Functions and Mappings

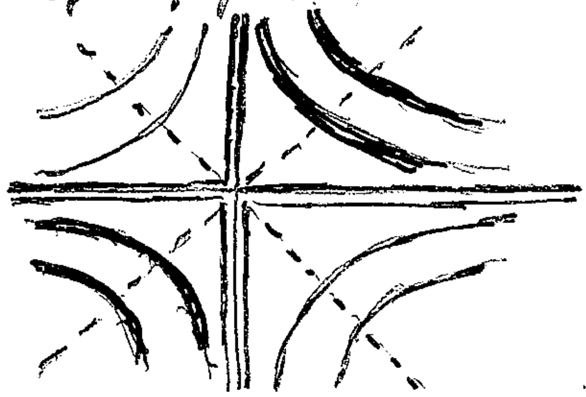
$w = \sqrt{z} \iff z = w^2$.

Such an equation makes points of the w plane correspond to points of the z plane. On either plane we can indicate the values of the other variable.

Let $z = x+iy$, $w = u+iv$.

$x+iy = (u+iv)^2 = u^2-v^2 + i.2uv$.

Thus, in the w-plane the curves $2uv = k$ correspond to $y = k$, points in the z plane with a fixed imaginary part.



ws plane

If $w = re^{i\theta}$, $z = r^2e^{i2\theta}$

~~w will be in upper half plane $\Im(w)>0$ if~~

~~$0<\theta<\pi$~~ z will be in upper half plane if $0 < 2\theta < \pi$ i.e. $0<\theta<\frac{\pi}{2}$. So first quadrant of ws plane maps to $\Im(z)>0$.

If $\frac{\pi}{2} < \theta < \pi$, then $\pi < 2\theta < 2\pi$.

So second quadrant of w-plane maps to $\Im(z)<0$.

~~Curves drawn in ink~~. Thick $\to$ upper half plane.

In third quadrant, u, v both minus, so $2uv$ positive and $\Im(z)>0$. Drawn in pencil

In fourth quadrant, u +, v − and $2uv<0$.

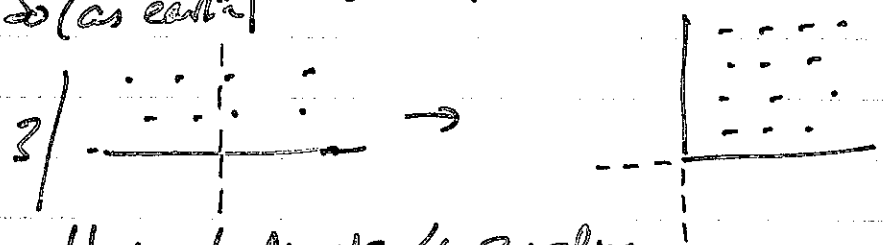
As we would expect, each value of z appears for 2 values of w. The whole w plane is mapped twice to the z plane.

FM2
C42

$w=\sqrt{z}$ If $z=re^{i\theta}$ $w=\sqrt{r}\,e^{i\theta/2}$

As $\arg z$ goes from 0 to π
$\arg w$ goes from 0 to $\pi/2$

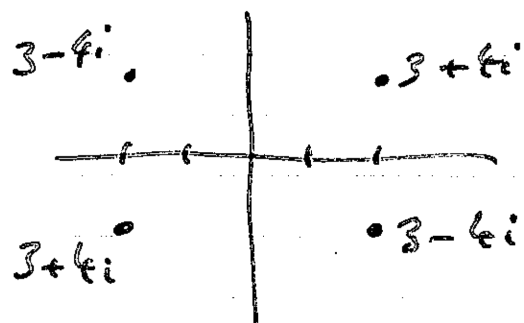
So (as earlier)



