NOTES ON A BRIDGE FROM CLASSICAL TO MODERN ANALYSIS.

Fréchet has pointed out that it is impossible to define a metric appropriate to the function space in which $f_n \to f$ when $f_n(x) \to f(x)$ for each x. The reason is that in any metric space "the limit of a limit is a limit", that is, if $a_1, a_2, a_3, \ldots\ldots$ are limit points of a set S and $a_n \to a$, then a is a limit point of S. But Baire showed that this does not hold for functions with limit defined as above. Let S be the set of continuous functions. Functions in Baire Class 1 are limits of continuous functions. Functions in Baire Class 2 are limits of functions in Baire Class 1 but are not limits of any sequence of continuous functions.

For example let $f_n(x)$ be 1 when $2^n x$ is an integer, 0 otherwise. It is possible to find a sequence of continuous functions tending to f_n. Let $f(x)$ be 1 when x, multiplied by any positive integral power of 2, gives an integer, 0 otherwise. Then $f_n \to f$ but (as will be proved below) no sequence of continuous functions tends to f.

The idea of Baire category, of a "sparse set", appears for the first time in Baire's demonstration of this result. The idea of category generalizes without any difficulty at all, from the real line to any metric space, and plays an important role in metric space theory.

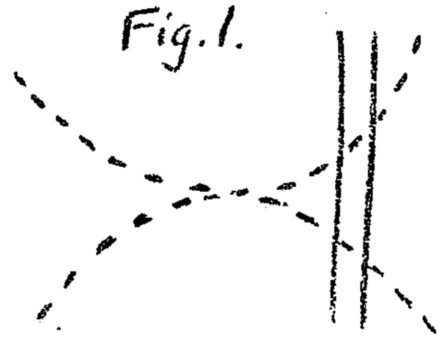
The present notes give an account of Baire's proof, and may serve the following purposes; (1) to show whence the concept of Baire category came, (2) to explain Fréchet's remark that there are topological spaces (i.e. spaces where limit is defined) for which no metric can be found, (3) to show a way of approaching and visualizing proofs in analysis that may sometimes be found useful.

Since Baire was working on the classification of discontinuous functions, we begin by recalling some terminology related to discontinuities.

Terminology.

Consider a partifular example of a discontinuous function, f, where $f(a) = a^2$ for rational a and $-a^2$ for irrational a. The graph, as shown here, consists of 2 dotted parabolas. Consider the values of f(x) when x is in an interval $[a-b, a+b]$ and b is small. Whether a is rational or irrational, the upper bound will be $(a+b)^2$ and the lower bound $-(a+b)^2$, if a and b are positive. As $b \to 0$, the bounds tend to a^2 and $-a^2$; for any function, the corresponding limits are denoted by

Fig. 1.



M(a) and m(a). The difference, $w(a) = M(a) - m(a)$, which in our example is $2a^2$, may be called the jump at a.

If a is rational, the corresponding point on the graph is on the upper parabola. If we take values of x near to a, we may find $f(x)$ much less than $f(a)$, because we have gone to the lower parabola, but it is impossible for $f(x)$ to be much larger than $f(a)$. Here f is called <u>upper semi-continuous</u> and $f(a) = M(a)$.

In the same way, if a is irrational, it is impossible for a small variation to produce a large decrease in the value of the function. In this case $f(a) = m(a)$ and the function is <u>lower semi-continuous</u> at this point.

At the origin, $f(a) = M(a) = m(a)$ and the jump $w(a)=0$. Here the two halves of continuity have got together to produce continuity.

Fréchet has pointed out, in Les Espaces Abstraits, that the length of a curve provides a simple and natural example of semi-continuity. In Figure 2, graph A is a smooth curve, which we may think of as a thread. In order to make sure that the curve is only altered "a little" we enclose the thread in a narrow tube. By wrinkling the thread I can obtain a much longer curve, X, without going outside the tube, but by tightening the thread I can only make the length inside the tube a little shorter. So the function curve → length is lower semi-continuous, but not continuous.

