STRINGS
A$ + B$ gives A$ followed by B$

110 STRING$(B% - A%, ""A") + A$
makes a string as long as B$ by putting the appropriate number of zeros (B% - A%) in front of A$
B% = LEN(B$)

MID$(A$, L%, 1) gives string consisting of one symbol.
L% $= L$ in A$.
(L% tells where to look; I say how many to pull)

VAL MID$(A$, L%, 1) takes the numerical value of this, of it is a number, while it is
C% in the number carried. T% begins by being
X + 9 + C%, The units C% is 1 if the sum $>$ 10.
Then T% $= T$ rem 10. Its units digit is sum
160 T% $= STR$(T$% + 1$). It puts the T% junk
found in front of the numbers already found.
L% goes down 1. Then if there is carry, I put
in front. 180, 90.

DEFADD, 70-290. DEFSUBTRACT 28-330
DEFFN$trip$. Tack initial 28 to the DEFFNmultiply 410-570
DEFFN$PROCIV$(A, B) PROCIV 820-940
PROCIV$=div$ 0 subtracts in the 58-680
690-810 FINET - Purpose?
960-1090 1
1100-
The divisor is $B$ with $b$ digits.

$R_{bc}$ is the number represented by the first $b$ digits of $R$.

Let $R = 10Y$

Is $R_b > B$? No

Take $R = R_{b+1}$

Yes

Take $R = R_b$

$Y = R$ with $R_b$ erased

Take $B$ away from $R$ as many times as you can without getting a negative number. If it goes $x$ times, and leaves remainder $r$, put $x$ in front of $Y$, put $r$ in front of $Y$. 
Osgood. Lehrbuch der Funktionentheorie
For example, a series that cannot be interpreted in power
\[ a_n(x) = S_n(x) - S_{n-1}(x). \]
\[ S_n(x) = n x e^{-nx^2}. \]
He has given geometrical description of 5.61
earlier.

R. Courant. Vorlesung über Differential- und
Integralrechnung (Göttingen 1930)
\[ \text{VIII 83}. \]
\[ f(x) = x^2 + \frac{x^2}{(1 + x)^2} + \frac{x^2}{(1 + x^2)^2}. \]
\[ x \to 0, \quad f(x) = \frac{x^2}{1 - 1/x^2} = 1 + 2^{-}. \]
\[ \lim_{x \to 0} f(x) = 0 \]
\[ \text{in figure will change, VIII 84.} \]
Example 2
At any stage there is a remainder. If $R$ is
$123456789987654321$
This is written as
$R(5) = 123\quad R(4) = 456\quad R(3) = 789$
$R(2) = 987\quad R(1) = 654\quad R(0) = 321.$
The degree of $R(1)$ is shown as $N$, here $N = 5$.

It is sometimes necessary to use numbers larger
than $1000$, just as in ordinary long division we may
need numbers bigger than $10$. e.g. $2117 \div 22$
2 (over degree than 2)
21 same degree but less than 22
211 $9 \times 23 + 4,$

The divisor will always go into a number of
higher degree. If the divisor is less than $1000$, we
will be sufficient to have a 9-figure number
in a stage.

It is not satisfactory to carry our division
by repeated subtraction. If at some stage we have
999, this may come from 999999999 with
\[ \frac{1}{1000} \]
partial quotient 999, or from 999999999 with
\[ \frac{1}{1000} \]
partial quotient 500.

Suppose $1000a + b$ is the first part of the
dividend. However large the divisor, $a$ may
number beginning 1000 that will be larger than the
dividend. Quickly by this will give us an understanding
of the partial quotients
\[ \frac{x}{1000a+b} = \frac{x}{1000a+b} = \frac{x}{(1000a+b)(100a+b-1)} \]

Use $x < 1000000$, this is certainly less than $1$.
So it will be sufficient to divide by $1000a+b$,
and then find whether the quotient may be $\frac{1}{100}$ or not.

Let the number we are dividing by be
1001 $R(1) = 1001$ with degree $M = 1$, or $\frac{R(0)}{1001}$
with degree $M = 0$. 
At some stage we have to consider

\[ X = 1000R(N) + R(N-1) \quad \text{which we have checked} \]

\[ \text{to be an error as long as} \quad Y = 1000B(I) + B(0). \]

\[ \text{Let} \quad Z = Y + 1. \]

\[ \text{Let} \quad C = \text{INT}(X/Z) \]
Continued fractions.

An expression such as

\[ a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ddots}}} \]

To express a number as a continued fraction.

Example, \( \frac{23}{18} \).

\[ \frac{23}{18} = 1 + \frac{5}{18} \]
\[ \frac{5}{18} = \frac{25}{90} = 1 + \frac{5}{9} \]
\[ \frac{5}{9} = \frac{15}{27} = 1 + \frac{5}{9} \]
\[ \frac{5}{9} = \frac{25}{45} = 1 + \frac{5}{9} \]

\[ \therefore \frac{23}{18} = 1 + \frac{1}{9 + \frac{1}{5}} \]

More interesting, express \( \sqrt{2} \) as a continued fraction.

\[ \sqrt{2} = 1 + (\sqrt{2}-1) = 1 + \frac{1}{\sqrt{2}+1} \]
\[ \sqrt{2}+1 = 2 + (\sqrt{2}-1) = 2 + \frac{1}{\sqrt{2}+1} \]
\[ \sqrt{2}+1 = 2 + (\sqrt{2}-1) \] From now on, we get 2 at each step.

\[ \sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \ddots}}} \] for ever.

It is of course necessary to check that the R.H.S. does converge to a definite limit.

We can break \( \frac{3}{2} \), or \( 1 + \frac{1}{2} \), or \( \frac{1}{2} + \frac{1}{2} \) etc.

\[ 1 + \frac{1}{2} = 1 + \frac{1}{2} = 1 + \frac{1}{2} = 1 + \frac{1}{2} \]

more \( \frac{2}{3} = 9 \times 4 = 24 \frac{1}{4} \), \( \frac{7}{25} = \frac{42}{25} = 2 \frac{1}{25} \)

The sequence continues \( \frac{9}{12}, \frac{40}{29}, \frac{99}{70}, ... \)

\[ \frac{12}{17} = \frac{229}{144} = 2 + \frac{1}{1 + \frac{1}{144}} \]
\[ \frac{144}{29} = \frac{1681}{841} = 2 + \frac{1}{1 + \frac{1}{841}} \]
\[ \frac{841}{99} = \frac{8400}{9900} = 2 + \frac{1}{4900} \]

It seems that we get ever better approximations.

