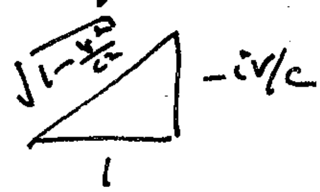
Lorentz Transformation

$x^2+y^2+z^2-c^2t^2=0$ is equation for propagation of light. This is to be invariant. Let $\tau = ict$.
$x^2+y^2+z^2+\tau^2$ to be invariant.

$\bar{x} = x\cos\theta - \tau\sin\theta \qquad \bar{y}=y \quad \bar{z}=z.$
$\bar{\tau} = x\sin\theta + \tau\cos\theta$

$\bar{x}=0$ represents a point at rest in bar system.
It corresponds to $0 = x\cos\theta - \tau\sin\theta$. $x = \tau\tan\theta$
$x = ict\tan\theta.$
So velocity of motion of one system relative to the other is $v = ic\tan\theta \qquad \tan\theta = -iv/c$

$\sin\theta = -\frac{iv/c}{\sqrt{}}, \cos\theta = \frac{1}{\sqrt{}}$



$$\bar{x} = x\cos\theta - ict\sin\theta$$
$$\bar{x} = \frac{x-vt}{\sqrt{1-v^2/c^2}} \qquad (1)$$
$$\bar{t} = -\frac{i}{c}\bar{\tau} = -\frac{i}{c}\left[\frac{-\frac{iv}{c}x}{\sqrt{}}+\frac{ict}{\sqrt{}}\right]$$
$$\bar{t} = \frac{-xv/c^2+t}{\sqrt{}} \qquad (2)$$

(1) + v(2) $\qquad \bar{x}+v\bar{t} = \frac{x(1-v^2/c^2)}{\sqrt{1-v^2/c^2}}$

So $\quad x = \frac{\bar{x}+v\bar{t}}{\sqrt{1-v^2/c^2}}$

$\frac{v}{c^2}$(1) + (2) $\qquad \frac{v\bar{x}}{c^2}+\bar{t} = \frac{t(1-v^2/c^2)}{\sqrt{1-v^2/c^2}}$

$$t = \frac{\frac{v\bar{x}}{c^2}+\bar{t}}{\sqrt{1-v^2/c^2}}$$

N49

L5

Lorentz transformation.

$x^2+y^2+z^2-c^2t^2$ to be invariant.

Take $y^*=y$ $z^*=z$. Let $u=ict$

Then x^2+u^2 is to be invariant.

$$x^* = x\cos\theta - u\sin\theta$$
$$u^* = x\sin\theta + u\cos\theta \quad ?$$

So

$$x^* = x\cos\theta - ict\sin\theta = \cos\theta\,(x - i\tan\theta . ct)$$

Let $v = ic\tan\theta$ so bracket is $x - vt$, which we expect for system in which $x^*=0$ corresponds to $x=vt$.

Then $\tan\theta = -iv/c$. $\sec^2\theta = 1-\frac{v^2}{c^2}$

$$\cos\theta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad \sin\theta = \tan\theta\cos\theta = \frac{-iv/c}{\sqrt{1-v^2/c^2}}$$

Thus. $$x^* = \frac{x-vt}{\sqrt{1-v^2/c^2}}$$

$$ict^* = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(-\frac{iv}{c}x + ict\right)$$

$$t^* = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(t - \frac{vx}{c^2}\right).$$

Note $$x^*+vt^* = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\,x\left(1-\frac{v^2}{c^2}\right)$$

So $$x = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\,(x^*+vt^*)$$

So $x=0$ corresponds to $x^* = -vt^*$.

See also Vibrations

METI
NS4
M-6

Sketch of approach to Maxwell's Equations.
In traditional symbolism.

$\underline{E}$ = electric intensity $\underline{H}$ = magnetic intensity.
In vacuum
$\operatorname{div} E = 0$ $\operatorname{div} H = 0$, the first because no electric charges are present, the second because free magnetic poles never exist.

The EMF induced in an electric circuit is $-dN/dt$ where N is the magnetic induction.

$$\oint \underline{E}\,d\underline{s} = -\iint \frac{\partial H}{\partial t}\,d\underline{S}$$

$$\oint E\,ds = \iint \operatorname{curl} E\,d\underline{S}. \quad E \text{ is E.M.U. For}$$

E in ESU we have $\quad \operatorname{curl} E = -\frac{1}{c}\frac{\partial H}{\partial t}$.

Work done in threading a current i with unit magnetic pole is $4\pi i$, so $\oint \underline{H}.ds = 4\pi\int \underline{j}\,d\underline{S}$
leading to $\operatorname{curl} H = 4\pi j$ for closed currents in conducting circuits.
When electrons etc are moving about in space, difficulty as some surfaces are crossed by moving electrons, others are not. $\operatorname{div}\operatorname{curl} H = 0$
Maxwell introduces displacement current $\frac{1}{4\pi}\frac{\partial E}{\partial t}$
and $\operatorname{div}\left(j + \frac{1}{4\pi}\frac{\partial E}{\partial t}\right) = 0$. He suggested $\operatorname{curl} H = 4\pi\left(j + \frac{1}{4\pi}\frac{\partial E}{\partial t}\right)$
In vacuum, $j = 0$ and $\operatorname{curl} H = \frac{\partial E}{\partial t}$. E in EMU.
Thus

$$\operatorname{curl} H = \frac{1}{c}\frac{\partial E}{\partial t} \quad \text{when } E \text{ in ESU.}$$

So we have

$$\operatorname{curl} H = \frac{1}{c}\frac{\partial E}{\partial t} \qquad \operatorname{curl} E = -\frac{1}{c}\frac{\partial H}{\partial t}$$

$$\operatorname{div} E = 0 \qquad \operatorname{div} H = 0.$$

Note $\quad 0 = \operatorname{div}\operatorname{curl} H = \frac{1}{c}\frac{\partial}{\partial t}(\operatorname{div} E) = 0$

$$0 = \operatorname{div}\operatorname{curl} E = -\frac{1}{c}\frac{\partial}{\partial t}(\operatorname{div} H) = 0.$$

N.S 3

M 7

Maxwell's Equations for Vacuum.

$$\operatorname{curl} H = \frac{1}{c}\frac{\partial E}{\partial t} \qquad \operatorname{curl} E = -\frac{1}{c}\frac{\partial H}{\partial t}.$$

$$\operatorname{div} E = 0 \qquad \operatorname{div} H = 0.$$

Klein (p 62) puts $\tau = ict \qquad \mathcal{E} = iE.$

$$\frac{1}{c}\frac{\partial E}{\partial t} = \frac{i\partial E}{ic\partial t} = \frac{\partial \mathcal{E}}{\partial \tau}.$$

$$\frac{\partial H}{\partial \tau} = \frac{\partial H}{ic\partial t} = -\frac{1}{i}\operatorname{curl} E = i \operatorname{curl} E = \operatorname{curl} \mathcal{E}.$$

Thus we have

$$\operatorname{curl} H = \frac{\partial \mathcal{E}}{\partial \tau} \qquad \operatorname{curl} \mathcal{E} = \frac{\partial H}{\partial \tau}$$

$$\operatorname{div} \mathcal{E} = 0 \qquad \operatorname{div} H = 0.$$

$$0 = \quad \cdot \quad -\frac{\partial H_3}{\partial y} + \frac{\partial H_2}{\partial z} + \frac{\partial \mathcal{E}_1}{\partial \tau}$$

$$0 = \frac{\partial H_3}{\partial x} \quad \cdot \quad -\frac{\partial H_1}{\partial z} + \frac{\partial \mathcal{E}_2}{\partial \tau}$$

$$0 = -\frac{\partial H_2}{\partial x} + \frac{\partial H_1}{\partial y} \quad \cdot \quad \frac{\partial \mathcal{E}_3}{\partial \tau}$$

$$0 = -\frac{\partial \mathcal{E}_1}{\partial x} - \frac{\partial \mathcal{E}_2}{\partial y} - \frac{\partial \mathcal{E}_3}{\partial z}$$

$$0 = \qquad -\frac{\partial \mathcal{E}_3}{\partial y} + \frac{\partial \mathcal{E}_2}{\partial z} + \frac{\partial H_1}{\partial \tau}$$

$$0 = \frac{\partial \mathcal{E}_3}{\partial x} \qquad -\frac{\partial \mathcal{E}_1}{\partial z} + \frac{\partial H_2}{\partial \tau}$$

$$0 = -\frac{\partial \mathcal{E}_2}{\partial x} + \frac{\partial \mathcal{E}_1}{\partial y} \qquad + \frac{\partial H_3}{\partial \tau}$$

$$0 = -\frac{\partial H_1}{\partial x} - \frac{\partial H_2}{\partial y} - \frac{\partial H_3}{\partial z}$$

Traditional Electromagnetic Theory.

E.S.U. chosen so that force $= ee'/r^2$ for charges e, e' at distance r cms. Force in dynes.

Magnetic pole, unit chosen so that force $= mm'/r^2$.

The inverse square law was originally guessed (for gravitation) as for flow of incompressible fluid. We expect the same for electric and magnetic force.

~~If velocity is v_1 v_2 v_3~~
If ρ is density of fluid, $\rho(x,y,z)$ and velocity is v_1 v_2 v_3, condition that no change of density occurs is $\operatorname{div} \rho v = 0$

If we identify ρv with force $\frac{\mathbf{r}}{r^3}$, condition for fluid to be incompressible is $\operatorname{div} \frac{\mathbf{r}}{r^3} = 0$, which in fact is so.

For $\frac{\mathbf{r}}{r^3} = \left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\right)$. $r^2 = x^2 + y^2 + z^2$

$\therefore 2r\frac{\partial r}{\partial x} = 2x$ $\frac{\partial r}{\partial x} = \frac{x}{r}$ Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\operatorname{div} \frac{\mathbf{r}}{r^3} = \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)$$

$$= \left(\frac{1}{r^3} - \frac{3x}{r^4}\frac{\partial r}{\partial x}\right) + (\quad) + (\quad)$$

$$= \left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right) + \left(\frac{1}{r^3} - \frac{3y^2}{r^5}\right) + \left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right)$$

$$= \frac{3}{r^3} - \frac{3(x^2+y^2+z^2)}{r^5} = 0.$$

The condition being linear, it holds for sum of charges.

If we assume $\rho = 1$ initially for all x, y, z and $\operatorname{div} \underline{v} = 0$, then ρ will remain equal to 1.

and we may picture force as corresponding to velocity of incompressible fluid.

For electricity. Gauss Theorem flux $E = \sum 4\pi e$
and $\operatorname{div} E = 4\pi\rho$.

For magnetism $\operatorname{div} H = 0$.