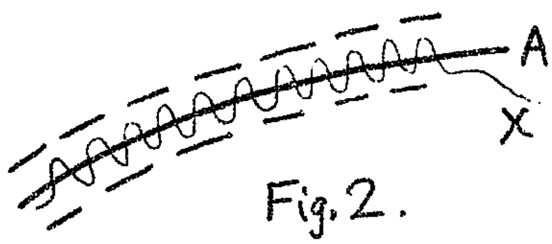


Fig. 2.

<u>The union of nowhere dense sets.</u>

At a point x where f is continuous, $w(x) = 0$. Points of discontinuity have $w(x) > 0$. Baire first uses the concept of sparse set ("set of first category") in discussing the discontinuities of a function. We will not go into the details of this theorem, but will merely indicate how the concept of sparse set enters the argument. As just mentioned, at a point of discontinuity, $w(x)>0$. We may use S_1 to describe the set of points where $w(x)>1$, S_2 for the set where $w(x) > 1/2$ and generally S_n for the set of points with $w(x) > 1/n$. Each set includes the earlier ones but that is not particularly relevant. The set of all discontinuities is $\cup S_n$, the union of all these sets as n runs through the natural numbers. In certain circumstances each of the sets S_n is nowhere dense. $\cup S_n$ is thus the union of $\aleph_0$ nowhere dense sets; we condense this by saying that $\cup S_n$ is a sparse set. Baire proves that no sparse set can contain all the points in an interval of the real line.

The proof is very easy. S_1 is nowhere dense. This means that in any interval I_0 we can find a closed interval I_1 free from points of S_1.

We now operate in I_1 instead of I_0 ; the effect is that we have got rid of S_1 and it has not cost us anything, for in this work one interval is as good as another. No doubt I_1 is shorter than I_0 but none of the properties we are dealing with is altered by a change of scale. So we continue ; within I_1 we find an interval I_2 free from points of S_2 , and so on. The nested closed intervals $I_1, I_2, I_3 \ldots$ are bound to have at least one common point, which will be in none of the sets S_n, hence not in $\cup S_n$. So $\cup S_n$ does not contain all the points of I_0 . Q.E.D.

This proof appears to show that there is one point not in $\cup S_n$, but in fact there will be very many. For instance, we could equally well carry out the above construction within any sub-interval of I_0 and find a point in it not in $\cup S_n$.

The concept of sparse set will be seen in action in the proof below of the main theorem. This theorem depends on the concept of <u>total discontinuity</u>. A function f is called totally discontinuous if there is an interval <u>at every point of which</u> the jump $w(x)>K$, where of course $K > 0$. (There are other, equivalent definitions.) The function graphed in Figure 1 for example is totally discontinuous. So is the function f mentioned in the second paragraph of page 1 above. So is the well known function with value 1 for x rational, 0 for x irrational.

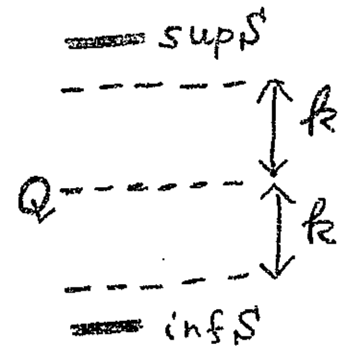
Baire proved that, if $f_1, f_2, f_3 \ldots$ is a sequence of continuous functions, and f a bounded function such that $f_n(x) \to f(x)$ for each x in some interval, then f cannot be totally discontinuous.

<u>A Lemma : Appearance and Reality.</u>

Baire starts his proof with a lemma, which may serve to illustrate the general principle that proofs in analysis usually look much harder than they are. When you glance at the proof of this lemma, you see 9 inequations. Your mind is perhaps already tired with what you have read, and the lemma may loom in your way as a substantial obstacle. In fact, once you have seen what the lemma means, its proof becomes almost a single mental act. The lemma assumes that there is some set S of real numbers, for which $\sup S - \inf S > 2k$. It states that if <u>a</u> is any real number whatever, there exists a member of S whose distance from <u>a</u> exceeds <u>k</u> .

