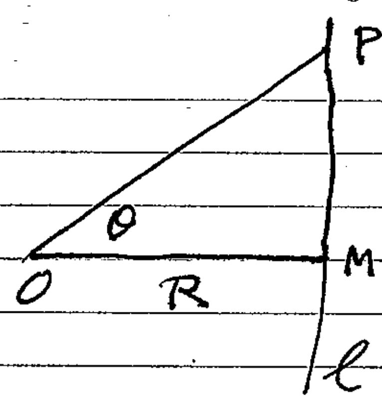
Changing the radius of the circle of inversion only produces a change of scale, so C inverted in any circle, centre O, goes to a circle.

C4

C21

Problem l is a line at perpendicular distance R from a point O. What do we get if we invert l in the circle centre O, radius R?

Suggestion: Find OQ when $\angle POM$ is θ



(C22)

$$w = \frac{az+b}{cz+d} \quad \text{where } ad-bc \neq 0.$$

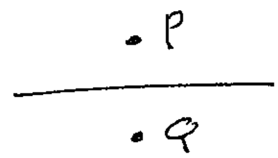
Just one value of w corresponds to a given value of z.

$cwz + wd = az + b \quad \therefore \; z = \dfrac{wd-b}{-cw+a}.$

This gives a single value for z when w is known
[Question. What goes wrong with this argument if
$ad - bc = 0$?]

"Kreisverwandschaft."

<u>Inversion</u> Reflection in a line is

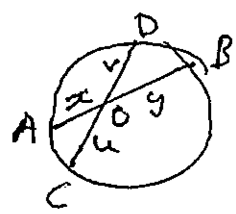


$P \to Q$, $Q \to P$

Inversion may be thought of as reflection in a circle.

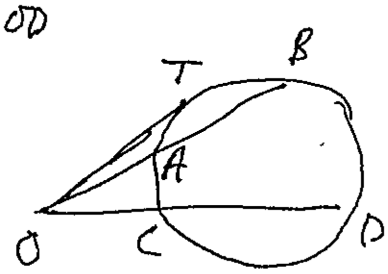
$OP.OQ = R^2.$

Important property of a circle
$x.y = u.v$
(lengths)



$OA.OB = OC.OD$

This still holds if O is outside the circle.
For the tangent OT,
$A \to T$, $B \to T$.



So $OA.OB = OC.OD = OT^2.$

If we invert with respect to the circle, centre O, radius OT $\quad A \to B$, $B \to A$

So the circle TBDCA reflects to itself.

If we invert with respect to a circle, centre O, but a different radius, we have similar effect with a change of scale: the circle goes to a different circle.

There is an exception, if O lies on the circle BDCA.

1) $f(z) = f(x+iy)$ is analytic in a region if there is a continuous derivative $f'(z)$, independent of direction in which z is changing.

$$df(z) = f'(z)\,dz.$$

Let $f(x+iy) = u+iv$. $\quad f'(z) = p+iq$

$$du + i\,dv = (p+iq)(dx+i\,dy)$$
$$= p\,dx - q\,dy + i(q\,dx + p\,dy)$$

$$du = p\,dx - q\,dy \quad \therefore \frac{\partial u}{\partial x} = p \quad \frac{\partial u}{\partial y} = -q.$$

$$dv = q\,dx + p\,dy \quad \therefore \frac{\partial v}{\partial x} = q \quad \frac{\partial v}{\partial y} = p.$$

Thus $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$. Cauchy Riemann.

These derivatives must exist and be continuous.

2) Geometrical meaning of Cauchy Riemann equations.

Suppose a point (x,y) starts at $(x_0\ y_0)$ and moves a distance s at angle θ to OX.

Then $x = x_0 + s\cos\theta \qquad y = y_0 + s\cdot\sin\theta$

Rate of change of $u(x,y)$ w.r.t. s is then

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta.$$

Suppose v changes as a result of x, y, moving in direction $\theta + \frac{\pi}{2}$. $\quad \cos(\theta+\frac{\pi}{2}) = -\sin\theta$, $\sin(\theta+\frac{\pi}{2}) = \cos\theta$

$$\frac{dv}{ds} = -\frac{\partial v}{\partial x}\sin\theta + \frac{\partial v}{\partial y}\cos\theta$$
$$= \frac{\partial u}{\partial y}\sin\theta + \frac{\partial u}{\partial x}\cos\theta \quad \text{if C-R hold.}$$

Thus rate of change of u in direction θ
= rate of change of v in direction $\theta + \frac{\pi}{2}$.