The condition $\rho = 1$ is for an imaginary fluid ~~that~~ helps us to visualize the electrostatic field. 9

In $\operatorname{div} E = 4\pi\rho$, ρ is actual charge density.

Later, when we consider charges moving around we have, for v the actual velocity,

$$\operatorname{div} \rho \underline{v} = -\frac{\partial \rho}{\partial t}$$

Potentials

Potential is the work required, per unit charge (or magnetic strength) to bring charge (magnet pole) to point. Work done in a displacement = decrease of potential.

Thus $X\,dx + Y\,dy + Z\,dz = -\,d\varphi$

$$\operatorname{grad} \varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}\right)$$

so $(X, Y, Z) = -\operatorname{grad} \varphi$.

Electrostatic potential.

$$\operatorname{grad}\left(\frac{e}{r}\right) = \left(\frac{\partial}{\partial x}\left(\frac{e}{r}\right), \frac{\partial}{\partial y}\left(\frac{e}{r}\right), \frac{\partial}{\partial z}\left(\frac{e}{r}\right)\right)$$

$$= \left(-\frac{e}{r^2}\frac{\partial r}{\partial x}, \quad , \quad \right)$$

$$= \left(-\frac{ex}{r^3}, -\frac{ey}{r^3}, -\frac{ez}{r^3}\right)$$

Thus electric intensity $= -\operatorname{grad} V$

with $V = \sum \frac{e}{r}$.

Similarly, for magnetism $\sum \frac{m}{r}$ gives potential.

Vector Potential Since magnetic poles always occur in pairs (dipoles) the sum of magnetic poles in any region is 0. Hence $\operatorname{div} H = 0$.

This means a vector $\underline{A}$ exists such that $\operatorname{curl} \underline{A} = H$. A is called a vector potential. ("a", not "the", as $\underline{A}$ is so far not uniquely defined)

(A, V) is important in relativity theory.

Derivation of Maxwell's Equations (Traditional form)

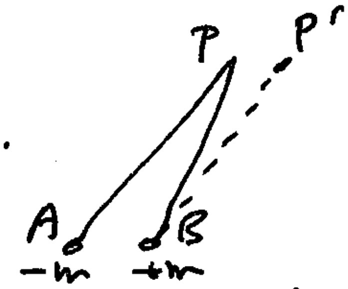
E.M.U. Magnetic poles exert force mm'/r^2
Force in dynes, r in cms.

A dash is shown with electrical quantities in EMU

Potential due to magnetic pole = work done in bringing unit pole to given position $= \int_r^\infty \frac{m}{x^2}\,dx$

$$= \left[-\frac{m}{x}\right]_r^\infty = \frac{m}{r}.$$

Potential of magnetic dipole.



Potential at $P = \frac{m}{BP} - \frac{m}{AP} = \frac{m}{BP} - \frac{m}{BP'} = -ml\frac{\partial}{\partial x}\left(\frac{1}{r}\right)$

AB being supposed of length l, in Ox direction

Potential $= M\frac{1}{r^2}\frac{\partial r}{\partial x}$ $r^2 = x^2+y^2+z^2$

$= \frac{Mx}{r^3}$. $2r\frac{\partial r}{\partial x} = 2x$

Let $\underline{\varepsilon}$ be unit vector in x direction. $x = \underline{r}.\underline{\varepsilon}$

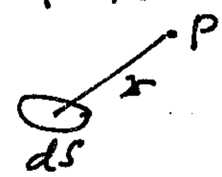
∴ Potential $= \frac{M\underline{\varepsilon}.\underline{r}}{r^3} = \frac{\underline{\mu}.\underline{r}}{r^3}$

where μ is vector representing moment of dipole and its direction.

Magnetic shell of strength φ. Every element $d\underline{S}$ of surface behaves like magnetic dipole with $\underline{\mu} = \varphi\, d\underline{S}$.

Potential it produces $= \iint \varphi\frac{\underline{r}.d\underline{S}}{r^3}$

$= \varphi\Omega$ where Ω is the solid angle subtended by surface at P.



E.M.U of current chosen so that a current i' is equivalent to a shell of strength $\varphi = i'$.

Work done when unit magnetic pole threads circuit carrying i' is thus $4\pi i'$.

Work $= \oint H\,ds = \iint \text{curl}\, H.\, d\underline{S}$

Current passing through loop $= \iint j'.d\underline{S}$

∴ $\iint \text{curl}\, H.d\underline{S} = 4\pi\iint j'\,d\underline{S}$ ∴ $\text{curl}\, H = 4\pi j'$.

Induction of electric currents by changing magnetic field.

When a magnetic pole moves near electric current, work is done on it. This work must come from the batteries maintaining the current.

Suppose we arrange that just enough extra EMF is provided so that current in coil stays constant. Pole m, current i'. If Ω is the solid angle subtended by the coil at the magnetic pole, the potential energy of the pole is $mi'\Omega$.

If pole moves so that Ω increases, it is going "uphill" so work is being done on it, $mi'd\Omega$. If this happens in time dt work done is $mi'\frac{d\Omega}{dt}dt$.

The charge passes through the coil in dt is $i'dt$, and if EMF is V', this charge has fallen through V' and work done is $V'i'dt$.

$$\therefore \quad V' = m\frac{d\Omega}{dt}$$

This is EMF required to keep current constant.

$\therefore$ Motion of magnet produces EMF $-m\frac{d\Omega}{dt}$.

The magnetic pole has $4\pi m$ lines of force coming from it, equally spread in all directions. Hence number of lines of force passing through coil is $m\Omega = N$, the flux through coil.

$$\therefore \text{ Induced EMF is } -\frac{dN}{dt} = -\iint \frac{\partial H}{\partial t} d\underline{S}$$

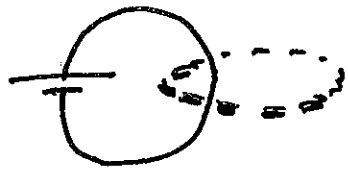
$$\text{EMF} = \oint E' ds = \iint \text{curl } E' d\underline{S}$$

$$\therefore \quad \text{curl } E' = -\frac{\partial H}{\partial t}$$

Now $E' = cE$ so $\text{curl } E = -\frac{1}{c}\frac{\partial H}{\partial t}$.

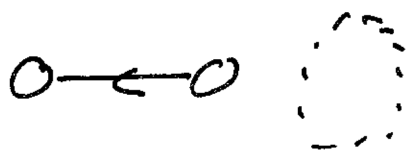
$j' = j/c$. so curl $H = \frac{4\pi}{c} j$.

Displacement current.



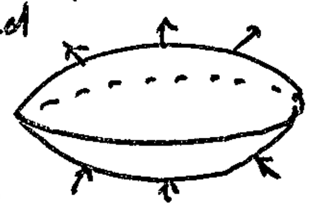
With a closed circuit, there is no uncertainty whether a loop threads it or not.

If we take two metal balls, charge one and then connect it to the other, a current flows in a line segment. How to decide whether a loop "threads" it or not?



With current in some conducting medium, charges flow in a way resembling an incompressible fluid

If two surfaces are bounded by the same loop, current crossing them is the same.

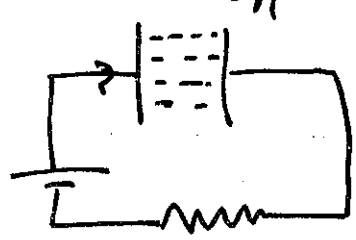


Mathematically this appears as div curl $H \equiv 0$.

$\boxed{\text{E.S.U}}$

Maxwell was led to the concept of displacement current by the need to overcome this difficulty.

We assume plates of a capacitor to have large area in relation to their distance apart, so that lines of force are parallel, straight lines. Area A, charge q per unit area.



Charge on one plate is $e = qA$.

Current $i = \frac{de}{dt} = A \frac{dq}{dt}$

$E = 4\pi q$ $\quad$ j = current density on plate. $i = Aj$

$\frac{\partial E}{\partial t} = 4\pi \frac{dq}{dt} = 4\pi \frac{i}{A} = 4\pi j$ so $\frac{1}{4\pi}\frac{\partial E}{\partial t}$ gives a continuation of current. If we have ordinary currents and changing electric field, we regard

$j + \frac{1}{4\pi}\frac{\partial E}{\partial t}$ as effective current.

We have $\operatorname{div} E = 4\pi\rho$ (Gauss)

$$\operatorname{div} j = -\frac{\partial\rho}{\partial t}$$

$$\therefore \operatorname{div}\left(j + \frac{1}{4\pi}\frac{\partial E}{\partial t}\right) = 0.$$

Thus $\operatorname{curl} H = \frac{4\pi}{c}\left(j + \frac{1}{4\pi}\frac{\partial E}{\partial t}\right)$

In space where there are no ordinary currents

$$\operatorname{curl} H = \frac{1}{c}\frac{\partial E}{\partial t}$$

Also. $$\operatorname{curl} E = -\frac{1}{c}\frac{\partial H}{\partial t}.$$

Maxwell's Equations and Relativity.

Two possible approaches (i) Put $T = ict$ and make $x^2+y^2+z^2-c^2t^2 = x^2+y^2+z^2+T^2$. (ii) Keep $x^2+y^2+z^2-c^2t^2$. In this case we need to distinguish covariant from contravariant. Only (i) done here.

(i) $T = ict$ method.

$$\frac{\partial E}{\partial T} = \frac{1}{ic}\frac{\partial E}{\partial t} = \frac{1}{i}\,\mathrm{curl}\,H \quad \text{Let } \mathcal{E} = iE; \ \mathrm{curl}\,H = \frac{\partial \mathcal{E}}{\partial T}$$

$$\frac{\partial H}{\partial T} = \frac{1}{ic}\frac{\partial H}{\partial t} = -\frac{\mathrm{curl}\,E}{i} = \mathrm{curl}\,iE = \mathrm{curl}\,\mathcal{E}.$$

Thus $\mathrm{curl}\,H = \dfrac{\partial \mathcal{E}}{\partial T} \qquad \mathrm{curl}\,\mathcal{E} = \dfrac{\partial H}{\partial T}$.