deliberately alive and licent.
Convergent.
If we break off \( a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \cdots \) all terms

points to \( p_m \)
\[
\frac{p_1}{q_1} = a_1; \quad \frac{p_2}{q_2} = a_1 + \frac{1}{a_2}; \quad \frac{p_3}{q_3} = a_1 \frac{a_2}{a_2} + a_1 + \frac{1}{a_3}
\]
we observe
\[
p_3 = a_3 p_2 + p_1 \quad \text{and} \quad q_3 = a_3 q_2 + q_1
\]
This suggests that perhaps
\[
p_n = a_0 p_{n-1} + p_{n-2} \quad \text{(IA)}
\quad q_n = a_0 q_{n-1} + q_{n-2} \quad \text{(IB)}
\]
**Proof by induction.** This holds for \( n = 3 \). If it holds as far as \( n = N \), \( p_{N+1}/q_{N+1} \) is formed by putting a \( a_{N+1} \) term on \( q_N + q_{N-1} \). This means that we replace
\[
a_1 + \frac{1}{a_2} + \frac{1}{a_3} \quad \text{by} \quad a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4 + a_2}
\]
so we get \( p_{N+1}/q_{N+1} \) by changing \( a_{N+1} \) to \( a_{N+1} + \frac{1}{a_{N+1}} \) in the equations for \( p_N/q_N \).
\[
\begin{align*}
p_{N+1} &= \frac{(a_{N+1} + \frac{1}{a_{N+1}}) p_N + p_{N-2}}{q_N + \frac{1}{a_{N+1}} p_N + q_{N-2}} \\
                 &= \frac{a_{N+1} p_N + \frac{1}{a_{N+1}} p_{N-2} + \frac{1}{a_{N+1}} q_N - \frac{1}{a_{N+1}} q_{N-2}}{a_{N+1} q_N - \frac{1}{a_{N+1}} q_{N-2} + q_{N-1}} \\
                 &= \frac{a_{N+1} p_N + p_{N-1} \frac{1}{a_{N+1}} q_N - \frac{1}{a_{N+1}} q_{N-2} + q_{N-1}}{a_{N+1} q_N + q_{N-2} + q_{N-1}}
\end{align*}
\]
Thus, if the formula is correct up to \( n = N \), it is correct up to \( n = N + 1 \).
The convergents (i.e. $p_n/q_n$) to $\sqrt{2}$ are:

\[
\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \ldots
\]

\[-\frac{1}{2} + \frac{1}{2} = \frac{1}{2} ; \frac{-2}{5} + \frac{7}{5} = \frac{-1}{2} ; \frac{-7}{12} + \frac{17}{12} = \frac{1}{60} ; \frac{-17}{12} + \frac{29}{29} = \frac{-1}{12 \times 29} \cdot \frac{29 - 17}{29} + \frac{99}{70} = \frac{1}{29 \times 70} \]

These results suggest (1) the numerator is always $\pm 1$, (2) the denominator is the product of the denominators of the last two convergents (i.e. $q_n = q_{n-1} q_{n-2}$). (IIa)

Proof by induction. Suppose this is true for $n$. Then:

\[q_n = q_{n-1} q_{n-2} \quad \text{and} \quad p_n = p_{n-1} q_{n-1} + p_{n-2} q_{n-2} \]

Proof by induction. If this is true for an $n = k$ then the next expression is:

\[p_{k+1} q_k = p_k + p_{k-1} q_{k-1} = (k_{k-1} q_{k-1} + k_{k-2} q_{k-2}) q_k - (k_{k-1} q_{k-1} + k_{k-2} q_{k-2}) p_k\]

\[= p_{k-1} q_k - p_k q_{k-1} = (-1)^{n+1} \]

\[q_{n+1} = q_n q_{n-1} \quad \text{and} \quad p_{n+1} = p_n q_n + p_{n-1} q_{n-1} \]

\[= (-1)^n \]

Corollary. The convergents, as formed by the rule, (II) are in their lowest terms. If $p_n$ and $q_n$ had a common factor, it would be a factor of $(-1)^n$.

The equation $p_n - p_{n-1} = (-1)^n$. Show that:

If $n$ is even, $\frac{p_n}{q_n} > \frac{p_{n-1}}{q_{n-1}}$; if $n$ is odd, $\frac{p_n}{q_n} < \frac{p_{n-1}}{q_{n-1}}$.

Corollary. If $p, q$ are two integers with no common factors, we can find integers $p', q'$ such that $p/q = (p'q' - p'q')$.

Proof. Express $p/q$ as a continued fraction. Let $p'/q'$ be the convergent before the final $p/q$. 
It follows from Ia and IIb that \( p_n + p_{n-1} \) and \( q_n \geq q_{n-1} \).

\[ (V) \quad p_n q_{n-2} - p_{n-2} q_n = (-1)^{n-1} a_n. \]

**Proof**

From I

\[ p_n q_{n-2} - p_{n-2} q_n = (a_n p_{n-1} + p_{n-2}) q_{n-2} - p_{n-2} (a_n q_{n-1} + q_{n-2}) \]

\[ = a_n (p_{n-1} q_{n-2} - p_{n-2} q_{n-1}) = a_n (-1)^{n-1}. \]

\[ (VII) \quad \text{It follows from (VI) that} \]

\[ \frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = (-1)^{n-1} \frac{a_n}{q_n q_{n-2}}. \]

\[ (VIII) \quad \text{Hence the even convergents continuously decrease, while the odd convergents continuously increase.} \]

\[ (IX) \quad \text{Hence even } p_n/q_n \text{ decrease to a limit; the odd } p_n/q_n \text{ increase to a limit.} \]

\[ (X) \quad \text{These limits are the same, for} \]

\[ \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = (-1)^n \frac{a_n}{q_n q_{n-1}} \quad \text{and RHS} \to 0 \quad \text{by (V).} \]
A complex variable \( w = f(z) \), \( z = x + iy \), \( \bar{z} = x - iy \) is called analytic in a region if at each point of the region the derived function \( f'(z) \) exists, not depending on the direction of \( dr + id\phi \).

For a simple treatment it is helpful to make certain assumptions about the continuity of the partial derivatives (these assumptions can be relaxed).

We require then \( du + idv = f'(x + iy)(dr + id\phi) \)

\[
\text{Let } f'(x + iy) = p + iq.
\]

Then \( du + idv = (p + iq)(dr + id\phi) \)

\[
= p dr - q d\phi + i (q dr + pd\phi)
\]

\[
\therefore \quad du = pdx - q dy, \quad dv = q dx + pdy
\]

\[
\therefore \quad \frac{\partial u}{\partial x} = p \quad \frac{\partial v}{\partial y} = q \quad \frac{\partial v}{\partial x} = q \quad \frac{\partial u}{\partial y} = -p
\]

Candy-Riemann.