In particular, if u is constant for $\angle\theta$, v is constant for $\angle\theta + \frac{\pi}{2}$. The curves $u = c$, $v = k$ cross at right angles.

Integration in 2 dimensions.

To define $\int X(x,y)\,dx + Y(x,y)\,dy$ along a given curve, we suppose the curve given by $x = x(t)$, $y = y(t)$ $0 \le t \le 1$.

Then $X(x,y) = X(x(t), y(t))$ a fn of t.

$$dx = \frac{dx}{dt}\,dt = x'(t)\,dt.$$

We define the integral as

$$\int_0^1 X(x(t),y(t))x'(t) + Y(x(t),y(t))y'(t)\,dt.$$

Suppose for example we require $\int y\,dx + 2x\,dy$ along the line joining $(1,1)$ to $(3,4)$.

$x = 1+2t$, $y = 1+3t$ takes x, y from $(1,1)$ to $(3,4)$ as t goes from 0 to 1.

$$\int = \int_0^1 (1+3t)2 + (2+4t)3 \,.\, dt$$

$$= \int_0^1 8 + 18t\,dt = 8t+9t^2\Big]_0^1 = 17.$$

We would get a different result if we went from $(1,1)$ to $(3,1)$ and then $(3,1)$ to $(3,4)$

In first part $x = 1+2t$, $y = 1$ $x' = 2$, $y' = 0$

We have $\int_0^1 2 + 4t\,dt = 2t + 2t^2\Big]_0^1 = 4.$

In the second part $x = 3$, $y = 1+3t$. $x' = 0$ $y' = 3$

$$\int_0^1 6.3\,dt = 18t\Big]_0^1 = 18. \qquad \int = 22.$$

$\int X\,dx + Y\,dy$ is an expression for work done. This example corresponds to a non conservative system. Work could be obtained from it by following an appropriate loop.

The result will be independent of the path if $X\frac{dx}{dt} + Y\frac{dy}{dt} \equiv \frac{d}{dt}f(x,y)$ for some $f(x,y)$ For $\int_0^1 \frac{d}{dt} f(x,y)\,dt = f(x_1,y_1) - f(x_0,y_0)$.

Now $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

This case arises if there exists $f(x,y)$:-

$X = \frac{\partial f}{\partial x} \quad Y = \frac{\partial f}{\partial y}$.

We suppose X and Y have continuous derivatives.

Then $\frac{\partial}{\partial y}X = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\frac{\partial}{\partial y}f = \frac{\partial}{\partial x}Y$

$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}$ is the usual condition for $X\,dx + Y\,dy$ to be an exact differential.

[Psaggio. Appendix A. $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ if f is continuous, and has continuous first and second derivatives.]

If $\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}$ we can show that there is an f with the required property. $\int X\,dx + Y\,dy$ is then independent of the path between given end points, provided $\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}$ holds throughout the region. Stokes' theorem.

$$\oint X\,dx + Y\,dy = \iint \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\,.\,dx\,dy.$$

If $f(x+iy)$ satisfies the Cauchy-Riemann conditions, $\int_a^b f(z)dz$ does not depend on the path connecting given a, b.

Proof. Let $f(x+iy) = u + iv$.

$$\int (u+iv)(dx+idy) = \int u dx - v dy + i\int v dx + u dy.$$

$u dx - v dy$ is exact if $\frac{\partial}{\partial y}(u) = \frac{\partial}{\partial x}(-v)$

i.e. $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ This is a C-R condition.

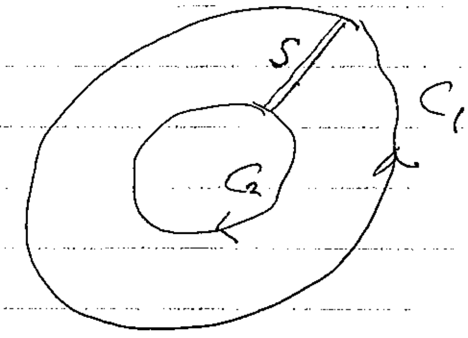
$v dx + u dy$ is exact if $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$, the other C-R condition. So we have

Cauchy's Theorem.

If $f(z)$ is analytic in some region, $\oint f(z)dz = 0$ for any smooth closed curve in that region, and for any curve consisting of a finite number of smooth curves.

