I3

A NOTE ON INTEGRATION BY PARTS.

Warwick Sawyer.

Integration by parts appears to be a trick specially designed for dealing with certain problems in integration. There is, however, a way of looking at it, by which it is seen not as a trick but as something that arises very naturally. You will meet this viewpoint in a few years time, if you go on to a particular kind of advanced mathematics (the theory of Stieljes integration). It is not for a moment suggested that the argument outlined below should replace the usual proof in the A-level textbooks; it does however throw an interesting sidelight on the nature of integration by parts.

The usual explanation is the following. The formula for differentiating the product $u(x)v(x)$ is $(uv)' = uv' + u'v$, so we have $uv' = (uv)' - u'v$.

If we integrate from a to b we get

$$\int_a^b uv'dx = [uv]_a^b - \int_a^b u'v\, dx$$

which we may write in the form

$$\int udv = [uv]_a^b - \int v\, du \;.......(I)$$

The first integral has to be taken from $v=v_a$ to $v=v_b$ and the last integral from $u=u_a$ to $u=u_b$, where $v_a=v(a)$, $v_b=v(b)$, $u_a=u(a)$ and $u_b=u(b)$. This follows from the usual rule for change of variable in integration. The integral, $\int udv$, can be visualized as the area under a curve. As x grows from a to b, $u(x)$ and $v(x)$ change continuously and on graph paper the point $(v(x) , u(x))$ will describe a curve as shown in Figure 1. The integral $\int udv$ represents the area under this curve, and we are interested in the part lying between v_a and v_b.

We can make a rough estimate of this area in the usual way. The rectangles under the dotted lines in the diagram have the total area

$$u_1(v_1-v_a)+u_2(v_2-v_1)+u_3(v_3-v_2)+u_4(v_4-v_3)+u_5(v_b-v_4)\quad ..(II)$$

The values u_1, u_2, u_3, u_4 and u_5 are for points chosen arbitrarily within the appropriate intervals.

The exact value of the integral is the limit of this sum when the number of intervals tends to infinity and the length of every interval tends to zero.

In its present form, expression (II) shows each u multiplied by a bracket containing two v-symbols. It could equally well be written so as to show each v multiplied by brackets containing u-symbols. It would then appear as

$$\left.\begin{array}{l} -v_1(u_2-u_1)-v_2(u_3-u_2)-v_3(u_4-u_3)-v_4(u_5-u_4) + \\ \\ \qquad + u_5v_b - u_1v_a \end{array}\right) \quad ....(III)$$

If we were concerned only to show in a general way that algebra suggests the existence of an operation such as integration by parts, we could stop here. The first line of (III) is clearly a rough estimate for some integral

I5

$-\int v\,du$, and the second line has products of the form uv taken at or near the ends of the interval.