Let $\mathcal{E} = (U, V, W)$ $\quad H = (L\ M\ N)$

$\mathrm{div}\,\mathcal{E} = 0 \qquad \mathrm{div}\,H = 0.$

$(x\ y\ z\ T) = (x^1\, x^2\, x^3\, x^4)$

I) $\mathrm{curl}\,H = \dfrac{\partial \mathcal{E}}{\partial x^4}$

$$\begin{array}{l|llll}
\dfrac{\partial U}{\partial x^4} = \dfrac{\partial N}{\partial x^2} - \dfrac{\partial M}{\partial x^3} & 0 = \ \cdot & -\dfrac{\partial N}{\partial x^2} & +\dfrac{\partial M}{\partial x^3} & +\dfrac{\partial U}{\partial x^4} \quad (1)\\
\dfrac{\partial V}{\partial x^4} = \dfrac{\partial L}{\partial x^3} - \dfrac{\partial N}{\partial x^1} & 0 = \dfrac{\partial N}{\partial x^1} & \cdot & -\dfrac{\partial L}{\partial x^3} & +\dfrac{\partial V}{\partial x^4} \quad (2)\\
\dfrac{\partial W}{\partial x^4} = \dfrac{\partial M}{\partial x^1} - \dfrac{\partial L}{\partial x^2} & 0 = -\dfrac{\partial M}{\partial x^1} & +\dfrac{\partial L}{\partial x^2} & \cdot & +\dfrac{\partial W}{\partial x^4} \quad (3)\\
0 = \dfrac{\partial U}{\partial x^1} + \dfrac{\partial V}{\partial x^2} + \dfrac{\partial W}{\partial x^3} & 0 = -\dfrac{\partial U}{\partial x^1} & -\dfrac{\partial V}{\partial x^2} & -\dfrac{\partial W}{\partial x^3} & \cdot \quad (4)
\end{array}$$

The minus signs in last equation preserve antisymmetry.

II)

$$0 = \ \cdot \ -\frac{\partial W}{\partial x^2} + \frac{\partial V}{\partial x^3} + \frac{\partial L}{\partial x^4} \qquad (1)$$

$$0 = \frac{\partial W}{\partial x^1} \ \cdot \ -\frac{\partial U}{\partial x^3} + \frac{\partial M}{\partial x^4} \qquad (2)$$

$$0 = -\frac{\partial V}{\partial x^1} + \frac{\partial U}{\partial x^2} \ \cdot \ + \frac{\partial N}{\partial x^4} \qquad (3)$$

$$0 = -\frac{\partial L}{\partial x^1} - \frac{\partial M}{\partial x^2} - \frac{\partial N}{\partial x^3} \ \cdot \qquad (4)$$

Material on electromag. Soft Hraph N.82

M15

TET 1, 2. Inverse square law. $\text{div}\, E = 4\pi\rho$ $\text{div}\, H = 0$.
$E = -\text{grad}\, V$ $V = \sum \frac{e}{r}$. $H = \text{curl}\, A$.

Z1-3 Establishes $H = \oint i\, ds \wedge \frac{x}{r^3}$. Not needed for direct route to invariance under Lorentz.
I-III. Vector potential of magnetic particle.
N41 Verification of $H = \oint i\, ds \wedge x / r^3$.

N40 TEM 1. Covered by TET.
N39. TEM 2. Potential due to magnetic shell, leads to work through current is $4\pi i$. ASSUMES POTENTIAL of magnetic particle.
N38 Covered by I-III.
N37 Note on solid angle, re N39.
N36 –
N35 Dielectrics.
N34 Displacement current.
N32 Induced EMF $= -dN/dt$. N = induction through circuit.
N31 $E = -\frac{\partial A}{\partial t} - \text{grad}\, V$.
N30 $\text{curl}\, H = 4\pi j'$.
N29 $\rho v_x/c$, $\rho v_y/c$, $\rho v_z/c$, ρ is 4-vector.
N28. Potential of magnetic particle is $\mu \cdot x / r^3$.
N27-N24 K_{ij} and Maxwell's equations
N23-21 Maxwell's eqs.
N20. Behaviour of $\partial p/\partial y - \partial q/\partial x$ in $\mathbb{R}^2$, under transformation.
N19-18. Operations that give tensors.
N16 Maxwell's eqns.
N14-13. Klein p60. Maxwell's eqns.
N12 Maxwell's eqs with left axes.
N11 much like N29.
N10 $\square(A, V)$ shows A, V is 4 vector.
N9 curl curl leads to $\square$
N8-N7 Displacement current.
N6 Displacement current.
N3 E-M theory.

N41 $H = \oint i\, ds \wedge x / r^3$ N42-44 Vector potential for mag. particle
N45-47 $H = \oint i\, ds \wedge x / r^3$ and vector potential
N49 Lorentz transformation N50, N51 TET. Trad electromag. HY.
N53. Maxwell eqns as for Klein. N59. Maxwells eqs as from F_{ij}.

$x \quad y \quad z \quad t$

$\xi \quad \eta \quad \zeta \quad \tau$

$$\tau = \beta\left(t - \frac{v}{c^2}x\right) \qquad \eta = y \quad \zeta = z$$

$$\xi = \beta(x - vt) \qquad \beta = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$\xi = 0$ is fixed in Greek system.
It corresponds to $x = vt$ in Roman.

$E = (X\ Y\ Z) \qquad H = (L\ M\ N) \qquad x\,y\,z\,t$ system

$(X'\,Y'\,Z') \qquad (L'\,M'\,N') \qquad \xi\,\eta\,\zeta\,\tau$ system

$$X' = X \qquad Y' = \beta\left(Y - \frac{v}{c}N\right) \qquad Z' = \beta\left(Z + \frac{v}{c}M\right)$$

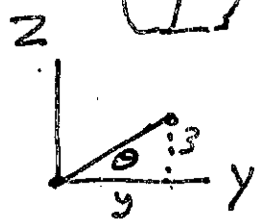
$$L' = L \qquad M' = \beta\left(M + \frac{v}{c}Z\right) \qquad N' = \beta\left(N - \frac{v}{c}Y\right)$$

Suppose in $x\,y\,z\,t$ system we have charge density σ per unit length along OX, at rest.

$$4\pi\sigma = 2\pi r E$$

$$E = \frac{2\sigma}{r}.$$

Force $\frac{2\sigma}{r}$ away from O.



$$X = 0 \qquad Y = \frac{2\sigma}{r}\cdot\frac{y}{r} \qquad Z = \frac{2\sigma}{r}\cdot\frac{z}{r}$$

$$X = 0 \qquad Y = \frac{2\sigma y}{y^2 + z^2} \qquad Z = \frac{2\sigma z}{y^2 + z^2}.$$

$L = M = N = 0$

$$X' = 0 \qquad Y' = \frac{2\beta\sigma y}{y^2 + z^2} \qquad Z' = \frac{2\beta\sigma z}{y^2 + z^2}$$

$$L' = 0 \qquad M' = \frac{\beta v}{c}\cdot\frac{2\sigma z}{y^2 + z^2} \qquad N' = -\frac{\beta v}{c}\cdot\frac{2\sigma y}{y^2 + z^2}$$

$$M' = \frac{2\beta\sigma v}{c}\cdot\frac{\zeta}{\eta^2 + \zeta^2} \qquad N' = -\frac{2\beta v\sigma}{c}\,\frac{\eta}{\eta^2 + \zeta^2}$$

$$= \frac{2\beta\sigma v}{c}\,\frac{\sin\theta}{r} \qquad = -\frac{2\beta\sigma v}{c}\,\frac{\cos\theta}{r}$$

$$\mathbf{j} = (\rho v_x, \rho v_y, \rho v_z)$$

Thus $\left(\frac{\mathbf{j}}{c}, \rho\right)$ is 4-vector.

$$\Box \mathbf{A} = -\frac{4\pi \mathbf{j}}{c}$$

$$\Box V = -4\pi\rho$$

Hence it is consistent to suppose that $(\mathbf{A}, V)$ is a 4-vector.

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$m^{£}$18

Charge density ρ, with velocity v_x, v_y, v_z.

$\rho\frac{v_x}{c}, \rho\frac{v_y}{c}, \rho\frac{v_z}{c}, \rho$ is a vector.

Co-ordinates $x_1\ x_2\ x_3\ x_4 \qquad x_4 = ct$

$x \quad y \quad z \quad ct$

$dx, dy, dz, c\,dt$ is a vector.

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \text{ is invariant}$$
$$= dt^2\left(c^2 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2\right)$$
$$= c^2dt^2(1 - v^2/c^2)$$
$$\therefore ds = c\,dt\sqrt{1 - v^2/c^2}$$

$\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}, c\frac{dt}{ds}$ is a vector.

Hence $\frac{dx}{dt}\cdot\frac{dt}{ds}, \frac{dy}{dt}\cdot\frac{dt}{ds}, \frac{dz}{dt}\frac{dt}{ds}, c\frac{dt}{ds}$ is vector

$$\frac{v_x}{c\sqrt{1-\frac{v^2}{c^2}}}, \frac{v_y}{c\sqrt{1-\frac{v^2}{c^2}}} \frac{v_z}{c\sqrt{1-\frac{v^2}{c^2}}} \frac{1}{\sqrt{1-v^2/c^2}}.$$

Let ρ_0 be charge density in the system in which charge is at rest, i.e. in system with velocity $(v_x\ v_y\ v_z)$ relative to $xyzt$ system.

$$dx_0\,dy_0\,dz_0 = \beta\,dx\,dy\,dz.$$
$$\rho_0\,dx_0\,dy_0\,dz_0 = \rho\,dx\,dy\,dz$$
$$\therefore \quad \rho = \beta\rho_0 = \frac{\rho_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

Multiply vector above by ρ_0. We get

$\rho\frac{v_x}{c}, \rho\frac{v_y}{c}, \rho\frac{v_z}{c}, \rho$ is vector.

Vector potential and the 4-vector (A, V)

Free electric charges exist but free magnetic poles do not. Accordingly we have (in empty space) $\operatorname{div} H = 0$. Hence, $\exists A :- H = \operatorname{curl} A$.

[Prov. Jeans § 441 $A_1 = \int H_2 dz$, $A_2 = -\int H_1 dx$, $A_3 = 0$ will do.]

A is not yet uniquely defined.

$$\operatorname{curl} E = -\frac{1}{c}\frac{\partial H}{\partial t} = -\frac{1}{c}\operatorname{curl}\frac{\partial A}{\partial t}$$

$$\therefore \operatorname{curl}\left(E + \frac{1}{c}\frac{\partial A}{\partial t}\right) = 0$$

Hence $E + \frac{1}{c}\frac{\partial A}{\partial t} = -\operatorname{grad} V$ for some V

$$E = -\frac{1}{c}\frac{\partial A}{\partial t} - \operatorname{grad} V. \qquad (1)$$

If it should happen that $\partial A/\partial t = 0$, i.e. no magnetic change occurring, $E = -\operatorname{grad} V$ so we identify V with the electrostatic potential.

$$(2)\quad \operatorname{curl} H = \frac{1}{c}\left(4\pi j + \frac{\partial E}{\partial t}\right) \qquad \operatorname{curl} E = -\frac{1}{c}\frac{\partial H}{\partial t}\quad (3)$$

With $H = \operatorname{curl} A$ we have

$$\frac{1}{c}\left(4\pi j + \frac{\partial E}{\partial t}\right) = \operatorname{curl}\operatorname{curl} A = \operatorname{grad}\operatorname{div} A - \nabla^2 A$$

$$E = -\frac{1}{c}\frac{\partial A}{\partial t} - \operatorname{grad} V.$$

$$\therefore \frac{1}{c}\frac{\partial E}{\partial t} = -\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \operatorname{grad}\frac{\partial V}{\partial t}$$

$$\therefore \nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} + \frac{4\pi j}{c} = \operatorname{grad}\left[\operatorname{div} A + \frac{1}{c}\frac{\partial V}{\partial t}\right]$$

So far, A has been restricted only by $\operatorname{curl} A = H$. We are free to make the second condition

(4) $\operatorname{div} A + \frac{1}{c}\frac{\partial V}{\partial t} = 0$, and this is usually done.