Figure 3 shows the situation. The only information we have about the set relates to sup S and inf S. For all we know, the set may consist of only 2 numbers, one at sup S and one at inf S. If sup S is not in S, there must be members of the set just below it. If we mark on our figure a narrow interval, with its ceiling at sup S, we can be sure this interval is inhabited by elements of S. In the same way we can mark an inhabited interval rising from inf S. The fact that both intervals are inhabited is all the information at our disposal. The assertion in the lemma is now obvious. In the figure, is <u>a</u> is below Q, its distance from an inhabitant of the upper interval must exceed <u>k</u> ; if it is at or above Q, its distance from the lower interval exceeds <u>k</u> .

Fig. 3.



Clearly, Baire did not regard this lemma as expressing any great or surprising truth. It was just a detail he wanted out of the way, so the reader would not be distracted by it in the middle of the main proof. Personally, having seen what this lemma amounted to, I would not even bother to check Baire's 9 inequalities. These cannot amount to anything more than putting into formal shape the visual argument just outlined.

<u>Proof of the Main Theorem.</u>

If you make a few attempts to construct a sequence of continuous functions with the function graphed in Figure 1 as a limit, you will not find it hard to believe that this task is in fact impossible, as the theorem implies. The functions are required to leap about in a very violent way. Baire's proof sets out to establish a contradiction between the following statements ; -

1.) f is bounded and totally discontinuous.
2.) Each f_n is continuous.
3.) $f_n(x) \rightarrow f(x)$ for each <u>x</u> in I.

The break eventually comes in statement (3). Baire finds an <u>x</u> for which $\{f_n(x)\}$ is not a Cauchy sequence, so it does not converge at all, and hence cannot tend to f(x).

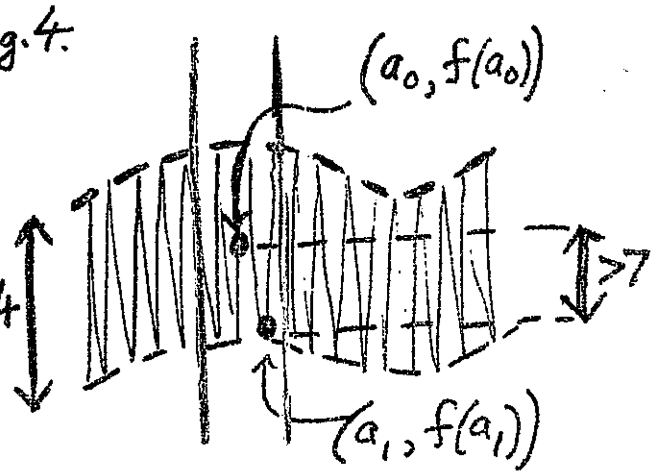
Accordingly our concern is with the tail of the sequence $f_p, f_{p+1}, f_{p+2}, \ldots$ We allow an opponent to select the number p.

As f is totally discontinuous, in some interval I at every point $w(x) > K$.

For convenience of illustration, we choose a particular K, say K = 14. This means that, in any vertical strip, however thin, we can find points in the graph of f with vertical separation exceeding 14.

Within I we let the opponent choose any interval I_o and any a_o in I_o. By the lemma, we can find a_1, as near as we like to a_o, with $|f(a_1)-f(a_o)| > 7$. (See Figure 4.)