If we look at 4 z values

w	z
$2+i$	$3+4i$
$2-i$	$3-4i$
$-2+i$	$3-4i$
$-2-i$	$3+4i$

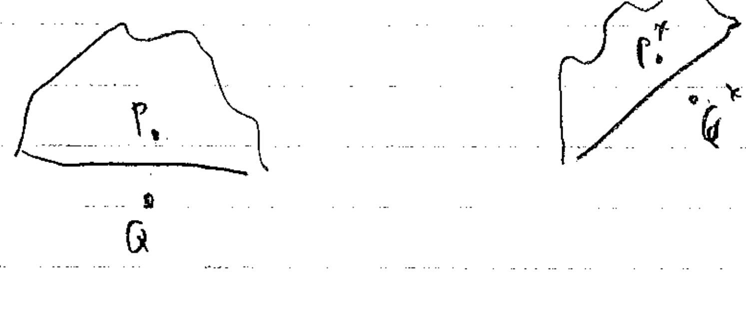
w plane
z-values written



The values are as shown by 2 mirrors that change i to $-i$.

It will be seen later that, in a case such as this, what happens in the whole plane can be deduced by the reflection principle from what happens in the first quadrant.

It is often useful to examine what region the upper half plane, $f(z)$ maps to.



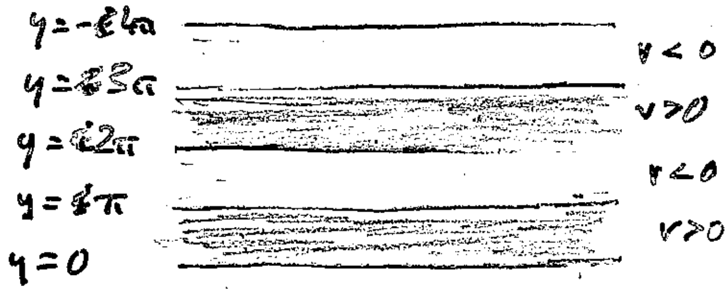
$w = e^{z}$ $\qquad z = \ln w$.

$z = x+iy \qquad u+iv = e^{x+iy} = e^{x}\cos y + i\, e^{x}\sin y$

$u = e^{x}\cos y \qquad v = e^{x}\sin y$.

Cosine and sine are periodic, so $y + 2n\pi$ will give the same value for w as y.

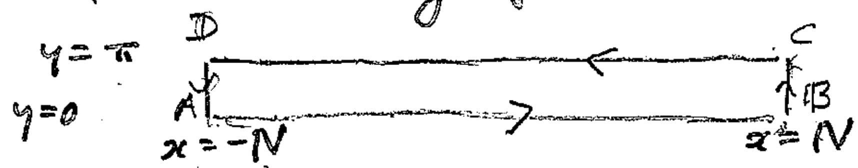
Upper half plane for w has $v>0$.
$v>0$ if $\sin y>0$
so $0<y<\pi$
$2\pi<y<3\pi$ etc



The regions of the z plane that correspond to ~~lower~~ upper half of w plane are shaded.

If we fix y, the factor e^{x} rises from 0 for $x=-\infty$ to ∞ for $x=+\infty$.