Accordingly
$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\frac{4\pi j}{c}. \qquad (5)$$

Also $\operatorname{div} E = 4\pi\rho$; from (1) we have

$$-4\pi\rho = \frac{1}{c}\frac{\partial}{\partial t}\operatorname{div} A + \nabla^2 V = \nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} \text{ from (4)}$$

$$\therefore \quad (6)\quad \nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -4\pi\rho.$$

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curl curl A.

$(H_1\ H_2\ H_3) = \text{curl } A = \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}, \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}, \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$

$\text{curl curl } A = \text{curl } H_\gamma = \left(\frac{\partial H_3}{\partial y} - \frac{\partial H_2}{\partial z}, \quad , \quad \right)$

$$\frac{\partial H_3}{\partial y} - \frac{\partial H_2}{\partial z} = \frac{\partial^2 A_2}{\partial x \partial y} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} + \frac{\partial^2 A_3}{\partial x \partial y}$$

$$= \frac{\partial}{\partial x}\left\{\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right\} - \frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2}$$

This is the first component of grad div A − $\nabla^2 A$. Cyclic permutation of indices gives the result for y and z components.

Hence curl curl A = grad div A − $\nabla^2 A$.

(A is a vector. By $\nabla^2 A$ we understand $(\nabla^2 A_1, \nabla^2 A_2, \nabla^2 A_3)$.)

It is convenient to use H and A here, but of course A could stand for <u>any</u> vector.

Covariant and contravariant.

$$x^i = t^i{}_j X^j \qquad \text{contravariant} \qquad t^i{}_j = \frac{\partial x^i}{\partial X^j}$$

Let $u_i = \frac{\partial \varphi}{\partial x_i^i} \qquad \frac{\partial \varphi}{\partial x^i} = \frac{\partial \varphi}{\partial X^j}\frac{\partial X^j}{\partial x^i} = U_j \frac{\partial X^j}{\partial x^i}$

$$m^i{}_l = M^j{}_m \frac{\partial x^i}{\partial X^j}\frac{\partial X^m}{\partial x^l}$$

$$m^i{}_i = M^j{}_m \frac{\partial x^i}{\partial X^j}\frac{\partial X^m}{\partial x^i} \qquad \text{Contraction}$$

$$\frac{\partial X^m}{\partial x^i}\frac{\partial x^i}{\partial X^j} = \frac{\partial X^m}{\partial X^j} = \delta^m{}_j$$

$$m^i{}_i = M^j{}_j$$

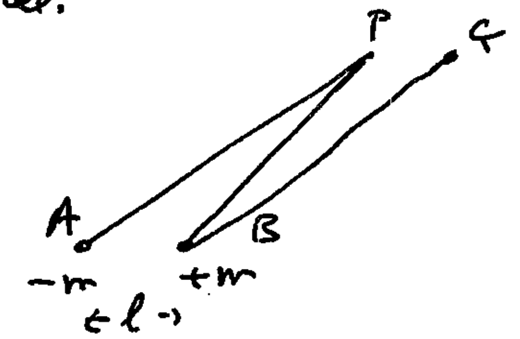
U mg4

N. 47

Z. 1. $\quad m^{423}$

Field of magnetic particle.

potential



$$\varphi_P = \frac{m}{BP} - \frac{m}{AP}$$

$$= \frac{m}{BP} - \frac{m}{BQ}$$

$$= -m.PQ.\,\mathrm{grad}\,\frac{1}{r}$$

$mPQ = \mu$ moment of magnet.

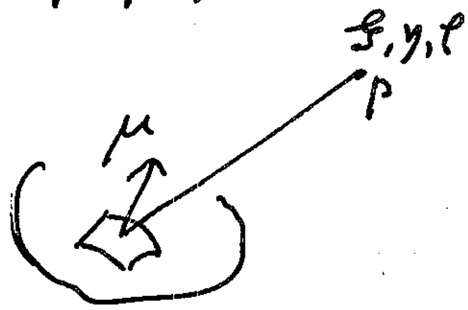
$$= -\mu\cdot\mathrm{grad}\,\frac{1}{r}$$

$$\mathrm{grad}\,\frac{1}{r} = \frac{\partial}{\partial x}\frac{1}{r}, \frac{\partial}{\partial y}\frac{1}{r}, \frac{\partial}{\partial z}\frac{1}{r}$$

$$= \left(-\frac{1}{r^2}\cdot\frac{x}{r}, -\frac{1}{r^2}\frac{y}{r}, -\frac{1}{r^2}\cdot\frac{z}{r}\right)$$

$$= -\frac{\mathbf{r}}{r^3}$$

$$\therefore \quad \varphi_P = \frac{\mu\cdot\mathbf{r}}{r^3}.$$



Let μ represent moment of element dS at $x\ y\ z$. $\quad$ P is (ξ, η, ζ)

Take $\mathbf{r} = (\xi - x, \eta - y, \zeta - z)$.

~~Field at~~ Potential at P is

$$\frac{\mu_1(\xi - x) + \mu_2(\eta - y) + \mu_3(\zeta - z)}{r^3} = \varphi$$

$$H_1 = -\frac{\partial\varphi}{\partial\xi}$$

$$= \mu_1\left[-\frac{1}{r^3} + \frac{3(\xi-x)^2}{r^5}\right] + \mu_2\frac{3(\xi-x)(\eta-y)}{r^5} + \mu_3\frac{3(\xi-x)(\zeta-z)}{r^5}$$

$$= \mu_1\frac{2(\xi-x)^2 - (\eta-y)^2 - (\zeta-z)^2}{r^5} + \mu_2\frac{3(\xi-x)(\eta-y)}{r^5} + \mu_3\frac{3(\xi-x)(\zeta-z)}{r^5}$$

$$H_2 = \mu_1\frac{3(\xi-x)(\eta-y)}{r^5} + \mu_2\frac{2(\eta-y)^2 - (\xi-x)^2 - (\zeta-z)^2}{r^5} + \mu_3\frac{3(\eta-y)(\zeta-z)}{r^5}$$

$$H_3 = \mu_1\frac{3(\xi-x)(\zeta-z)}{r^5} + \mu_2\frac{3(\eta-y)(\zeta-z)}{r^5} + \mu_3\frac{2(\zeta-z)^2 - (\xi-x)^2 - (\eta-y)^2}{r^5}$$

We would expect H to depend only on current loop, not on surface bounded by it. Z.2 N.46

The coefficients of μ_1, μ_2, μ_3 in H_1 are the same as those of μ_1 in H_1, H_2, H_3. These last are the components of H for the case $\mu_1 = 1, \mu_2 = 0, \mu_3 = 0$. (Tz 24 M)

It is known that $\underline{A}$ exists :- $H = \text{curl } A$ since $\text{div } H = 0$.

Now the components are of the form $f(\xi - x, \eta - y, \zeta - z)$ so $\frac{\partial f}{\partial x} = -\frac{\partial f}{\partial \xi}$.

div H = 0 is for variation of the point P so

$$0 = \frac{\partial H_1}{\partial \xi} + \frac{\partial H_2}{\partial \eta} + \frac{\partial H_3}{\partial \zeta} = -\left(\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} + \frac{\partial H_3}{\partial z}\right)$$

Thus H, considered as a function of x, y, z, equals curl A for some A, i.e.

$$\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = \frac{2(\xi - x)^2 - (\eta - y)^2 - (\zeta - z)^2}{r^5} \quad (1)$$

$$\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} = \frac{3(\xi - x)(\eta - y)}{r^5} \quad (2)$$

$$\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} = \frac{3(\xi - x)(\zeta - z)}{r^5} \quad (3)$$

We do not require the most general solution of these equations. A simple solution can be found by taking $A_1 = 0$. Now $r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$

$$\therefore \; 2r\frac{\partial r}{\partial x} = 2(x - \xi) \qquad 2r\frac{\partial r}{\partial y} = 2(y - \eta)$$

$$\therefore \; \frac{\partial r}{\partial x} = \frac{x - \xi}{r} \qquad \frac{\partial r}{\partial y} = \frac{y - \eta}{r}$$

Hence, from (2) $\quad -\frac{\partial A_3}{\partial x} = \frac{3(\eta - y)}{r^4}\frac{\partial r}{\partial x} \qquad A_3 = \frac{\eta - y}{r^3}$

$$\frac{\partial A_2}{\partial x} = \frac{3(\zeta - z)}{r^4}\frac{\partial r}{\partial x} \qquad A_2 = -\frac{\zeta - z}{r^3}$$

Thus, since same expressions occur in H_1 equation

$$H_1 = \mu_0 \,\text{curl}\left(0, \frac{z - \zeta}{r^3}, -\frac{y - \eta}{r^3}\right)$$

Z.3

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As $\mu = i\,d\underset{\sim}{S}$, for shell as a whole we have

$$H_1 = \iint i\,d\underset{\sim}{S} \wedge \operatorname{curl}\left(0, \frac{\zeta-\varsigma}{r^3}, -\frac{y-\eta}{r^3}\right)$$

$$= \oint i\,d\underset{\sim}{s} \cdot \left(0, \frac{\zeta-\varsigma}{r^3}, -\frac{y-\eta}{r^3}\right)$$

$$= \oint i\left\{\frac{\zeta-\varsigma}{r^3}\,dy - \frac{y-\eta}{r^3}\,d\zeta\right\}$$

The bracket is the first component of

$$(dx, dy, d\zeta) \wedge \left(\frac{x-\xi}{r^3}, \frac{y-\eta}{r^3}, \frac{\zeta-\varsigma}{r^3}\right)$$

$$= \oint i\,d\underset{\sim}{s} \wedge \frac{\underset{\sim}{r}}{r^3}.$$

The result can be proved similarly for H_2, H_3.