So \( \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \)

These are the equations satisfied by the potential and stream function for an incompressible flow in 2 dimensions with a velocity potential. Current in a copper sheet may be taken as an example of this.

The equations fail at point where current enters or leaves. These correspond to singularities.

Stereographic projection.

If we suppose a sphere placed with its centre at or above the origin, and point \( P \) of the complex plane projected to the correspond point \( P' \), it is found that we obtain \( w = f(z) \) for an incompressible flow on the sphere. This removes the special solution of 8, which was seen at the point \( N \).

Circles in the plane project to circles on the sphere.
The equations for the stereographic projection of \( N = 3 = 0 \) onto the complex plane:

\[ x^2 + y^2 + z^2 = 1 \]

where \( (1, 0, 0) \) is \( N \).

\( a + bi \) is represented by the point \( (a, b, -1) \) on the sphere. We need to find where the line \( NP \) meets the sphere.

The vector \( N \vec{P} \) is \( (a, b, -1) \). The point \( N + k \vec{N} \vec{P} \) has coordinates \( (a, b, 1 - k) \). It moves from \( N \) to \( P \), in a straight line, as \( k \) goes from 0 to 1.

It is on the sphere when \( a^2 b^2 + (1 - k)^2 = 1 \)

\[ k^2 (a^2 + b^2 + 1) = 2 \]

\[ k^2 = \frac{2}{2} \]

\[ k = \pm \frac{1}{2} \]

The point is then \( \frac{2a}{a^2 + b^2 + 1}, \frac{2b}{a^2 + b^2 + 1}, \frac{a^2 + b^2 + 1}{a^2 + b^2 + 1} \)

Inverse points on same line on sphere:

\[ \frac{1}{a + bi} = \frac{a - bi}{a^2 + b^2} = a + bi \]

Thus \( a = \frac{a}{a^2 + b^2} \)

\[ \beta = \frac{-b}{a^2 + b^2} \]

The point corresponding to \( a + \beta \) on the sphere is

\[ \frac{2a}{a^2 + b^2 + 1}, \frac{2b}{a^2 + b^2 + 1}, \frac{a^2 + b^2 + 1}{a^2 + b^2 + 1} \]

\[ a^2 + b^2 = (\frac{a}{a^2 + b^2})^2 + (\frac{b}{a^2 + b^2})^2 = \frac{1}{a^2 + b^2} \]

Thus \( \frac{2a}{a^2 + b^2 + 1}, \frac{2a}{a^2 + b^2 + 1}, \frac{a^2 + b^2 + 1}{a^2 + b^2 + 1} \)

\[ \frac{2b}{a^2 + b^2 + 1} = \frac{-2b}{a^2 + b^2 + 1} = \frac{-2b}{a^2 + b^2 + 1} \]

\[ \frac{a^2 + b^2 + 1}{a^2 + b^2 + 1} = -\frac{1}{a^2 + b^2 + 1} = -\text{previous value} \]
Thus, \( y + i z \) maps to \((x, y, z)\) on sphere.
\[ x + iy \] maps to \((-x, -y, -z)\).

There are connected by a rotation of \(180^\circ\) about \(Oz\).

Circle of convergence.

If \( f(z) \) is analytic in a region containing the origin, and \( z = k \) is the nearest singularity, or a pole, to \( z = 0 \), we can expand \( f(z) \) in an absolutely and uniformly convergent series
\[ f(z) = \sum_{n=0}^{\infty} a_n z^n \]
provided \(|z| < k\), in a circle, centre 0, with nearest singularity or circumference.

For example, \( f(z) = \frac{1}{z-3} \) has only one
singularity, \( z = 3 \).

\[
\frac{1}{z-3} = 1 + \frac{3}{z-3} + \frac{3^2}{z-3} + \cdots \quad (I)
\]

convex for \(|z| < 1\), diverges for \(|z| > 1\).

If we want a series valid for \(|z| > 1\), we put
\[ f(z) = \frac{1}{z-3} = \frac{-1}{z-1} = \frac{-1}{z-1} \]

\[ = -\frac{1}{3} \left(1 + \frac{3}{z-1} + \frac{3^2}{z-1} + \cdots\right) \]

\[ = -\frac{1}{3} - \frac{1}{3} - \frac{3}{3} - \frac{3^3}{3} - \cdots \quad (II)
\]

If \( z = 3 \), \( f(z) = -1 - t^2 - t^4 - \cdots \).

If \( \alpha \to \infty \), \( \beta \to \infty \). We call the above
series the series about \( z = \alpha \). It converges if \(|z| < 1\).

\( z \) is changed to \( p \)
by a rotation of \(180^\circ\)
about \(Oz\).

Thus the upper hemisphere

When the point is inside the circle \(|z| < 1\).

It corresponds to all \( \beta \) in the hemisphere.

The point outside \(|z| = 1\).

Thus we may think of the point outside a

circle as being inside a circle, centre \(Oz\).
An important property of integrals analytic in the region:

\[ \oint_C f(z) \, dz = 0 \]

for any closed path lying entirely in the region.

\[ 0 = \oint_C f(z) \, dz = \oint_{ABCD} f(z) \, dz \]

\[ = \oint_{ABCD} f(z) \, dz + \oint_{CDA} f(z) \, dz = \oint_{ABC} f(z) \, dz - \oint_{ADC} f(z) \, dz \]

\[ = \oint_{ABCD} f(z) \, dz = \oint_{ABC} f(z) \, dz \]

The value of the integral is not altered if the path is deformed, the end points staying the same, if it is assumed that the path does not pass over any

If \( f(z) \) is analytic in the region between the two closed curves, then

\[ \oint_{C_1} f(z) \, dz = \oint_{C_2} f(z) \, dz \]

If \( f(z) \) \( \sim \frac{1}{z} \) near \( z = 0 \), for \( C \), a

circle around origin,

\[ \oint_{C} f(z) \, dz \sim \oint_{C} \frac{1}{z} \, dz \]

\[ \oint_{C} \frac{1}{z} \, dz = \int_{0}^{2\pi} k \, i \, e^{i\theta} \, d\theta = k \cdot 2\pi i \]
Moment function

\[ f(b) = \int_{a}^{b} \frac{w(t)}{3 - r} \, dt \quad \text{where } w(t) \text{ is real, } > 0 \]

is called a moment function.

\[ f(b) = \int_{a}^{b} \frac{w(t)}{3 (1 - \frac{r}{3})} \, dt = \int_{a}^{b} \frac{w(t)}{3} \left(1 + \frac{r}{3} + \frac{r^2}{9} + \cdots \right) \]

\[ = \int_{a}^{b} w(t) \, dt \left(\frac{t}{3} + \frac{t^2}{9} + \cdots \right) \]

Coefficient of \( \frac{1}{3} \) is \( \int_{a}^{b} w(t) \, dt \), the total mass.