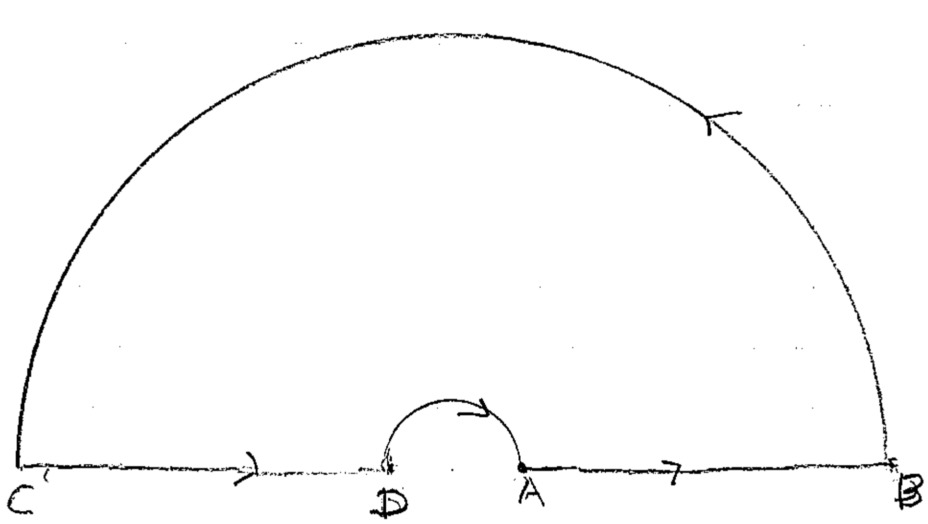
If we take a z-region



the left end maps to $e^{-N}(\cos\theta + i\sin\theta)$ $\quad 0\le\theta\le\pi$

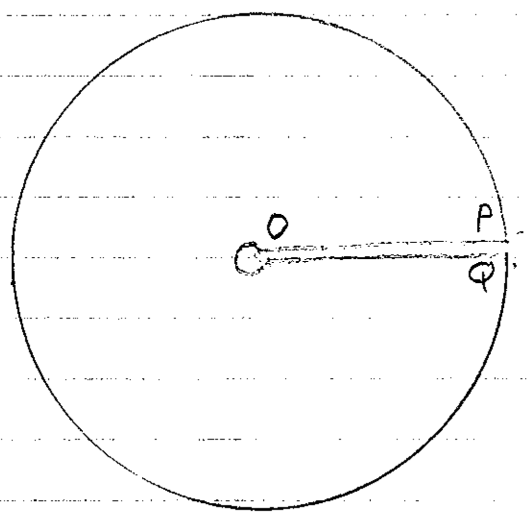
right hand to $e^{N}(\cos\theta + i\sin\theta)$

i.e to a small semicircle and a large one



On this region in the w-plane we could write at each point a value of z from the region ABCD.

If we take y from 0 to 2π, we get a complete circle (except for a puncture at the centre) and on each point of this circle we can write a value of $\ln w$ with imaginary part in $(0, 2\pi)$.



However on OP $y = 0$
on OQ $y = 2\pi$
so we cannot join them.

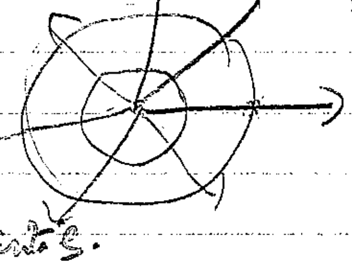
If we want to have a sheet on which we can write each value of $\log w$ just once, we need an infinite number of such pieces, and we must join OQ to OP on the sheet above, and OP to OQ on the sheet below. The resulting sheet is called the Riemann surface for $\ln$.

The path of a point on this surface can be seen more easily on the z plane. A point that moves vertically up the z plane goes round and round the origin in the w plane or Riemann surface.

If $w = re^{i\theta}$, $z = \ln r + i\theta$. Thus x = constant gives circles, y = constant lines through the origin. We can interpret the circles as contour lines, the straight lines as lines of flow.

$\ln w = \infty$ if $w = 0$ or ∞.

Note that fluid emerges from $w = 0$ and disappears at ∞. Generally, ~~space~~ places where fluid is created or destroyed correspond to singularities.



Observe how fluid leaving origin gives increase in stream fn. for circuit of O.