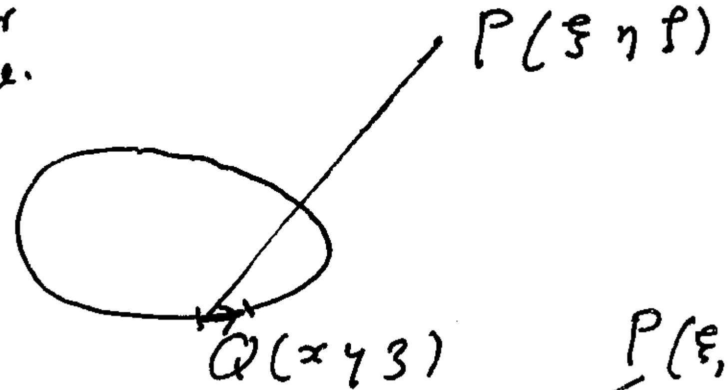
So

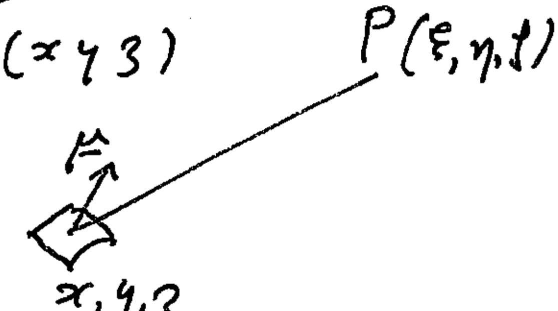
$$H = \oint i\,d\underset{\sim}{s} \wedge \frac{\underset{\sim}{r}}{r^3}.$$

N44 (I)

M26

Vector potential for magnetic particle. (not for shell)





Let moment of magnet ~~corresponding to dS~~ be μ.

Potential at P is $\frac{\mu \cdot r}{r^3}$

$$= \frac{\mu_1(\xi - x) + \mu_2(\eta - y) + \mu_3(\zeta - z)}{r^3} = \varphi$$

$$H_1 = -\frac{\partial\varphi}{\partial\xi} \qquad H_2 = -\frac{\partial\varphi}{\partial\eta} \qquad H_3 = -\frac{\partial\varphi}{\partial\zeta}$$

As $r^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2$

$2r\frac{\partial r}{\partial \xi} = 2(\xi - x)$ so $\frac{\partial r}{\partial \xi} = \frac{\xi - x}{r}$.

Consider (skew) ~~coefficient of μ_1~~ in H_1, H_2, H_3

$$H_1 = \mu_1\left[-\frac{1}{r^3} + \frac{3(\xi - x)^2}{r^5}\right] + \mu_2[\ \ ] + \mu_3[\ \ ]$$

$$H_2 = \frac{3\mu_1(\xi - x)(\eta - y)}{r^5} + \dots + \dots$$

$$H_3 = \frac{3\mu_1(\xi - x)(\zeta - z)}{r^5} + \dots + \dots$$

? goes wrong way?

So
$$H_1 = \frac{\mu_1[2(\xi-x)^2-(\eta-y)^2-(\zeta-z)^2]}{r^5} + \ldots + \ldots$$

$$H_2 = \frac{3\mu_1(\xi-x)(\eta-y)}{r^5} + \ldots + \ldots$$

$$H_3 = \frac{3\mu_1(\xi-x)(\zeta-z)}{r^5} + \ldots + \ldots$$

Now $\operatorname{div} H = 0$, so $H = \operatorname{curl} A$ for some A. Here div means $\frac{\partial H_1}{\partial \xi} + \frac{\partial H_2}{\partial \eta} + \frac{\partial H_3}{\partial \zeta} = 0$ as we are concerned with the variation of P. But all functions are of the form $f(\xi - x, \eta - y, \zeta - z)$, so
$\frac{\partial f}{\partial x} = -\frac{\partial f}{\partial \xi}$, $\frac{\partial f}{\partial y} = -\frac{\partial f}{\partial \eta}$, $\frac{\partial f}{\partial z} = -\frac{\partial f}{\partial \zeta}$
so $\operatorname{div} H = 0$ is equally true if we take x, y, z are the quantities that vary, and $H = \operatorname{curl} A$ for some A is valid. Let $A = \mu_1 \mathcal{A} + \ldots + \ldots$
We require

$$\frac{\partial \mathcal{A}_3}{\partial y} - \frac{\partial \mathcal{A}_2}{\partial z} = \frac{2(\xi-x)^2-(\eta-y)^2-(\zeta-z)^2}{r^5} \quad (1)$$

$$\frac{\partial \mathcal{A}_1}{\partial z} - \frac{\partial \mathcal{A}_3}{\partial x} = \frac{3(\xi-x)(\eta-y)}{r^5} \quad (2)$$

$$\frac{\partial \mathcal{A}_2}{\partial x} - \frac{\partial \mathcal{A}_1}{\partial y} = \frac{3(\xi-x)(\zeta-z)}{r^5} \quad (3)$$

These do not determine $\mathcal{A}$ uniquely. We can find a solution with the additional assumption $\mathcal{A}_1 = 0$.
Then $\frac{\partial \mathcal{A}_2}{\partial x} = \frac{3(x-\xi)(z-\zeta)}{r^5}$ $\quad \frac{\partial \mathcal{A}_3}{\partial x} = -\frac{3(x-\xi)(y-\eta)}{r^5}$

As $r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$ $\quad \frac{\partial r}{\partial x} = \frac{x-\xi}{r}$.

Thus $\frac{\partial \mathcal{A}_2}{\partial x} = \frac{3(z-\zeta)}{r^4}\frac{\partial r}{\partial x}$ so $\mathcal{A}_2 = -\frac{z-\zeta}{r^3}$ is soln.

$\frac{\partial \mathcal{A}_3}{\partial x} = -\frac{3(y-\eta)}{r^4}\frac{\partial r}{\partial x}$ leads to $\mathcal{A}_3 = \frac{y-\eta}{r^3}$.

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$$\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = \left[\frac{1}{r^3} - \frac{3(y-\eta)^2}{r^5}\right] + \left[\frac{1}{r^3} - \frac{3(z-\zeta)^2}{r^5}\right]$$

$$= \frac{2(x-\xi)^2 - (y-\eta)^2 - (z-\zeta)^2}{r^5}$$ in agreement with I.

The coefficients of μ_2 and μ_3 can we written down by symmetry.

$$A_1 = \qquad \frac{\mu_2(z-\zeta)}{r^3} - \frac{\mu_3(y-\eta)}{r^3}$$

$$A_2 = -\frac{\mu_1(z-\zeta)}{r^3} \qquad + \frac{\mu_3(x-\xi)}{r^3}$$

$$A_3 = \frac{\mu_1(y-\eta)}{r^3} - \frac{\mu_2(x-\xi)}{r^2}$$

Verification of $H = \oint \frac{i\,ds \wedge r}{r^3}$

Two complications. (1) $\oint \underline{v}.\underline{ds} = \iint \text{curl}\, v . \underline{ds}$ involves $\underline{v}.\underline{ds}$ not $v \wedge ds$. (2) Formula (1) relates to single quantity: H is a vector.

Suppose we consider H_1, component of magnetic field at origin, due to $i\,ds$ at $x\ y\ z$. $r = (-x, -y, -z)$

$$H_1 = i \oint \frac{-z\,dy + y\,dz}{r^3}$$

$$\begin{matrix} dx & dy & dz \\ -x & -y & -z \end{matrix}$$

$$= i \oint (dx, dy, dz) \cdot \left(0, -\frac{z}{r^3}, \frac{y}{r^3}\right)$$

$$= i \iint \underset{\sim}{dS} . \text{curl} \left(0, -\frac{z}{r^3}, \frac{y}{r^3}\right)$$

$$\begin{matrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{matrix}$$

$$\text{curl}\left(0, -\frac{z}{r^3}, \frac{y}{r^3}\right) =$$

$$\left(\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) - \frac{\partial}{\partial z}\left(-\frac{z}{r^3}\right), \frac{\partial}{\partial z} 0 - \frac{\partial}{\partial x}\left(\frac{y}{r^3}\right), \frac{\partial}{\partial x}\left(-\frac{z}{r^3}\right) - \frac{\partial}{\partial y}(0)\right)$$

$$= \left(\frac{1}{r^3} - \frac{3y^2}{r^5} + \frac{1}{r^3} - \frac{3z^2}{r^5}, \frac{3xy}{r^5}, \frac{3xz}{r^5}\right)$$

$$= \left(\frac{2x^2 - y^2 - z^2}{r^5}, \frac{3xy}{r^5}, \frac{3xz}{r^5}\right)$$

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Traditional electromagnetic theory.

Two reasons for studying. (1) All papers until ? 1939 written in this symbolism. (2) The treatment is simpler and explains the assumptions in modern theory.

The traditional approach used the idea of a magnetic pole. Now magnetism seems to be due to electricity in motion. We meet positive and negative electric charges (electrons, protons etc) but never a free magnetic pole. On the other hand, the magnetic pole is not a mere fiction. A long magnet only 1mm. wide has ends that behave very much like the theoretical poles.

The unit for magnetic pole is so chosen that poles of strength m and m' repel each other with a force mm'/r^2 dynes when r cm apart.

A magnetic particle consists of poles $+m, -m$ a small distance l apart. $l \to 0$ in such a way that $ml = \mu$, the moment of the magnet.

If we have $+m$ at $(a,0,0)$ and $-m$ at $(-a,0,0)$ the magnetic potential is $\dfrac{m}{r_1} - \dfrac{m}{r_2}$

where $\underset{\sim}{r_1} = (x-a, y, z)$,

$\underset{\sim}{r_2} = (x+a, y, z)$.

$$\frac{1}{r_2} - \frac{1}{r_1} = \frac{1}{\sqrt{(x+a)^2+y^2+z^2}} - \frac{1}{\sqrt{(x-a)^2+y^2+z^2}}$$

$$\doteqdot 2a \frac{\partial}{\partial x}\frac{1}{r}$$

$$\text{Potential} = -2am\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\mu\frac{\partial}{\partial x}\left(\frac{1}{r}\right)$$

$$= \frac{\mu x}{r^3}$$

x is component of r in direction of magnet

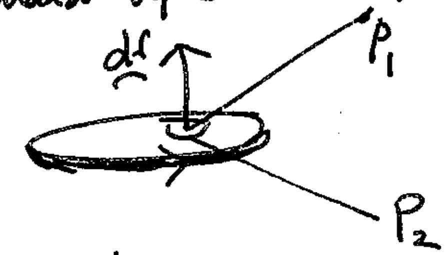
$$\therefore \text{Potential} = \frac{\underset{\sim}{\mu} \cdot \underset{\sim}{r}}{r^3}$$

$\underset{\sim}{\mu}$ being a vector in direction of magnet, size μ.

Magnetic Shell.

A shell of strength φ is a surface such that each element $d\underline{S}$ is a magnetic particle of moment $\varphi\, d\underline{S}$. The potential is thus $\int \frac{\varphi\, \underline{r}.d\underline{S}}{r^3}$

$$\int \frac{\underline{r}.d\underline{S}}{r^3} = \text{solid angle subtended by shell at } P.$$



Note that this is positive when P is as shown here at P_1; negative at P_2.