Corollary If C_1 and C_2 are two closed curves, C_1 inside C_2, and a region in which $f(z)$ is analytic contains C_1 and C_2.

then $\oint_{C_1} = \oint_{C_2}$



Proof Consider region bounded by C_2, C_1 and strip S.

This is bounded by C_1, S, C_2, S. The two parts on S cancel and we have $\int_{C_1 - C_2} f(z)dz = 0$.

Residue Theorem. $f(z)$ is analytic on a region containing curve C. A point a is inside C.

Consider $\oint_C \frac{f(z)\,dz}{z-a}$. By previous theorem this equals ~~val~~ value for a small circle, centre a.

On this circle $f(z) \doteq f(a)$. In the limit we have $f(a)\oint \frac{dz}{z-a}$, integral around small circle.

On the small circle $z = a + re^{i\theta}$ where r is radius of circle, $0 \le \theta \le 2\pi$.

$$dz = re^{i\theta} i\,d\theta.$$

$$\oint \frac{dz}{z-a} = \int_0^{2\pi} \frac{re^{i\theta} i\,d\theta}{re^{i\theta}} = \int_0^{2\pi} i\,d\theta = 2\pi i$$

$$\therefore \quad 2\pi i\, f(a) = \oint_C \frac{f(z)\,dz}{z-a}$$

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)\,dz}{z-a}$$

Given $f(z)$ on the boundary, this gives an explicit solution for values of $f(z)$ inside the curve.

C28

$I = \int x^{-3/2}(1-x)^{-3/2}\,dx$ $\quad$ let $y^2 = x - x^2$

If $y = tx$ $\quad t^2x^2 = x - x^2$

$x = \frac{1}{1+t^2}$

$1-x = \frac{t^2}{1+t^2}$

$dx = \frac{-2t}{(1+t^2)^2}$

$t^2 = \frac{x}{1-x}$

$t = \sqrt{\frac{x}{1-x}}$

$$I = \int \left[\frac{t^2}{(1+t^2)^2}\right]^{-3/2} \frac{-2t}{(1+t^2)^2}\,dt$$

$$= \int \frac{(1+t^2)^3}{t^3}\,\frac{-2t}{(1+t^2)^2}\,dt$$

$$= -2\int \frac{1+t^2}{t^2}\,dt$$

$$= -2\left[t - \frac{1}{t}\right]$$

$$= 2\left[\sqrt{\frac{1-x}{x}} - \sqrt{\frac{x}{1-x}}\right]$$

$$= \frac{2}{\sqrt{x(1-x)}}\left[1-x-x\right]$$

$$= \frac{2(1-2x)}{\sqrt{x(1-x)}}$$

Check $\frac{d}{dx}(2-4x)x^{-1/2}(1-x)^{-\frac{1}{2}}$

$= \cancel{-4x^{\frac{1}{2}}}$ $\ln I = \ln 2 + \ln(1-2x) - \frac{1}{2}\ln x - \frac{1}{2}\ln(1-x)$

$$\frac{I'}{I} = \frac{-2}{1-2x} - \frac{1}{2x} + \frac{1}{2(1-x)}$$

$$= \frac{-4x(1-x) - (1-x)(1-2x) + x(1-2x)}{2x(1-x)(1-2x)}$$

Numer $= \quad -4x + 4x^2$

$\quad -1 + 3x - 2x^2 \quad = 1.$

$\quad\quad + x - 2x^2$

$$I' = \frac{I}{2x(1-x)(1-2x)} = \frac{\cancel{2}\cancel{(1-2x)}}{\sqrt{x(1-x)}}\cdot\frac{1}{\cancel{2}x(1-x)\cancel{(1-2x)}}$$

$$= \frac{1}{\sqrt{x^3(1-x)^3}}$$

$$w(z) = \frac{2(1-2z)}{\sqrt{z(1-z)}}$$

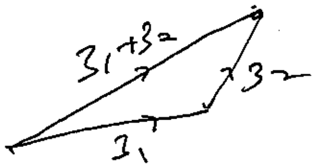
If z is real between 0 and 1

w is real and falls from $+\infty$ to $-\infty$.