FMS

C 45

An interesting function is $w = \sin^{-1} z$. Its mapping will be studied first by elementary means, with use of everything we know about sine functions. Later it will be shown that, by using certain principles, most of the facts can be deduced very simply from quite limited information.

$$w = \sin^{-1} z \quad \therefore \quad z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}$$

$$w = u + iv, \quad z = \frac{e^{-v+iu} - e^{v-iu}}{2i}$$

$$= \frac{1}{2i}\left[e^{-v}(\cos u + i\sin u) - e^{v}(\cos u - i\sin u)\right]$$

$$= \frac{1}{2i}\left[\cos u\,(e^{-v} - e^{v}) + i\sin u\,(e^{-v} + e^{v})\right]$$

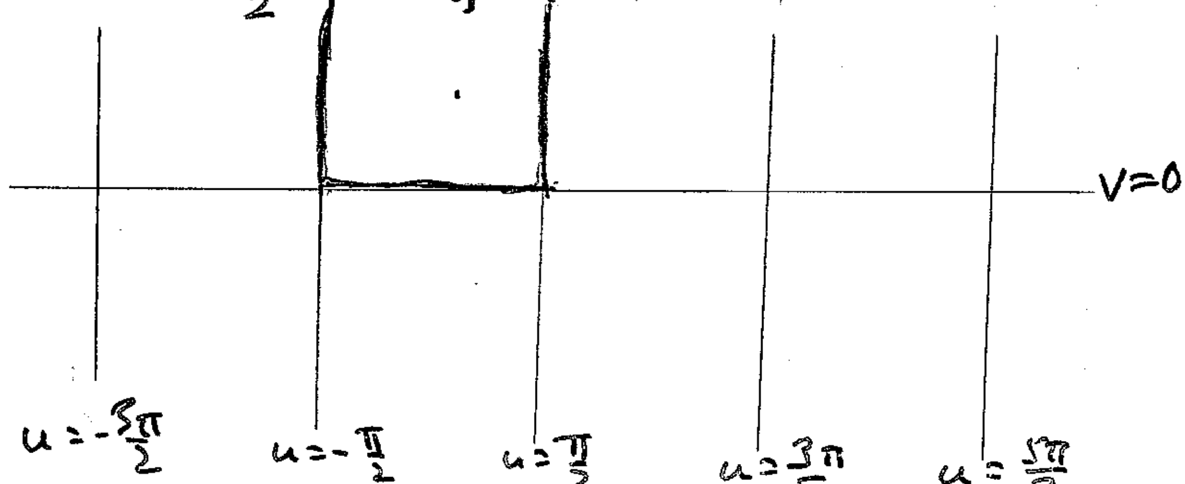
So $\quad x + iy = \sin u \cosh v + \cancel{i\sin u}\; i\cos u \sinh v.$

$x = \sin u \cosh v \qquad y = \cos u \sinh v.$

We consider how upper half plane of z maps to w.
i.e $\Im(z) > 0$, $y > 0$.

$y = 0$ where $\cos u = 0$ or $\sinh v = 0$.

Thus $u = \frac{\pi}{2} + n\pi$ or $v = 0$.



Consider $-\frac{\pi}{2} < u < \frac{\pi}{2}$. For these values $\cos u > 0$.

So $y > 0 \Leftrightarrow v > 0$. For positive values of v, $\sinh v$ runs from 0 to $+\infty$. $\quad y = \cos u \sinh v \Leftrightarrow \sinh v = \frac{y}{\cos u}$
and for any $y > 0$, there will be a +ve value of v that satisfies this equation.

For a given value of y, v will ~~be least~~ when $\cos u$ is greatest, i.e when $u = 0$. So each v curve has