At point just above disc, solid angle is 2π, at point just below, -2π.

Hence work done on unit pole in a path that threads the boundary curve once is $4\pi\varphi$.

Electro-magnetic unit of current.

Experimentally it is found that a current in a closed loop produces the same magnetic effects as a shell bounded by that loop. If the unit of current is correctly chosen, $\varphi = i$.

It follows that the work done in taking a unit magnetic pole round a path that threads the current once is $4\pi i$.

Vector potential for magnetic particle.

div $H = 0$ so $\exists A$:— $H = \text{curl } A$.

Consider $\mu = (1, 0, 0)$ $V = \frac{\mu . r}{r^3} = \frac{x}{r^3}$

$$H_x = -\text{grad } V = -\left(\frac{1}{r^3} - \frac{3x^2}{r^5}, -\frac{3xy}{r^5}, -\frac{3xz}{r^5}\right)$$

$$\therefore H = \left(\frac{2x^2-y^2-z^2}{r^5}, \frac{3xy}{r^5}, \frac{3xz}{r^5}\right)$$

If $H_x = \text{curl } A$,
$$\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = \frac{2x^2-y^2-z^2}{r^5} \quad (1)$$
$$\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} = \frac{3xy}{r^5} \quad (2)$$
$$\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} = \frac{3xz}{r^5}. \quad (3)$$

These equations do not determine A uniquely.
We can get a solution by taking $A_1 = 0$.
Then $A_2 = -\frac{z}{r^3}$ $A_3 = \frac{y}{r^3}$ are suggested by (2) & (3), and do satisfy (1).

If the particle is at $(x' y' z')$ these become

$$A_1 = 0 \quad A_2 = -\frac{z-z'}{R^3} \quad A_3 = \frac{y-y'}{R^3}$$

where $R^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$

As these are all of form $f(x-x', y-y', z-z')$
$$\frac{\partial f}{\partial x} = -\frac{\partial f}{\partial x'} \quad \frac{\partial f}{\partial y} = -\frac{\partial f}{\partial y'} \quad \frac{\partial f}{\partial z} = -\frac{\partial f}{\partial z'}$$

If we take $\text{curl}' A = \left(\frac{\partial A_3}{\partial y'} - \frac{\partial A_2}{\partial z'}, \quad , \quad \right)$

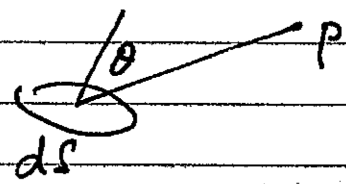
We must change sign of A. to get H_x

$$H_x = \text{curl}'\left(0, \frac{z-z'}{R^3}, -\frac{y-y'}{R^3}\right)$$

~~At the first lesson I said the course was for arithmetic, but we would make trips and only do any arithmetic that became necessary.~~

Magnetic shell.

Strength φ. Element $d\underline{S}$ behaves like magnet of moment $\varphi\, d\underline{S}$.

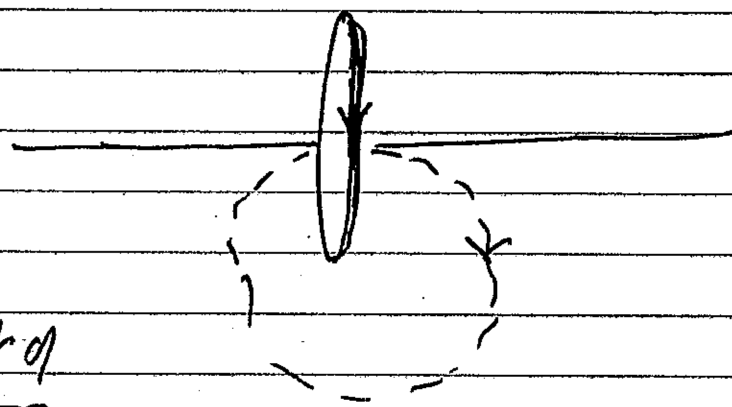


$$\varphi \frac{\mathbf{r} \cdot d\mathbf{S}}{r^3} = \varphi\, d\omega$$

ω being solid angle at

This is true if P lie on side of normal, negative for other side.

For an actual loop. $V = \varphi \Omega$.



A magnetic pole would move in broken line, the potential steadily decreasing.

Thus, just to right of disc, $V = 2\pi\varphi$.

To left $V = -2\pi\varphi$.

Change of potential $= 4\pi\varphi$.

Field of a dipole. $+m$ at $x=a$, $-m$ at $x=-a$

Force at x, for $x>a$ is

$$\frac{m}{(x-a)^2}-\frac{m}{(x+a)^2}$$

$$=\frac{m\cdot 4ax}{(x^2-a^2)^2}$$ If $a\to 0$, $2ma=\mu$, this is $\frac{2\mu}{x^3}$

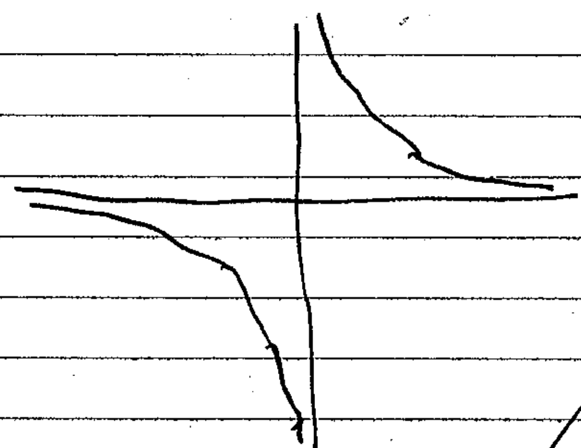
For $x<a$, force is $\cancel{m}$ $-s \quad -a \quad 0$

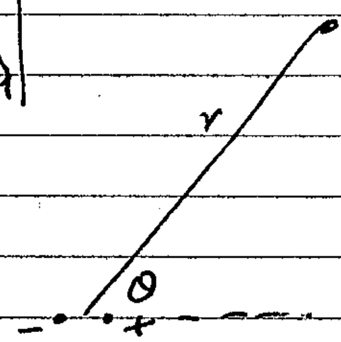
$x=-s$ $$\frac{m}{(s-a)^2}-\frac{m}{(s+a)^2}=\frac{4mas}{(s^2-a^2)^2}$$

$$\longrightarrow \frac{2\mu}{s^3}=-\frac{2\mu}{x^3} \text{ positive.}$$

Potential $x>a$ $$\frac{m}{x-a}-\frac{m}{x+a}=\frac{2ma}{x^2-a^2}\Longrightarrow\frac{\mu}{x^2}$$

$x<a$ $x=-s$ $$\frac{m}{s+a}-\frac{m}{s-a}=\frac{-2ma}{s^2-a^2}=\frac{-\mu}{x^2}$$

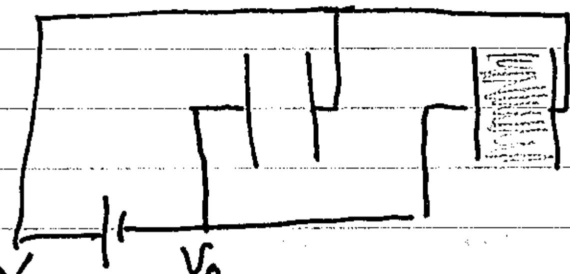




Potential $=\frac{\mu\cos\theta}{r^2}$

$=\frac{\mu\cdot x}{r^3}$

Dielectrics.
Jeans, Chap. V,

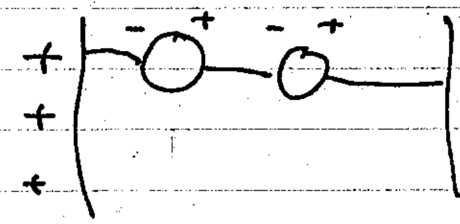


Charges per unit area are

$$\frac{V_1-V_0}{4\pi}\ \text{(air) and}\ K\frac{V_1-V_0}{4\pi}\ \text{(dielectric)}$$

Work done in taking unit charge across capacity is V_1-V_0 in each case. So intensity (i.e. $-\partial V/\partial x$) the same in both cases.

Molecular model
Molecules regarded as conductors, which become dipoles.

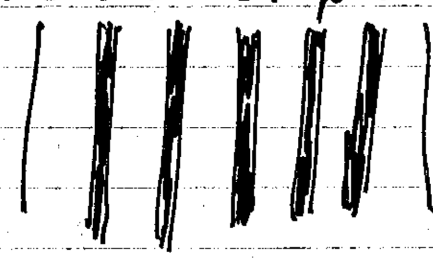


p47. Strength of a tube of force = charge at the end
Polarization P = aggregate strength of tubes per unit area.
Exptl result. Intensity $R = \frac{4\pi}{K}P$.

$P=(f,g,h)$ $\quad f=\frac{K}{4\pi}X \quad g=\frac{K}{4\pi}Y \quad h=\frac{K}{4\pi}Z$.

Gauss Theorem. $4\pi Q = \iint K N dS = -\iint K\frac{\partial V}{\partial n}dS$
$= -\iint K \,\mathrm{grad}\, V \,d\mathbf{S}$.

A very simplified model of a dielectric - put conducting pieces between plates



The $\frac{1}{c}$ business.

Work done in threading current j' is $4\pi i$

$$4\pi \iint j' dS = \cancel{4\pi \int j\,ds = 4\pi \iint \operatorname{curl} j\,ds}.$$

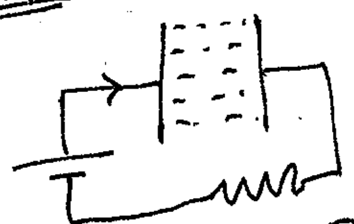
$$= \int H\,ds = \iint \operatorname{curl} H\,dS$$

j' in EMU

$$\operatorname{curl} H = 4\pi j'$$

$$\operatorname{curl} H = \frac{4\pi}{c} j$$

Displacement current. Use E.S.U.



Let q be charge density on plates of area A. $e = Aq$

$$i = \frac{de}{dt} = A\frac{dq}{dt}.$$

If E is electric intensity in capacitor $E = 4\pi q$

$$\text{so} \quad i = \frac{A}{4\pi}\frac{\partial E}{\partial t}$$

If current i spread evenly over area A, $i = Aj$

$$\text{so} \quad j = \frac{1}{4\pi}\frac{\partial E}{\partial t}$$

Thus we may regard $\frac{1}{4\pi}\frac{\partial E}{\partial t}$ as giving continuity to j. ~~div j~~

In space where actual currents exist and E is changing, we may regard $j + \frac{1}{4\pi}\frac{\partial E}{\partial t}$ as effective current.

$$\text{flux } E = 4\pi(\text{charge}) = 4\pi\rho \cdot d\tau$$

$$\operatorname{div} E = 4\pi\rho \qquad \frac{1}{4\pi}\operatorname{div} E = \rho.$$

$$\operatorname{div} j = -\frac{d\rho}{dt} \qquad \operatorname{div} \frac{1}{4\pi}\frac{\partial E}{\partial t} = \frac{\partial \rho}{\partial t}$$

Thus $\operatorname{curl} H = \frac{4\pi}{c}\left(j + \frac{1}{4\pi}\frac{\partial E}{\partial t}\right)$ is consistent.