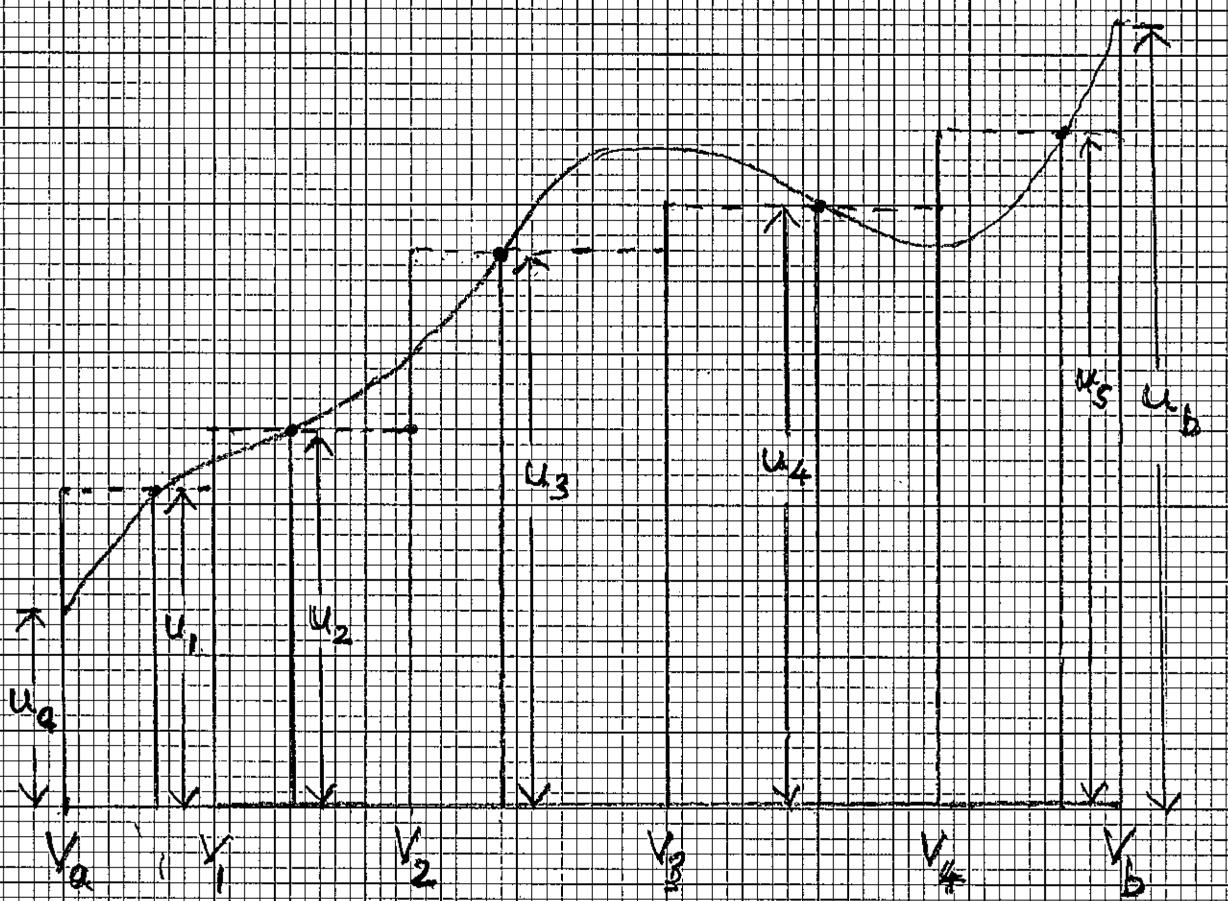
If we want to go into it a bit more closely, there are some loose ends that need to be tidied up. On the one hand, $[uv]_a^b$ ought to be $u_b v_b - u_a v_a$, which is not the second line in (III) as it stands at present . On the other hand, the changes of u in the top line involve only values lying between u_1 and u_5. We ought to have $u_1 - u_a$ at the beginning and $u_b - u_5$ at the end, if the sum is to correspond to an integral from u_a to u_b. However these two discrepancies cancel out. If we write the second line so that it starts with the correct expression for $[uv]_a^b$, and follow this with terms needed to give the expression as it is at present in the second row of (III), we obtain

$$(u_b v_b - u_a v_a) - v_a(u_1 - u_a) - v_b(u_b - u_5)$$

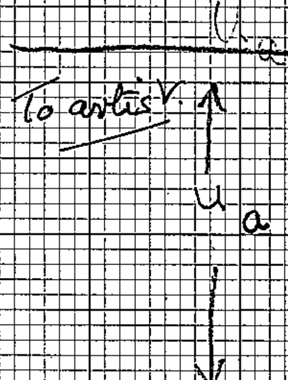
and we now have the terms with u-differences needed for estimating an integral from u_a to u_b.

A similar calculation could be made with any number of intervals, and the limit for indefinitely fine subdivisions would give equation (I).

I 6



To artist.



Note. Label on first upright line is u_a not u_0.

Label on last upright is u_b not u_6.