Fig.4



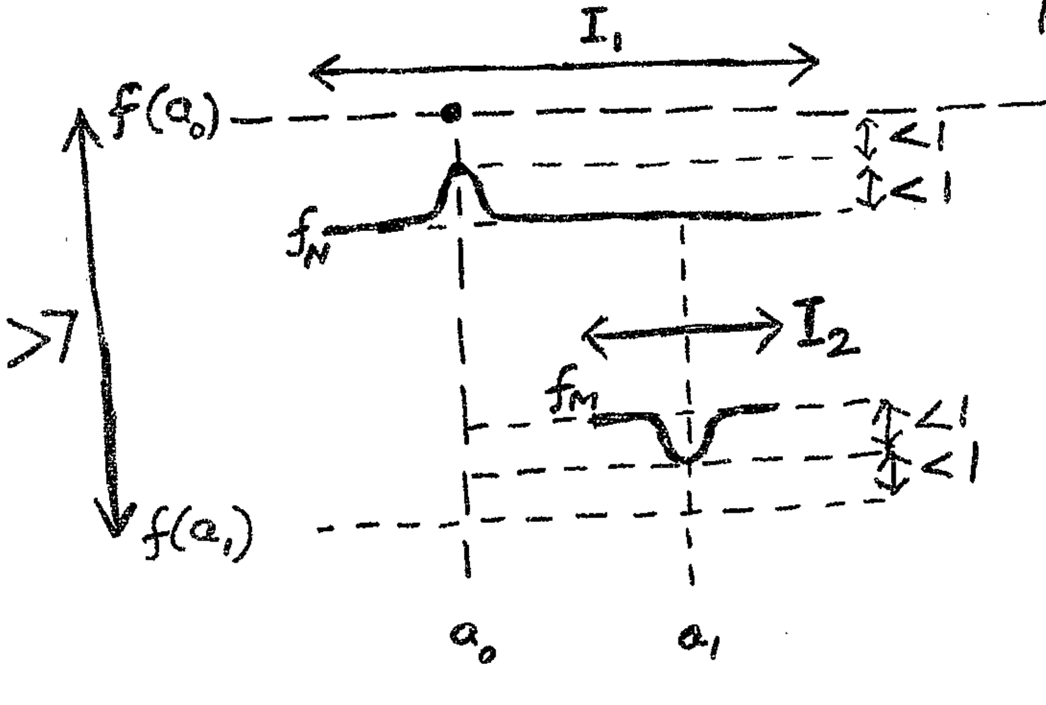
In drawing diagrams to illustrate arguments in analysis, we must always load the dice against ourselves. The figure must show the situation least favourable to our argument. For instance, in Figure 4, we want the difference between $f(a_1)$ and $f(a_o)$ to be <u>more</u> than 7. Accordingly $f(a_o)$ is shown half-way up the region occupied by the graph, for this is the place it is hardest to be remote from.

We do not proceed immediately to select a_1. We first look at the functions f_n assumed to tend to f. As $f_n(a_o) \rightarrow f(a_o)$ we can find $N > p$ so that

$|f_N(a_o) - f(a_o)| < 1.$

As f_N is continuous, we can choose an interval I_1, within I_o, throughout which $|f_N(x) - f_N(a_o)| < 1.$ (See Figure 5.)

Fig.5



It is now that we choose a_1, inside I_1, so that $|f(a_1) - f(a_0)| > 7$. Then we have two steps similar to those done a little earlier. We choose $M > p$ so that $|f_M(a_1)-f(a_1)| < 1$. As f_M is continuous, there is an interval I_2 throughout which $|f_M(x)-f_M(a_1)| < 1$. (See the lower part of Figure 5.)

It is evident that, in I_2, $|f_M(x) - f_N(x)| > 3$.

Now if $\{y_n\}$ is a convergent sequence, the differences between the terms should become indefinitely small, so certainly less than 3. We shall say the sequence has "settled down after p terms" if $M > p$, $N > p$ implies $|y_M - y_N| < 3$.

Our work above shows that in any interval I_0 chosen by the opponent we can find an interval I_2 in which the sequence $\{f_n(x)\}$ has not settled down by the pth term. If S_p denotes the set of x for which $\{f_n(x)\}$ has settled down after p terms, this means that the set S_p is nowhere dense.

If x is not in the set S_p, it does not follow that the sequence $\{f_n(x)\}$ diverges. It still has time for a deathbed repentance ; it may settle down for some number larger than p. But to converge, it must settle down at some stage. Now the choice of p was left to the opponent. We have shown that for each p, S_p is nowhere dense. If the sequence converges, it must settle down for some value of p : that is, x must lie in $U\, S_p$. But $U\, S_p$ is a sparse set, and cannot fill the interval I.

At any point not in the sparse set, the sequence $f_n(x)$ fails to converge, and we have a contradiction.

Note. It can now be seen why Figure 5 was drawn as it was. We are trying to establish that $f_M(x)$ and $f_N(x)$ are far apart. All the inequalities are therefore drawn in such a way that f_M and f_N are made to approach each other as much as possible - the case most unfavourable to our argument.