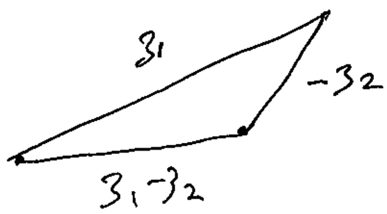
z real, > 1. z is pure imaginary.

z real < 0 z is pure imaginary

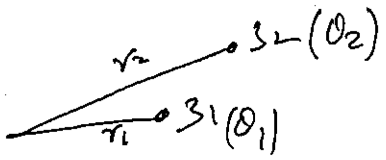
$|z|$ is length of OP, P being point that represents z.



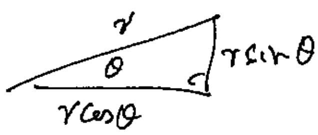
$|z_1+z_2| \leq |z_1|+|z_2|$



$$\begin{matrix}|z_1|-|z_2| \\ |z_2|-|z_1|\end{matrix} \leq |z_1-z_2| \leq |z_1|+|z_2|$$



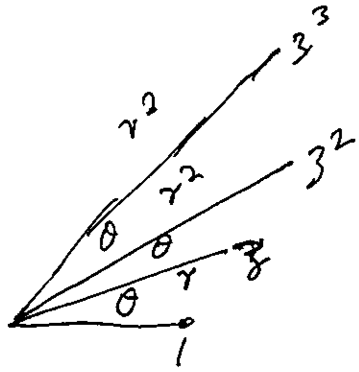
z_1z_2 $\quad r_1r_2 : \theta_1+\theta_2$



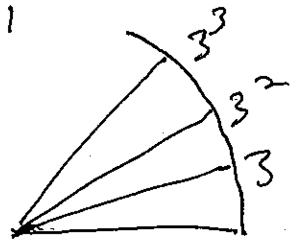
$r(\cos\theta + i\sin\theta)$ is number at r, θ.

$$r_1(\cos\alpha + i\sin\alpha) \cdot r_2(\cos\beta + i\sin\beta)$$
$$= r_1r_2\left[(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)\right]$$
$$= r_1r_2\left[\cos(\alpha+\beta) + i\sin(\alpha+\beta)\right]$$

Powers of a number



Special case $r=1$



Root extraction.

$z \quad r, \theta \qquad z^n \quad r^n, n\theta$

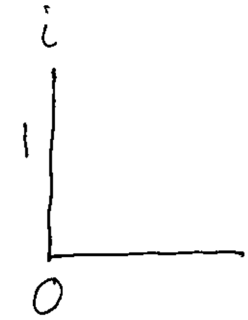
If $z^n = w$ where w is at R, Θ

$r^n = R \qquad n\theta = \Theta$

$r = \sqrt[n]{R} \qquad \theta = \Theta / n$

Particularly simple case, $R = 1$. Then $r = 1$.

Find $\sqrt{i}$.



$\sqrt{i}$ will be at distance 1, angle $\pi/4$ (45°)

$$\sqrt{i} = \cos 45^\circ + i \sin 45^\circ = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \frac{1+i}{\sqrt{2}}$$

Check $\left(\frac{1+i}{\sqrt{2}}\right)^2 = \frac{1-1+2i}{2} = i$.

Above we gave one value for n^{th} root of R, Θ

There should be n roots, as eqn $z^n = w$ is of degree n.

R, Θ is the same point as $R, \Theta + 2s\pi$

So $\theta = \frac{\Theta + 2s\pi}{n}$ will do

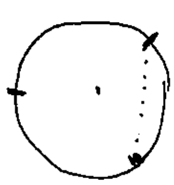
$$= \frac{\Theta}{n} + s\left(\frac{2\pi}{n}\right)$$

$s = 0, 1, 2 \ldots n-1$ give different positions.

$z^3 = -1$. -1 is at distance 1, angle π

So z is at $1, \frac{\pi + 2s\pi}{3}$

$$\frac{180^\circ + 2s.180^\circ}{3} = 60^\circ + s.120^\circ$$



Application to Differential Equations.

$$\frac{d^2y}{dx^2} + y = 0$$

Stock method. Try $y = e^{mx}$. Then $\frac{d^2y}{dx^2} = m^2 e^{mx}$

so $m^2 + 1 = 0$

$m = i$ or $-i$

Solutions e^{ix}, e^{-ix}

$e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

General solution

$$y = P(\cos x + i\sin x) + Q(\cos x - i\sin x)$$

$$= (P+Q)\cos x + i(P-Q)\sin x$$

$$= A\cos x + B\sin x.$$

If $w^2 = z$, there are two values of w corresponding to each value of z. Sometimes we can write $w = \pm\sqrt{z}$, when we do not care particularly which one is meant. There are other times when it is important to know which root we are dealing with.