Statement of electromagnetic theory.
Start with N30 leading to curl $H = 4\pi j'$.
Displacement current N6, N7.
N32. curl $E' = -\partial H/\partial t$.

N. 10 $\Box A = -4\pi j/c$
$\Box V = -4\pi\rho$.

See Jeans p 511 for $\frac{1}{c}$ in displacement current.

Jeans p452.

When a magnetic pole moves near electric currents work is done on it. This must come from the batteries maintaining the current.

Suppose extra EMF arranged so that currents remain constant. Pole m, current i. If ω is solid angle, the potential energy of the pole is $mi\omega$.

* In time dt work done on pole is $mi \frac{d\omega}{dt} dt$.

The current corresponds to a charge $i dt$ so the work done in keeping current fixed corresponds to EMF $m\, d\omega/dt$. Thus the pole must have set up an extra EMF $-m\frac{d\omega}{dt}$.

The number of tubes coming from the magnet is $4\pi m$, of which $m\omega$ pass through the circuit.

Thus induced EMF is $-dN/dt$, where N is number of tubes of induction through circuit.

THIS IS IN EMU.

$$\oint E' ds = -\iint \frac{\partial H}{\partial t} d\underset{\sim}{S}$$

$$\therefore \quad \text{curl } E' = -\frac{\partial H}{\partial t}.$$

$E' = cE$

p515

* The sign is correct. If $\frac{d\omega}{dt} > 0$, the current is pushing the pole "uphill".

~~curl H = 1/c ∂E/∂t~~

$$\text{curl } E = -\frac{1}{c}\frac{\partial H}{\partial t} \qquad E \text{ in ESU.}$$

$$H = \text{curl } A$$

$$\text{curl } E = -\frac{1}{c}\,\text{curl}\,\frac{\partial A}{\partial t}$$

$$\therefore \quad E = -\frac{1}{c}\frac{\partial A}{\partial t} - \text{grad } V$$

since grad V is the most general function with curl( ) $= 0$.

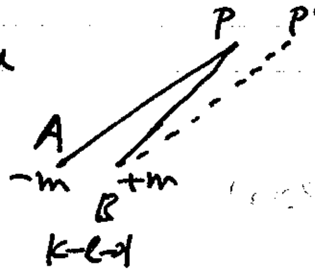
If no magnetic charges, $\frac{\partial A}{\partial t} = 0$

and $E = -\text{grad } V$ so we identify V with electrostatic potential.

E.M.U.

Force $= mm'/r^2$. Potential $= \int_r^\infty \frac{m}{x^2}\,dx = -\frac{m}{x}\Big]_r^\infty = \frac{m}{r}$.

Dipole



Potential at P $= \frac{m}{BP} - \frac{m}{AP} = \frac{m}{BP} - \frac{m}{BP'}$

$\varphi = -ml\frac{\partial}{\partial x}\left(\frac{1}{r}\right)$

$= M \cdot \frac{1}{r^2}\cdot\frac{x}{r} = Mx/r^3.$

$r^2 = x^2+y^2+z^2 \quad 2r\frac{\partial r}{\partial x} = 2x \quad \frac{\partial r}{\partial x} = \frac{x}{r}$ $\quad = \frac{M}{r^2}\cos\theta$

M = moment of magnetic particle.

<u>Magnetic shell of strength φ.</u> dS behaves like particle of moment $\varphi\, dS$.

Potential $V = \iint \varphi \frac{\underline{r}\cdot d\underline{S}}{r^3} = \varphi\Omega \quad \Omega$ = solid angle

[It can be deduced that force exerted at P can be taken as $\frac{\varphi\, d\underline{s} \wedge \underline{r}}{r^3}$ where $\underline{r}$ is vector from ds to P.]

Unit of current chosen so that $\varphi = i'$.

From solid angle it follows that work done in threading a circuit is $4\pi i$ for unit magnetic pole

So $\oint H\, dS = 4\pi \iint \underset{\sim}{j'}\, d\underset{\sim}{S}$

j is current in E.M.U.

$\therefore \quad \text{curl } H = 4\pi \underset{\sim}{j'}.$

4 vector

N 29

M 41

Coordinates $x_1\ x_2\ x_3,\ x_4 = ct.$

dx^1, dx^2, dx^3, dx^4 is a vector.

$ds^2 = dx_4^2 - dx_1^2 - dx_2^2 - dx_3^2$ is invariant.

$\therefore \dfrac{dx^1}{ds}, \dfrac{dx^2}{ds}, \dfrac{dx^3}{ds}, \dfrac{dx^4}{ds}$ is a vector.

If velocity is v with components

$v_x = \dfrac{dx^1}{dt}, v_y = \dfrac{dx^2}{dt}, v_z = \dfrac{dx^3}{dt}$

$$ds^2 = c^2dt^2 - (v_x^2 + v_y^2 + v_z^2)dt^2$$
$$= dt^2(c^2 - v^2)$$
$$= c^2dt^2\left(1 - \frac{v^2}{c^2}\right)$$
$$ds = c\,dt\sqrt{1-\frac{v^2}{c^2}}$$
$$dx^1 = v_x\,dt$$
$$\therefore \frac{dx^1}{ds} = \frac{v_x}{c\sqrt{1-\frac{v^2}{c^2}}}$$

dx^1 not dx' !

Similarly for $\dfrac{dx^2}{ds}$ and $\dfrac{dx^3}{ds}$.

$$\frac{dx^4}{ds} = \frac{c\,dt}{ds} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

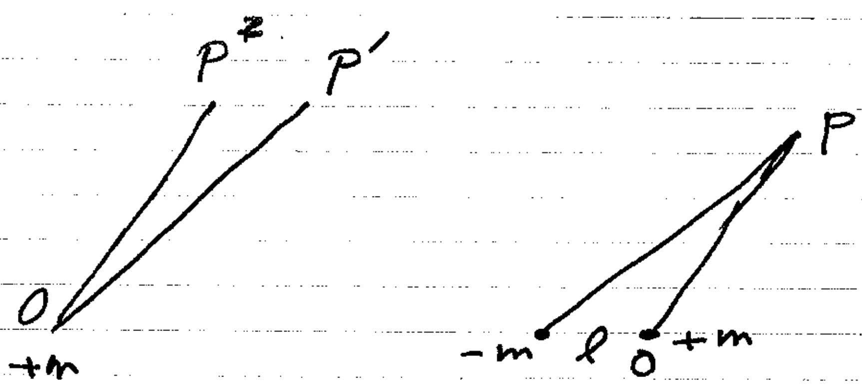
$$\therefore \frac{v_x}{c\sqrt{1-\frac{v^2}{c^2}}}, \frac{v_y}{c\sqrt{1-\frac{v^2}{c^2}}}\ \frac{v_z}{c\sqrt{1-\frac{v^2}{c^2}}}\ \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

is a vector.

Volumes shrink in ratio $\sqrt{1-\frac{v^2}{c^2}}$ so $\rho = \rho_0\Big/\sqrt{1-\frac{v^2}{c^2}}$

Multiply vector above by ρ_0.

We find $\dfrac{\rho v_x}{c}, \dfrac{\rho v_y}{c}, \dfrac{\rho v_z}{c}, \rho$ is a vector



+m produces potential $\frac{m}{OP}$ at P

The potential −m produces at P is the same as the potential −m at O would produce at P', distant l from P

Thus combined potential is $\frac{m}{OP} - \frac{m}{OP'}$

$$= -l\frac{\partial}{\partial x}\left(\frac{m}{r}\right) = -ml\frac{\partial}{\partial x}\left(\frac{1}{r}\right).$$

∴ Potential of magnetic particle of moment μ is $-\mu\frac{\partial}{\partial x}\left(\frac{1}{r}\right)$

$$r^2 = x^2+y^2+z^2 \qquad 2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Potential $= -\mu\left(-\frac{1}{r^2}\right)\frac{x}{r} = \frac{\mu x}{r^3}$.

N=7 M 43

$$\frac{\partial K_{ij}}{\partial x^k} + \frac{\partial K_{jk}}{\partial x^i} + \frac{\partial K_{ki}}{\partial x^j}.$$

$i=1,\ j=2,\ k=3$

$$\frac{\partial K_{12}}{\partial x^3} + \frac{\partial K_{23}}{\partial x^1} + \frac{\partial K_{31}}{\partial x^2}$$

$$= -\frac{\partial H_3}{\partial z} - \frac{\partial H_1}{\partial x} - \frac{\partial H_2}{\partial y} = 0.$$

$i=1,\ j=2,\ k=4$

$$\frac{\partial K_{12}}{\partial x^4} + \frac{\partial K_{24}}{\partial x^1} + \frac{\partial K_{41}}{\partial x^2}$$

$$= -\frac{\partial H_3}{ic\,\partial t} + \frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} = 0$$

$$\cancel{\frac{1}{c}\frac{\partial H_3}{\partial t} = i\left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}\right) = -\left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}\right)}$$

$$\frac{1}{c}\frac{\partial H_3}{\partial t} = i\left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}\right) = -\left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}\right)$$

as for $\frac{1}{c}\frac{\partial H}{\partial t} = -\operatorname{curl} E$.