Coefficient of \( \frac{1}{3^2} \) is \( \int_{a}^{b} f_{1} w(t) \, dt \), the moment about \( a \).

Coefficient of \( \frac{1}{3^3} \) is \( \int_{a}^{b} f_{2} w(t) \, dt \), the moment about \( a \). The moment about \( a \) is known as the 1st moment.

Analytic nature of \( f(b) \)

1. If \( b \) is any point in the interval \( [a, b] \), \( f(b) \) is analytic at \( b \).

Proof: \( f(b) = \frac{w(t)}{3 - r} \) if \( 3 - r > 0 \)

\[ = \int_{a}^{b} \frac{w(t)}{3 - r} \, dt \]

\[ = \int_{a}^{b} \frac{w(t)}{3 - r} \, dt \]

As \( r \) is not on \( [0, b] \), \( 3 - r \), the distance of \( 3 \) from \( r \) is a minimum greater than \( 2r \), so \( \frac{1}{(3 - r)^2} \) will be finite for \( 0 < r \leq 1 \), and the integral will be satisfactory.

There will be a discontinuity across the curve \( (a, b) \) if \( 3 \) is real, \( 0 < 3 < 1 \), we shall get different values for \( f(b) \) depending on whether we approach \( 3 \) from below or above.
If \( z \) is above the real axis, displace the path to \( P_1 \) will not cause the curve \( P_1 \rightarrow z \rightarrow b \) to pass over any singularity, since \( 3 - 1 \neq 0 \) in this region.

Then \( f(z) = \int_{z}^{b} \frac{w(t) dt}{3 - t} \), where \( t \rightarrow b \) is the value of \( f(b) \) found by continuation from above.

Similarly \( f(z) = \int_{z}^{b} \frac{w(t) dt}{3 - t} \).

Now the difference of these is integrated around

\[
\oint_{z} \frac{w(t) dt}{3 - t}
\]

\[
= -2\pi i w(b)
\]

the radius of the circle having \( b \) as center.

\[ f(z) - f(b) = -2\pi i w(b) \]

Note: The above proof uses \( w(t) \) being analytic. The theorem holds true in many cases where this is not so, for instance if \( w(t) \) is finite such as

\[ \frac{1}{t} \]

This theorem is used if we know \( f(b) \) and wants to find \( w(t) \) to fit on.
In the course of my work I came across the function

\[ \varphi(a) = \int_0^1 \frac{-\ln v}{a+v} \, dv \]

\[-\ln v > 0 \text{ for } 0 < v \leq 1.\]

Suppose \( a > 0. \)

\[ \int_0^1 \frac{-\ln v \, dv}{a+v} < \frac{1}{a} \int_0^1 -\ln v \, dv \]

\[ = -\frac{1}{a} \left[ v \ln v - v \right]_0^1 
\]

\( v \ln v \to 0 \text{ as } v \to 0, \text{ so that } \frac{1}{a} = \frac{1}{a}. \)

Thus \( \varphi(a) \) is finite when \( a > 0. \)

If \( a = 0, \)

\[ \int_0^1 \frac{-\ln v \, dv}{v} = -\frac{1}{2} (\ln 1)^2 = +\infty. \]

\[ \varphi(a) \to +\infty \text{ as } a \to 0. \]

If \( a \) increases, clearly \( \frac{1}{a+v} \) decreases.

Hence \( \varphi(a) \) decreases steadily as \( a \)

\[ \frac{1}{a+v} = \frac{1}{a(1+\frac{v}{a})} \]

\[ = \frac{1}{a} \left( 1 - \frac{v}{a} + \frac{v^2}{a^2} - \frac{v^3}{a^3} - \cdots \right) \]

\[ \varphi(a) = \int_0^1 -\ln v \cdot \frac{v^n}{a^{n+1}} \, dv 
\]

\[ \int_0^1 v^n \ln v \, dv = \left[ \frac{v^{n+1} \ln v}{n+1} \right]_0^1 + \int_0^1 \frac{v^{n+1}}{v} \, dv 
\]

\[ = \frac{1}{a^{n+1}} \int_0^1 \frac{v^{n+1}}{v} \, dv 
\]

\[ = \frac{1}{a^{n+1}} \left[ \frac{v^n}{(n+1)\cdot n} \right]_0^1 = \frac{1}{(n+1)^2} \]

Thus

\[ \varphi(a) = \sum_{n=0}^\infty \frac{(-1)^n v^n}{a^{n+1}} \]

(1)

This sense converges for \( |a| > 1, \)

diverges if \( |a| < 1. \)
It diverges everywhere inside the circle $|a| = 1$ and converges everywhere outside it. This indicates that there must be a singularity somewhere on $|a| = 1$.

For $|a| = 1$, the series defined by $f(z) = \sum \frac{a^n}{n!} z^n$ for $|z| > |a|$. If $|a| > 1$, the region of convergence of the series $f(z)$ would have some part inside $|a| = 1$.

The graph of $\Phi(a)$ for $a > 0$ is.

There is no indication of a singularity at $a = -1$.

The singularity must be elsewhere.

Series valid for $0 < a < 1$.

\[
\Phi(a) = \int_0^1 \frac{1}{a + v} \, dv = \ln a \ln \left(1 + \frac{1}{a}\right)
\]

Thus part of $\Phi(a)$ is $-\ln a \ln \left(1 + \frac{1}{a}\right)$.

\[
\int_0^1 \frac{\ln x \, dx}{1 + x} = \ln a \ln \left(1 + \frac{1}{a}\right)
\]

\[
\int_0^1 \frac{\ln x \, dx}{1 + x} = \int_0^1 \frac{\ln x \, dx}{1 + x} + \int_0^1 \frac{\ln x \, dx}{1 + x}
\]

\[
\int_0^1 \frac{\ln x \, dx}{1 + x} = \int_0^1 \ln x \cdot \frac{\delta(-x)}{x} \, dx = \frac{\delta (-1)^x}{x(1+x)^2}
\]
\[ \int \frac{\ln a}{1 + u} \, du \quad \text{for } u = \frac{1}{w} \]

\[ \int = \int_{w=1}^{a} \frac{\ln w}{1 + \frac{1}{w}} (-\frac{dw}{w^2}) = \int_{a}^{1} \frac{\ln w}{w(1+w)} \, dw \]

\[ = \int_{a}^{1} \ln w \left[ \frac{1}{w} - \frac{1}{1+w} \right] \, dw \]

Finally, \[ \int_{a}^{1} \frac{-\ln w \, dw}{1+w} = \int_{0}^{a} \frac{-\ln w \, dw}{1+w} \]

\[ = \sum_{n=0}^{\infty} (-1)^n \left[ \left( \frac{w^{n+1}}{n+1} - \frac{w^{n+2}}{n+2} \right) \right]_{a}^{1} \]

\[ = \sum_{n=0}^{\infty} (-1)^n \left\{ -a^n \ln a - \frac{1-a^{n+1}}{(n+1)^2} \right\} \]

\[ = \ln a \sum_{n=0}^{\infty} (-1)^n \frac{a^n}{(n+1)^2} + \frac{(-1)^n (1-a^{n+1})}{(n+1)^2} \]

\[ = \ln a \left( \frac{-1}{(n+1)^2} + \sum_{n=0}^{\infty} \frac{(-1)^n (1-a^{n+1})}{(n+1)^2} \right) \]

In \( a = 0 \) we have \( -\ln a h(1+y) + h(a)^2 \).