Square roots are particularly simple in polar coordinates. If z is at (r, θ), $\sqrt{z}$ is at $(\sqrt{r}, \theta/2)$. This gives 2 solutions, because adding 360° to the angle for z makes no difference, but it adds 180° for w, which changes by a factor of -1. If we take $r = 1$, the diagrams are simple.

0° to 360°

360° to 720°

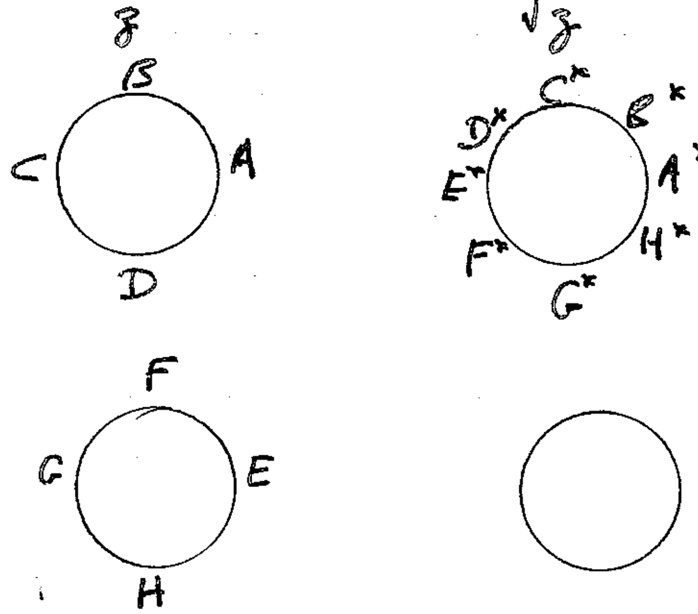


Fig 1

As we go round the circle for z, we start at A for $z = 1$, with $w = \sqrt{z}$ at A^*, $\sqrt{z} = 1$. As we continue around the circle, $\sqrt{z}$ always changing continuously, we get to E, $z = 1$, and $w = \sqrt{z}$ is now at E^*, -1.

Thus the z values proceed as

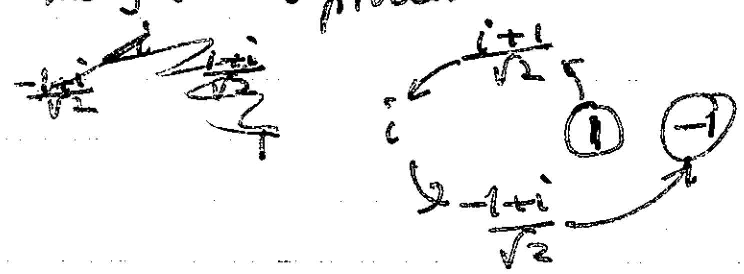


Fig 2

If z goes round the unit circle, with $\sqrt{z}$ changing in a continuous manner (this is an essential point of the theory), the effect of a complete circuit is to change one value of $\sqrt{z}$ to the other.

Riemann introduced the important idea of the Riemann surface, each point of which can be fixed by the values of z and w. Further, a small change in z and w must lead to a point not far away.

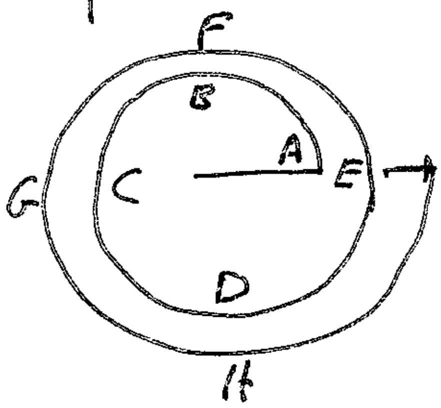


Fig 3

If we arrange points by their z-values, we obtain the curious shape in Figure 3. As a rotation of 720° brings us back to $w = +1$, the point A' in Fig. 3 must coincide with A. This seems strange. The point is that $z = x+iy$, $w = u+iv$ involves 4 numbers (x, y, u, v) and so that diagram should be in 4 dimensions, and naturally when we try to show this in 3 dimensions, we run into difficulties. Note that these do not appear if we arrange things corresponding to $\sqrt{z}$ as in the 4th diagram of Figure 1.