W26 M 44

$$\frac{\partial K_{ij}}{\partial x_j} = \frac{\partial K_{i1}}{\partial x_1} + \frac{\partial K_{i2}}{\partial x_2} + \frac{\partial K_{i3}}{\partial x_3} + \frac{\partial K_{i4}}{\partial x_4}$$

$$= \frac{\partial K_{i1}}{\partial x} + \frac{\partial K_{i2}}{\partial y} + \frac{\partial K_{i3}}{\partial z} + \frac{\partial K_{i4}}{ic\partial t}$$

We then have

(1) $$-\frac{\partial H_3}{\partial y} + \frac{\partial H_2}{\partial z} + \frac{\partial E_1}{ic\partial t} = 0$$

This is $$-\frac{\partial H_3}{\partial y} + \frac{\partial H_2}{\partial z} + \frac{1}{c}\frac{\partial E_1}{\partial t} = 0$$

so $$\frac{1}{c}\frac{\partial E_1}{\partial t} = \frac{\partial H_3}{\partial y} - \frac{\partial H_2}{\partial z}.$$

(2) $$\frac{\partial H_3}{\partial x} - \frac{\partial H_1}{\partial z} + \frac{\partial E_2}{ic\partial t} = 0$$

$$\frac{1}{c}\frac{\partial E_2}{\partial t} = \frac{\partial H_1}{\partial z} - \frac{\partial H_3}{\partial x}$$

(3) $$-\frac{\partial H_2}{\partial x} + \frac{\partial H_1}{\partial y} + \frac{\partial E_3}{ic\partial t} = 0$$

$$\frac{1}{c}\frac{\partial E_3}{\partial t} = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}$$

(4) $$+\frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z} = 0.$$

(N25)

$\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\frac{\partial}{\partial z}$

$A_1\ A_2\ A_3$

M 45

$$K_{12} = \frac{\partial s_1}{\partial x_2} - \frac{\partial s_2}{\partial x_1} = \frac{\partial A_1}{\partial y} - \frac{\partial A_2}{\partial x}$$
$$= -H_3$$

$$K_{13} = \frac{\partial s_1}{\partial x_3} - \frac{\partial s_3}{\partial x_1} = \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} = H_2$$

$$K_{14} = \frac{\partial s_1}{\partial x_4} - \frac{\partial s_4}{\partial x_1} = \frac{\partial A_1}{ci\partial t} - \frac{\partial iV}{\partial x}$$
$$= i\left[-\frac{\partial A_1}{c\partial t} - \frac{\partial V}{\partial x}\right]$$
$$= iE_1 = \mathcal{E}_1$$

$$K_{23} = \frac{\partial s_2}{\partial x_3} - \frac{\partial s_3}{\partial x_2} = \frac{\partial A_2}{\partial z} - \frac{\partial A_3}{\partial y} = -H_1$$

$$K_{24} = \frac{\partial s_2}{\partial x_4} - \frac{\partial s_4}{\partial x_2} = \frac{\partial A_2}{ci\partial t} - \frac{\partial iV}{\partial y}$$
$$= +i\left(-\frac{\partial A_2}{c\partial t} - \frac{\partial V}{\partial y}\right) = iE_2 = \mathcal{E}_2$$

$$K_{34} = \frac{\partial s_3}{\partial x_4} - \frac{\partial s_4}{\partial x_3} = \frac{\partial A_3}{ic\partial t} - \frac{\partial iV}{\partial z}$$
$$= +i\left[-\frac{\partial A_3}{c\partial t} - \frac{\partial V}{\partial z}\right] = iE_3$$

Thus

$$K_{ij} = \begin{pmatrix} \cdot & -H_3 & H_2 & \mathcal{E}_1 \\ H_3 & \cdot & -H_1 & \mathcal{E}_2 \\ -H_2 & H_1 & \cdot & \mathcal{E}_3 \\ -\mathcal{E}_1 & -\mathcal{E}_2 & -\mathcal{E}_3 & \cdot \end{pmatrix}$$

Let $E = \text{curl } S$

$$\text{curl } H = \frac{1}{c}\frac{\partial E}{\partial t} = \frac{1}{c}\,\text{curl}\,\frac{\partial S}{\partial t}$$

$$\therefore \quad \text{curl}\left(H - \frac{1}{c}\frac{\partial S}{\partial t}\right) = 0$$

$$\therefore \quad H - \frac{1}{c}\frac{\partial S}{\partial t} = \text{grad } \varphi$$

$$H = \frac{1}{c}\frac{\partial S}{\partial t} + \text{grad } \varphi.$$

$$-\frac{1}{c}\frac{\partial H}{\partial t} = \text{curl } E = \text{curl curl } S$$

$$= \text{grad div } S - \nabla^2 S$$

$$-\frac{1}{c^2}\frac{\partial^2 S}{\partial t^2} - \frac{1}{c}\,\text{grad}\,\frac{\partial \varphi}{\partial t} =$$

$$\nabla^2 S - \frac{1}{c^2}\frac{\partial^2 S}{\partial t^2} = -\text{grad}\left[\text{div } S + \frac{1}{c}\frac{\partial \varphi}{\partial t}\right)$$

Take $\text{div } S + \frac{1}{c}\frac{\partial \varphi}{\partial t} = 0.$

$$\nabla^2 S - \frac{1}{c^2}\frac{\partial^2 S}{\partial t^2} = 0$$

$$0 = \text{div } H = \frac{1}{c}\,\text{div}\,\frac{1}{c}\frac{\partial S}{\partial t} + \text{div grad } \varphi$$

$$= -\frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} + \nabla^2 \varphi.$$

Take (S, φ) to be 4-vector

Jeans p 474 with c inserted (N26) M 46

$$E = -\frac{1}{c}\frac{\partial A}{\partial t} - \text{grad } V$$

$$\text{div } A + \frac{1}{c}\frac{\partial V}{\partial t} = 0 \qquad \text{(These notes } N10\text{)}$$

$$\text{div } f + \frac{1}{c}\frac{\partial \varphi}{\partial t} = 0$$ Kopff p. 50.

$$H = -\text{curl } f$$

$$E = \frac{1}{c}\frac{\partial f}{\partial t} + \text{grad } \varphi$$

These agree if $f = -A$, $\varphi = -V$.

$\left(\frac{\rho V_x}{c}, \frac{\rho V_y}{c}, \frac{\rho V_z}{c}, \rho\right)$ is a 4 vector

for $\xi_1 = x, \xi_2 = y, \xi_3 = z, \xi_4 = ct$

Let $x_1 = \xi_1 \; x_2 = \xi_2 \; x_3 = \xi_3 \; x_4 = i\xi_4$.

This is transformation for a vector. Hence in the system with $x_1^2 + x_2^2 + x_3^2 + x_4^2 = s^2$

a vector is $\left(\frac{\rho V_x}{c}, \frac{\rho V_y}{c}, \frac{\rho V_z}{c}, i\rho\right)$.

p (N10).

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\frac{4\pi j}{c}$$

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -4\pi\rho$$

$$\therefore \nabla^2 iV - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} iV = -4\pi\rho i$$

$\therefore$ (A_1, A_2, A_3, iV) is a vector.

Let $s_1 = A_1 \; s_2 = A_2 \; s_3 = A_3 \; s_4 = iV$

Let $$K_{ij} = \frac{\partial s_i}{\partial x_j} - \frac{\partial s_j}{\partial x_i}$$

$$\frac{A_{r+2}}{A_r} = \cancel{\sum_{i=0} \frac{\lambda_{i,i}}{\lambda_{ii}}} + \frac{\lambda_{r+2,r+2}}{\lambda_{rr}} + \frac{\lambda_{r+1,r+1}}{\lambda_{r-1,r-1}} + \frac{\lambda_{r+1,r+2}}{\lambda_{rr}} \quad (g)$$

$$(d) \quad + \frac{\lambda_{r+1,r+1} + \lambda_{r,r+1}}{\lambda_{rr}} \cdot \sum_{0}^{r-1} \frac{\lambda_{i,i+1}}{\lambda_{ii}} + \sum\sum_{0 \le i < j \le r-1} \frac{\lambda_{i,i+1}}{\lambda_{ii}} \frac{\lambda_{j,j+1}}{\lambda_{jj}} \quad (e)$$

Take Δ:

$$(a) \quad \Delta \frac{A_{r+2}}{A_r} \quad (b) \quad \boxed{\frac{\lambda_{r+1,r+3}}{\lambda_{r+1,r+1}}} \quad (c) \quad \frac{\lambda_{r+3,r+3}}{\lambda_{r+1,r+1}} - \frac{\lambda_{r+1,r+1}}{\lambda_{r-1,r-1}}$$

covers top row.

$$(f) \quad \frac{\lambda_{r+2,r+3}}{\lambda_{r+1,r+1}} - \frac{\lambda_{r+1,r+2}}{\lambda_{rr}}$$

$$\begin{aligned} \Delta(a_r b_r) &= a_{r+1} b_{r+1} - a_r b_r \\ &= b_{r+1}(a_{r+1} - a_r) + a_r(b_{r+1} - b_r) \\ &= b_{r+1} \Delta a_r + a_r \Delta b_r \end{aligned}$$

$$\varphi(r) = \sum_{i=0}^{r} \frac{\lambda_{i,i+1}}{\lambda_{ii}}$$

$$\Delta \frac{\lambda_{r+1,r+1} + \lambda_{r,r+1}}{\lambda_{rr}} \varphi(r-1) = \frac{\lambda_{r+1,r+1} + \lambda_{r,r+1}}{\lambda_{rr}} \Delta\varphi_{r-1} + \varphi_r \Delta \text{———}$$

$$(d) \quad = \frac{\lambda_{r+1,r+1} + \lambda_{r,r+1}}{\lambda_{rr}} \frac{\lambda_{r,r+1}}{\lambda_{rr}} + \varphi(r) \Delta \frac{\lambda_{r+1,r+1} + \lambda_{r,r+1}}{\lambda_{rr}}$$

$$\Delta \sum_{0 \le i < j \le r-1} \frac{\lambda_{i,i+1}}{\lambda_{ii}} \frac{\lambda_{j,j+1}}{\lambda_{jj}}$$

If r increase, the only new terms are those with $j = r$

So $$\frac{\lambda_{r,r+1}}{\lambda_{rr}} \cdot \sum_{0}^{r-1} \frac{\lambda_{i,i+1}}{\lambda_{ii}} = \frac{\lambda_{r,r+1}}{\lambda_{rr}} \varphi(r) \quad (e)$$

$$\frac{\lambda_{r+1,r+1}}{\lambda_{rr}} = \frac{(2r+1)^2(2r+2)^2}{(4r+1)(4r+3)^2(4r+5)} \qquad \frac{\lambda_{r+2,r+2}}{\lambda_{r+1,r+1}} = \frac{(2r+3)^2(2r+4)^2}{(4r+5)(4r+7)^2(4r+9)}$$

$$\frac{\lambda_{r+2,r+2}}{\lambda_{rr}} = \frac{(2r+1)^2(2r+2)^2(2r+3)^2(2r+4)^2}{(4r+1)(4r+3)^2(4r+5)^2(4r+7)^2(4r+9)}$$

$$+ \quad \frac{\lambda_{r+3,r+3}}{\lambda_{r+1,r+1}} = \frac{(2r+3)^2(2r+4)^2(2r+5)^2(2r+6)^2}{(4r+5)(4r+7)^2(4r+9)^2(4r+11)^2(4r+13)}$$

$$(c) \quad - \quad \frac{\lambda_{r+1,r+1}}{\lambda_{r-1,r-1}} = \frac{(2r-1)^2(2r)^2(2r+1)^2(2r+2)^2}{(4r-3)(4r-1)^2(4r+1)^2(4r+5)^2(4r+7)}$$