After cancellation, we find

\[ \varphi(a) = \frac{1}{2} (\ln a)^2 + 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} + \sum_{n=0}^{\infty} \frac{(-a)^{n+1}}{(n+1)^2} \]
for $|a| > 1 \quad \Phi(a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2 a^{n+1}}$

How does this series behave if we take $a = 1$ on the unit circle?

There is a theorem that if $\sum C_n$ is absolutely convergent (i.e., $\sum |C_n|$ converges) then $\sum C_n$ is convergent.

The absolute series corresponding to $\Phi(a)$ is

$\sum_{n=0}^{\infty} \frac{1}{(n+1)^2 |a|^{n+1}}$.

If $a$ is on the unit circle, $|a| = 1$, and we have

$\sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$, which is convergent. It is in fact $\pi^2/6$. So $\Phi(a)$ is convergent everywhere on $|a| = 1$. How then does it manage to have a singularity on $|a| = 1$?

The clue was given by the graph for $a < 0$.

The graph of $\Phi(a)$ can be shown as $h$.

and the tangent at $a = -1$ is vertical.

In fact $\Phi(a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2 a^{n+1}}$,

$\Phi'(a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} \left[ - \frac{(n+1)a^{-n-2}}{a^{n+1}} \right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2 a^{n+2}}$.

$log(1 + \frac{1}{a}) = \frac{1}{a} - \frac{1}{2a^2} + \frac{1}{3a^3} - \frac{1}{4a^4} \cdots$

$\quad \Rightarrow \quad \Phi'(a) = -\frac{1}{a} log(1 + \frac{1}{a})$ which is $-\infty$ for $a = -1$. 


We are concerned with functions \( f : \mathbb{C} \to \mathbb{C} \), for which 
\[ f'(z) \text{ exists, i.e. for } w = f(z), \quad \frac{dw}{dz} = f'(z) \]

Consider a particular point \( z \), and variations \( dz \) from it. Then \( dw = f'(z) \, dz \). If \( f'(z) \) is some

particular complex number, multiplication by it

produces a rotation and change of scale, i.e. it leaves

angle and ratio unchanged. Such a transformation

is called conformal. If \( dw = f'(z) \, dz \) holds

only for differentials, this means that the geometry

is preserved only in the limit as we consider

neighborhoods of \( z \) and \( w \) close \( z \to \infty \). This

in particular means that the angle between the

tangents to two curves, at a place where they cross,

is not altered.

Note that multiplication produces a Blumming: it

cannot produce a Blumming over. Thus, if

\[ z = x + iy, \quad w = x - iy \]

the function \( z \to w \) cannot be analytic.

Always \( \frac{dz}{dz} \to \infty \) direction of

rotation is preserved.

Transformation, \( w = \frac{az+b}{cz+d} \), maps conformally

1-1 the entire plane \( z \) to the entire plane \( w \).

It preserves the essential behavior of functions

at each point, and can give a very useful way of

putting some structure in an easily understood form.

We suppose, of course, \( ad - bc \neq 0 \).
Such a transformation can be the combined effect of simpler transformations. For example

\[ \frac{123 + 22^2}{33 + 6} = 4 + \frac{1}{3} = 4 + \frac{1}{3(3+2)} \]

Thus \( q^x = \frac{123 + 22^2}{33 + 6} \) can be achieved by the following sequence of steps.

1. \( z_1 = z = 3 \) (1)
2. \( z_2 = z_1 \) (2)
3. \( z_3 = \frac{1}{z_2} \) (3)
4. \( z^x = z_3 + 4 \) (4)

Such a decomposition is always possible.

1 and 4 are simply translating,
2 is a change of scale,
3 is a new kind of transformation.

If \( z^x = \frac{1}{3} \) and \( z \) is \((1,0)\) then

\[ z^x = (\frac{1}{3}, -\frac{2}{3}) \]

If \( z = x + iy \), \( z^x = x - iy \), reflection in the real axis.

\( z^x \) is at distance \( r \) from the origin, \( z^x \) at distance \( \frac{1}{r} \).

\( z \rightarrow z^x \) is known as inversion in the unit circle.
Given any circle, radius $R$, centre $O$, inversion in it sends $P$ to $P'$ where $OPQ$ is straight and $OP \cdot OP' = R^2$.

This operation has much in common with reflection. If $P \rightarrow Q$, then $Q \rightarrow P$. If the radius $R$ become very large, it could be mistaken for reflection.

If reverses sense of rotation $O \rightarrow O'$, $g^* = \frac{1}{g}$ is analytic; it is equivalent to 2 operations, each of which reverses sense of rotation. It could not be equivalent to a single such operation.

Theorem: If we have a circle $C$, and if $OT$ is a tangent to it, inversion of $C$ in the circle, centre $O$, radius $OT$, makes $C \rightarrow C'$.

Proof: $OP \cdot OP' = OT^2$ is a known theorem.

Changing the radius of the circle of inversion only produces a change of scale, so $C$ inscribed in any circle, centre $O$, goes to a circle.
Problem: L is a line at perpendicular distance R from a point O.

What do we get if we insert L in the circle center O, radius R?

Suggestion: Find \( \angle LOM \) when \( \angle BOM \) is \( \theta \).
\[ w = \frac{ax + b}{cx + d} \text{ where } ad - bc \neq 0. \]

Just one value of \( w \) corresponds to a given value of \( g, \)
\[ cw^2 + wd = ax + b \quad \Rightarrow \quad w = \frac{wd - b}{cw + a}. \]

This gives a single value for \( g \) when \( w \) is known.

\[ \sum \text{Question: What goes wrong with this argument if } \]
\[ ad - bc = 0? \]

"Kreisverwandtschaft."

**Inversion**

**Reflection in a line is**

\[ P \rightarrow Q, \ Q \rightarrow P \]

**Inversion may be thought of as reflection in a circle.**

![Diagram of a circle with points A, B, and C labeled and a circle centered at O with radius OA = OB = OC = OD.]