$z=-1, w=i$

$z=-i, w=(-1+i)/\sqrt{2}$ $\quad$ $z=i, w=(1+i)/\sqrt{2}$

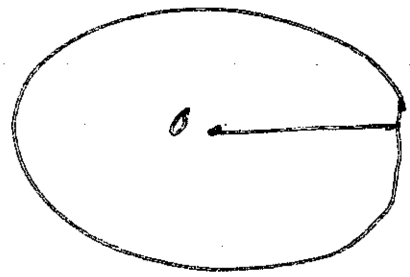
$z=1, w=-1$ $\quad$ $z=1, w=1$

$z=i, w=-(1+i)/\sqrt{2}$ $\quad$ $z=-i, w=\frac{1-i}{\sqrt{2}}$.

$z=-1, w=-i$

Exercise. In a similar way, consider the mappings and <u>Riemann</u> surface for $w = \log z$, $z = e^{w}$.

If we want to define a function for $w = \sqrt{z}$, it is necessary to have a cut, along the positive real axis for instance, to prevent z making a loop around the origin and so changing one $\sqrt{z}$ to the other.



If $z = 1$ has $\sqrt{z} = 1$ on the upper side of the cut, $\sqrt{z} = -1$ for $z = 1$ on the lower side of the cut.

If $z = (r, \theta)$, $\sqrt{z} = (\sqrt{r}, \theta/2)$ it being understood that θ goes from 0 to 2π so $\theta/2$ goes from 0 to π, i.e. w stays in the upper half plane, for $w = r(\cos\theta/2 + i\sin\theta/2)$, and as $0 < \theta < 2\pi$, $0 < \frac{\theta}{2} < \pi$, so $\sin\frac{\theta}{2} > 0$ and thus $\Im(w) > 0$ ~~in the~~ for z in the cut plane.

If we consider z moving along a path, and this path crosses the cut, then $\sqrt{z}$ must be regarded as having the value that is <u>not</u> given by the description above. This consideration is important in the work on $w = \sin^{-1} z$, defined as $\int_0^z \frac{dt}{\sqrt{1-t^2}}$.

In these symbols the Maxwell eqns are

(1)
$$
\begin{aligned}
0 &= \cdot + \frac{\partial K_{12}}{\partial x_2} + \frac{\partial K_{13}}{\partial x_3} + \frac{\partial K_{14}}{\partial x_4}\\
0 &= \frac{\partial K_{21}}{\partial x_1} \quad \cdot \quad \frac{\partial K_{23}}{\partial x_3} + \frac{\partial K_{24}}{\partial x_4}\\
0 &= \frac{\partial K_{31}}{\partial x_1} \quad \frac{\partial K_{32}}{\partial x_2} \quad \cdot \quad \frac{\partial K_{34}}{\partial x_4}\\
0 &= \frac{\partial K_{41}}{\partial x_1} + \frac{\partial K_{42}}{\partial x_2} + \frac{\partial K_{43}}{\partial x_3} \quad \cdot
\end{aligned}
$$

$$0 = \frac{\partial K_{ij}}{\partial x^j}$$

This is a tensor equation, that is accordingly valid in any system.

$$K_{ij} = \frac{\partial S_i}{\partial x_j} - \frac{\partial S_j}{\partial x_i}$$

$$\frac{\partial K_{ij}}{\partial x^k} = \frac{\partial^2 S_i}{\partial x_j \partial x_k} - \frac{\partial^2 S_j}{\partial x^k \partial x^i}$$

The second term is obtained from the first by cyclic permutation (ijk)

Hence $\frac{\partial K_{ij}}{\partial x^k} + \frac{\partial K_{jk}}{\partial x^i} + \frac{\partial K_{ki}}{\partial x^j} = 0.$

If two of i, j, k are equal, say $i=j$, LHS becomes

$$0 + \frac{\partial K_{ik}}{\partial x^i} + \frac{\partial K_{ki}}{\partial x^i} = \frac{\partial}{\partial x^i}(K_{ik}+K_{ki}) = \frac{\partial}{\partial x^i} 0.$$

We get substantial results only when i, j, k distinct. There are 4 ways of choosing 3 distinct ~~each~~ numbers from 1, 2, 3, 4.