Important property of a circle:

\[ x.y = u.v \]

(length)

So, \( OA \cdot OB = OC \cdot OD \).

This still holds if \( O \) is outside the circle.

For the point \( O \) to \( T, \)
\[ A \rightarrow T, \ B \rightarrow T. \]

So, \( OA \cdot OB = OC \cdot OD = \text{?}^2. \)

If we invert with respect to the circle, circle \( O, \)
radius \( OT \)
\[ A \rightarrow B, \ B \rightarrow A. \]

So the circle \( TBOA \) reflects to itself.

If we invert with respect to a circle, centered
at a different radius, we have similar effect with a change of scale: the circle goes
to a different circle.

There is an exception, if \( O \) lies on
the circle \( BODA. \)
1) \( f(z) = f(x+iy) \) is analytic in a region \( \Omega \) if there is a continuous, differentiable \( f'(z) \), independent of direction in which \( z \) is changed.

\[
\frac{df(z)}{dz} = f'(z) \frac{dz}{dt}.
\]

Let \( f(x+i0) = u + iv \). Then \( f'(z) = p + iq \)

\[
du + idv = \frac{du}{dx} dx + \frac{du}{dy} dy + i(\frac{dv}{dx} dx + \frac{dv}{dy} dy).
\]

\[
du = p \frac{dx}{dt} - q \frac{dy}{dt} = \frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y} = -q.
\]

\[
dv = q \frac{dx}{dt} - p \frac{dy}{dt} = \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y} = p.
\]

Thus \( \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \). Cauchy-Riemann.

These derivatives must exist and be continuous.

2) Geometrical meaning of Cauchy-Riemann equations.

Suppose a point \((x,y)\) starts at \((x_0, y_0)\)

and moves a distance \( s \) at angle \( \theta \) to \( \theta_0 \).

Then \( x = x_0 + s \cos \theta \)

\( y = y_0 + s \sin \theta \).

Rate of change of \( u(x,y) \) is \( \frac{du}{dt} \). Then

\[
\frac{du}{dt} = \frac{du}{dx} \frac{dx}{dt} + \frac{du}{dy} \frac{dy}{dt} = \frac{\partial u}{\partial x} s \cos \theta + \frac{\partial u}{\partial y} s \sin \theta.
\]

Suppose \( v \) changes as a result of \( x, y \), moving in direction \( \theta \).

\[
\frac{dv}{dt} = \frac{\partial v}{\partial x} s \cos \theta + \frac{\partial v}{\partial y} s \sin \theta \quad \text{CR hold.}
\]

Thus rate of change of \( u \) in direction \( \theta \)

\[
= \text{rate} \quad \text{of change of } v \quad \text{in } \text{direction } \theta + \frac{\pi}{2}.
\]

In particular, \( f \) is constant for \( \angle \theta \), \( u \) is constant for \( \angle \theta + \frac{\pi}{2} \). The curves \( u = c \), \( v = k \)

cross at right angles.
Integration in 2 dimensions.

To define \( \int X(x,y) \, dx + Y(x,y) \, dy \) along a given curve, we suppose the curve given by 
\[ x = x(t), \quad y = y(t) \quad 0 \leq t \leq 1. \]

Then \( X(x,y) = X(x(t), y(t)) \) and \( Y(x,y) = Y(x(t), y(t)) \).

\[
\frac{dx}{dt} = \frac{dx}{dt} \quad = x'(t) \, dt.
\]

We define the integral as
\[
\int_0^1 X(x(t), y(t)) \, x'(t) + Y(x(t), y(t)) \, y'(t) \, dt.
\]

Suppose, for example, we require \( \int y \, dx + 2x \, dy \) along the line joining \((1,1)\) to \((3,4)\).

\[ x = 1 + 2t, \quad y = 1 + 3t \]

takes \( x, y \) from \((1,1)\) to \((3,4)\) a \( t \) goes from 0 to 1.

\[
\int_0^1 (1+3t)2 + 2(1+3t)3 \, dt
\]

\[
= \int_0^1 8 + 18t - dt = 8t + 9t^2 \bigg|_0^1 = 17.
\]

We would get a different result if we went from \((1,1)\) to \((3,1)\) and then \((3,1)\) to \((3,4)\).

In first part \( x = 1 + 2t, \quad y = 1, \quad x' = 2, \quad y' = 0 \)
We have \( \int_0^1 2 + 4t \, dt = 1 + 2t^2 \bigg|_0^1 = 4 \).

In the second part \( x = 3, \quad y = 1 + 3t, \quad x' = 0, \quad y' = 3 \)

\[
\int_0^1 6.3 \, dt = 18t \bigg|_0^1 = 18. \quad \int = 22.
\]

\( \int X \, dx + Y \, dy \) is an expression for work along. This example corresponds to a non conservative system. Work could be obtained from it by following an appropriate loop.
The result will be independent of the path if
\[
\frac{dX}{dt} + \frac{dY}{dt} = \frac{\partial f}{\partial x} f(x, y)
\]
for some \( f(x, y) \) for \( f(x, y) = f(x, y) - f(x_0, y_0). \)

Now \( \frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \).

This case arises if there exists \( f(x, y) = \)
\[
X = \frac{df}{dx}, \quad Y = \frac{df}{dy}.
\]

We suppose \( X \) and \( Y \) have continuous derivatives.

Then \( \frac{\partial Y}{\partial x} = \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} = \frac{\partial f}{\partial y} \frac{\partial f}{\partial x} \) is the usual condition for \( f(x, y) \) to be an exact differential.

[Picard. Appendix A. \( \frac{df}{dx} \frac{dy}{dx} \)]

\[ \frac{df}{dx} \] is continuous, and has continuous first and second derivatives.

If \( \frac{df}{dy} = \frac{df}{dx} \) we can show that \( f(x, y) \) is

\[ f \] with the required property. \( f(x, y) \) is

then independent of the path between given end points, provided \( \frac{df}{dx} = \frac{df}{dy} \) holds throughout the region.

\[ \int X \, dx + Y \, dy = \int \frac{dy}{dx} - \frac{dx}{dy} \, dx \, dy. \]
If \( f(z) \) satisfies the Cauchy-Riemann conditions, \( \int f(z) \, dz \) does not depend on the path.

**Proposition:** \( f(z) = u + iv \).

\[
\int (u + iv)(dx + idy) = \int u \, dx - v \, dy + i \left( \int u \, dy + v \, dx \right).
\]

Since \( u \) and \( v \) are continuous, \( u \frac{\partial u}{\partial y} - v \frac{\partial v}{\partial x} = 0 \).

This is a C-12 condition.

\( u \frac{\partial u}{\partial y} - v \frac{\partial v}{\partial x} \) is zero if \( \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \). The other C-12 condition.

**Cauchy's Theorem.**

If \( f(z) \) is analytic in some region \( D \) and \( f(z) \) is analytic on any curve consisting of a finite number of smooth curves.

**Corollary:** If \( C_1 \) and \( C_2 \) are two closed curves, \( C_1 \) is contained in \( C_2 \), and a region in \( D \), then \( f(C_1) = f(C_2) \).

**Proof:** Consider region bounded by \( C_2 \), \( C_1 \), and \( S \), and \( K \).

This region is bounded by \( C_1 \), \( K \), \( S \), \( C_2 \). The two paths on \( S \) cancel, and we have \( \int \frac{f(z)}{z-a} \, dz = 0 \).

**Residue Theorem:** \( f(z) \) is analytic on a region containing curve \( C \) at point \( a \) inside \( C \).

Consider \( \int f(z) \, dz \). By Cauchy's theorem,

\[
\int_C f(z) \, dz = 2\pi i \text{Res}(f, a).
\]

This equals the value for a small circle, on \( a \).

On the circle \( f(z) = f(a) \). In the limit we have \( \int_C \frac{f(z)}{z-a} \, dz \), integrals around small circles.
On the small circle \( z = a + re^{i\theta} \) where \( r \) is the radius of circle, \( 0 \leq \theta \leq 2\pi \).

\[
\oint \frac{dz}{z-a} = \int_{0}^{2\pi} \frac{r e^{i\theta} i \, d\theta}{r e^{i\theta}} = \int_{0}^{2\pi} i \, d\theta = 2\pi i
\]

\[
2\pi i \ f(a) = \oint \frac{f(z) \, dz}{z-a}
\]

\[
f(a) = \frac{1}{2\pi i} \oint \frac{f(z) \, dz}{z-a}
\]

Given \( f(z) \) on the boundary, this gives an explicit solution for values of \( f(z) \) inside the curve.
\[ I = \int -s_2 \frac{2}{(1-x)^2} \, dx \, \, \, dq^2 = x - x^2 \]

\[ I = \int \left( \int \frac{t^2}{(1 - t^2)^2} \right) \frac{2t}{(1 - t^2)^2} \, dt \]

\[ = \int \frac{1 + t^2}{t^3} \, dt \]

\[ = -2 \left[ 1 - \frac{1}{t} \right] \]

\[ = -2 \left[ \sqrt{x} - \frac{1}{\sqrt{x}} \right] \]

\[ = \frac{2}{\sqrt{x(1-x)}} \left[ (1 - x) - x \right] \]

\[ C \text{ Ch.} \]

\[ \frac{d}{dx} (2 - 2x) x^{-1/2} (1-x) \]

\[ = -2x + ln I = \text{ln} 2 + \frac{1}{2} \ln (1 - 2x) - \frac{1}{2} \ln (1 - x) \]

\[ I' = \frac{-2}{1 - 2x} - \frac{1}{2x(1-x)} + \frac{1}{2x(1-x)} \]

\[ = \frac{2x(1-x)}{2x(1-x)(1-2x)} \]

\[ \text{Min} \]

\[ -4x + 4x^2 \]

\[ -1 + 3x - 2x^2 = 0 \]

\[ +x - 2x^2 \]

\[ I' = \frac{I}{2x(1-x)(1-x)} = \frac{2x}{\sqrt{x(1-x)}}, \frac{1}{2x(1-2x)(1-x)} \cdot \frac{1}{\sqrt{x(1-x)}} \]

\[ = \frac{1}{\sqrt{2x^3(1-x)^3}} \]
\[ \omega(s) = \frac{2(1-2s)}{\sqrt{3}(1-s^2)} \]

If \( s \) is real between 0 and 1

- \( \omega \) is real and goes from +\( \infty \) to -\( \infty \).
- \( s \) real, > 1. \( s \) is pure imagin.
- \( \omega \) real < 0 \( s \) is pure imagin.
\(|z|\) is length of \(OP\), \(P\) being point that represents \(z\).
\[
|z_1 - z_2| \leq |z_1| + |z_2|
\]

\[
|z_1| - |z_2| \leq |z_1 - z_2| \leq |z_1| + |z_2|
\]

\[
\sqrt{3} - (0, 0) \quad \sqrt{3, 2} \quad \eta, \zeta_2 \cdot \alpha_1 + \alpha_2
\]

\[
\eta \alpha_\psi \eta (\cos \theta + i \sin \theta) \text{ is number } \eta \alpha \psi.
\]

\[
\eta_1 (\cos \alpha + i \sin \alpha), \eta_2 (\cos \beta + i \sin \beta)
\]

\[
= \eta_1 \eta_2 \left[ (\cos \alpha \cos \beta - \sin \alpha \sin \beta) + i (\sin \alpha \cos \beta + \cos \alpha \sin \beta) \right]
\]

\[
= \eta_1 \eta_2 \left[ \cos (\alpha + \beta) + i \sin (\alpha + \beta) \right]
\]

\(\alpha \) is angle of a number

\[
\eta^3, \eta^2, \eta, 1
\]

Special case \(\eta = 1\)
Root extraction.

\[ 3 \sqrt[3]{\cos 30^\circ + i \sin 30^\circ} = \sqrt[3]{\cos \theta + i \sin \theta} \]

If \( 3^3 = w \) where \( w \) is at \( R, \theta \)

\[ r^n = R \quad n\theta = \theta \]

\[ r = \sqrt[n]{R} \quad \theta = \frac{\theta}{n} \]

Particularly simple case, \( R = 1 \). Then \( r = 1 \).

Find \( i \).

\[ \begin{array}{c}
| \hline
0 & 1 \\
\end{array} \]

\[ i \text{ will be at distance 1, angle } \frac{\pi}{4} \text{ (45°)} \]

\[ \sqrt{i} = \cos 45^\circ + i \sin 45^\circ = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \frac{1 + i}{\sqrt{2}} \]

Check \( \left( \frac{1 + i}{\sqrt{2}} \right)^2 = \frac{1 - 1 + 2i}{2} = i \).

Above we gave one value of \( \theta \) for the \( n \text{th} \) root of \( R, \theta \).

There should be \( n \) roots, so \( 3^3 = w \) is of degree \( n \).

\( R, \theta \) in the same power as \( R, \theta + 2\pi \)

so \( \theta = \frac{\theta + 2\pi}{n} \) will do

\[ = \frac{\theta}{n} + \frac{2\pi}{n} \]

\( s = 0, 1, 2 \ldots \) give different powers.

\[ 3^3 = -1. \] -1 is at distance 1, angle \( \pi \)

so \( 3 \) is at \( \frac{\pi}{3} + \frac{2\pi}{3} \)

\[ \frac{180^\circ + 2 \cdot 360^\circ}{3} = 60^\circ + 5 \cdot 120^\circ \]
Application to Differential Equation.

\[ \frac{d^2 y}{dx^2} + y = 0 \]

Stock method. Try \( y = e^{mx} \). Then \( \frac{d^2 y}{dx^2} = m^2 e^{mx} \)

so \[ m^2 + 1 = 0 \]

\( m = i \) or \(-i\)

Solve \( e^{ix}, e^{-ix} \)

\( e^{ix} = \cos x + i \sin x \)

\( e^{-ix} = \cos x - i \sin x \)

General solution

\[ y = P(\cos x + i \sin x) + Q(\cos x - i \sin x) \]

\[ = (P+Q)\cos x + i(P-Q)\sin x \]

\[ = A\cos x + B\sin x \]
If \( w^2 = 3 \), there are two values of \( w \) corresponding to each value of \( \theta \). Sometimes we can write \( w = \pm \sqrt{3} \), where we do not care particularly which one is meant. There are other times when it is important to know which root we are dealing with.

Square roots are particularly simple in polar co-ordinates. If \( z = \text{ab}(\gamma, \theta) \), \( \sqrt{z} = \text{ab}(\gamma/2, \theta/2) \). This gives 2 solutions, because adding 360° to the angle for \( z \) makes no difference, but doubles 180° for \( \sqrt{z} \), which changes by a factor of -1. If we take \( r = 1 \), the diagrams are simple.

![Diagram 1](image)

As we go round the circle for \( z \), we start at \( A \) for \( \theta = 0 \), with \( w = \sqrt{3} \) at \( A^* \). For \( \theta = 1 \), and we continue around the circle, \( \sqrt{3} \) always changing continuously, we get to \( B \), \( \theta = 1 \), and \( w = \sqrt{3} \) at \( B^* \). For \( \theta = 2 \), \( \sqrt{3} \) changes to \( -\sqrt{3} \) at \( C^* \).

Thus the \( \theta \) values proceed as

![Diagram 2](image)
If we go round the unit circle, with \( z \) changing in a continuous manner (this is an essential point of the theory), the effect of a complete circuit is to change one value of \( z^2 \) to the other.

Riemann introduced the important idea of the Riemann surface, each point of which can be fixed by the values of \( z \) and \( w \). Further, a small change in \( z \) and \( w \) must lead to a point very far away.

If we arrange points by their \( z \)-values, we obtain the curious shape in Figure 2.

At a distance of \( 720^\circ \) from \( A \), the point \( A' \) in Figure 3 must coincide with \( A \). This seems strange. The point in which \( z = w \) is real is called a branch. The branch cut starts at \( (x, y, z) \) and so that diagram should be in 4 dimensions, and naturally there is no difficulty. Note that these do not appear if we arrange things corresponding to \( F \) as in the \( z = w \) diagram of Figure 1.

\[
\begin{align*}
z &= -1, \quad w = (-1+i)/\sqrt{2} \\
&= (1+i)/\sqrt{2} \\
&= (1-i)/\sqrt{2} \\
&= (1+i)/\sqrt{2} \\
&= -i \quad w = -1\end{align*}
\]
Exercise. In a similar way, consider the mappings and Riemann surface for \( w = \log z, \ z = e^w \).

If we want to define a function for \( w = \sqrt[3]{z} \), it is necessary to have a cut, along the positive real axis, for instance, to prevent \( z \) making a loop around the origin and so changing \( \sqrt[3]{z} \) to the other.

If \( z = 1 \) has \( \sqrt[3]{z} = 1 \) on the upper side of the cut, \( \sqrt[3]{z} = -1 \) for \( z = 1 \) on the lower side of the cut.

If \( z = (\cos \theta, \sin \theta) \), \( \sqrt[3]{z} = (\sqrt[3]{\cos \theta}, \sqrt[3]{\sin \theta}) \).

It being understood that \( \theta \) goes from 0 to \( 2\pi \) so \( \theta/3 \) goes from 0 to \( \pi \), i.e., \( W \) stays on the upper half plane, for \( w = \sqrt[3]{\cos \theta} + i \sqrt[3]{\sin \theta/3} \), and as \( 0 < \theta < 2\pi \), \( 0 < \theta/3 < \pi \), so \( \sin \theta/3 > 0 \) and hence \( \sqrt[3]{z} > 0 \) i.e., for \( z \) in the cut plane.

If we consider \( z \) moving along a path, and this path crosses the cut, then \( \sqrt[3]{z} \) must be regarded as having the value that is not given by the description above. This consideration is important in the work on \( w = \sin^{-1} z \), defined as \( \int_{0}^{z} \frac{dt}{\sqrt{1-t^2}} \).
In these symbols the Maxwell eums are

\[
0 = \frac{\partial K_{12}}{\partial x_2} + \frac{\partial K_{13}}{\partial x_3} + \frac{\partial K_{14}}{\partial x_4}
\]

\[
0 = \frac{\partial K_{21}}{\partial x_1} + \frac{\partial K_{23}}{\partial x_3} + \frac{\partial K_{24}}{\partial x_4}
\]

\[
0 = \frac{\partial K_{31}}{\partial x_1} + \frac{\partial K_{32}}{\partial x_2} + \frac{\partial K_{34}}{\partial x_4}
\]

\[
0 = \frac{\partial K_{41}}{\partial x_1} + \frac{\partial K_{42}}{\partial x_2} + \frac{\partial K_{43}}{\partial x_3}
\]

\[
0 = \frac{\partial K_{ij}}{\partial x_j}
\]

This is a tensor equation, that is accordingly valid in any system.

\[
K_{ij} = \frac{\partial S_i}{\partial x_j} - \frac{\partial S_j}{\partial x_i}
\]

\[
\frac{\partial K_{ij}}{\partial x_k} = \frac{\partial^2 S_i}{\partial x_k \partial x_j} - \frac{\partial^2 S_j}{\partial x_k \partial x_i}
\]

The second term is obtained from the first by cyclic permutation (ijk).

Hence \( \frac{\partial K_{ij}}{\partial x_k} + \frac{\partial K_{ik}}{\partial x_j} + \frac{\partial K_{jk}}{\partial x_i} = 0 \).

If two of \( i, j, k \) are equal, say \( i = j \), LHS becomes

\[
0 + \frac{\partial K_{ii}}{\partial x_i} + \frac{\partial K_{ik}}{\partial x_i} = \frac{2}{\partial x_i} (K_{ik} + K_{ki}) = \frac{2}{\partial x_i} \cdot 0
\]

We get substantial results only when \( i, j, k \) distinct.

There are 4 ways of choosing 2 distinct letter numbers from 1, 2, 3, 4.