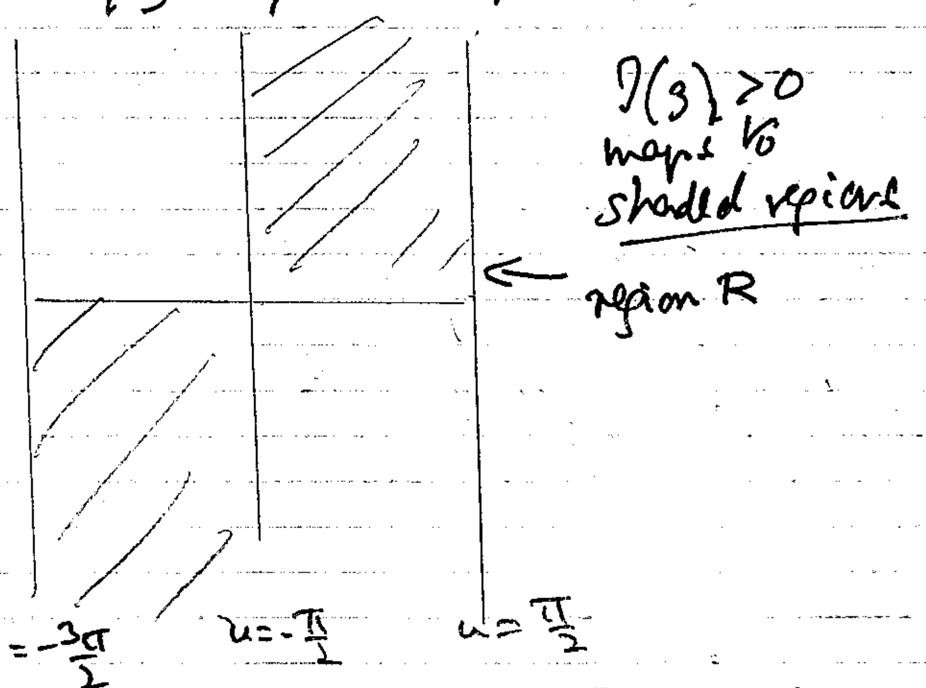
~~a minimum when u = 0.~~

$y = \cos u \sinh v$.

$y > 0$ if $\cos u$ & $\sinh v$ are both positive or both negative.

If $-\frac{\pi}{2} < u < \frac{\pi}{2}$, $\cos u > 0$, so upper halfplane of z (which has $y > 0$) maps to region with $\sinh v > 0$, i.e. $v > 0$.

If $-\frac{3\pi}{2} < u < -\frac{\pi}{2}$, $\cos u < 0$ so upper halfplane of z maps to region with $v < 0$.



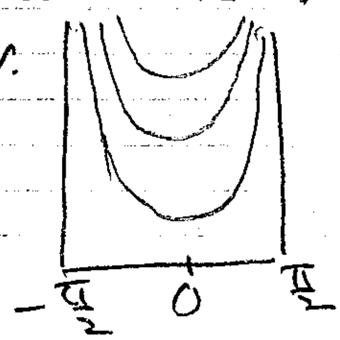
We now consider $-\frac{\pi}{2} < u < \frac{\pi}{2}$, $v > 0$.

$$\sinh v = \frac{y}{\cos u}.$$

Consider a curve y = constant. As u goes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, $\cos u$ rises from 0 at $-\frac{\pi}{2}$ to 1 at $u=0$, and then decreases to 0 at $\frac{\pi}{2}$.

Thus $\sinh v$ decreases ~~from~~ as u goes from $-\frac{\pi}{2}$ to 0, has a minimum at $u = 0$ and rises steadily as u increases from 0 to $\frac{\pi}{2}$. So v does same. As $\cos(-u) = \cos u$, the graph y = constant is symmetrical about the v-axis.

So graphs y = const. appear as here

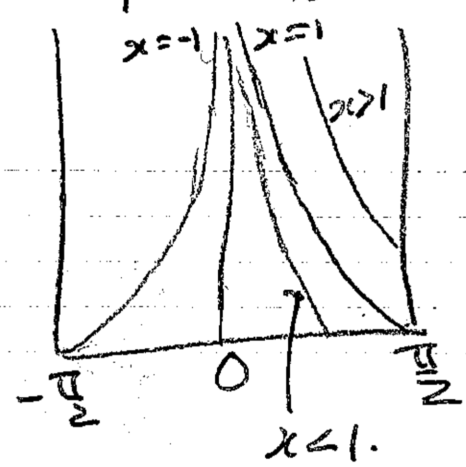


$x = \sin u \cosh v$.

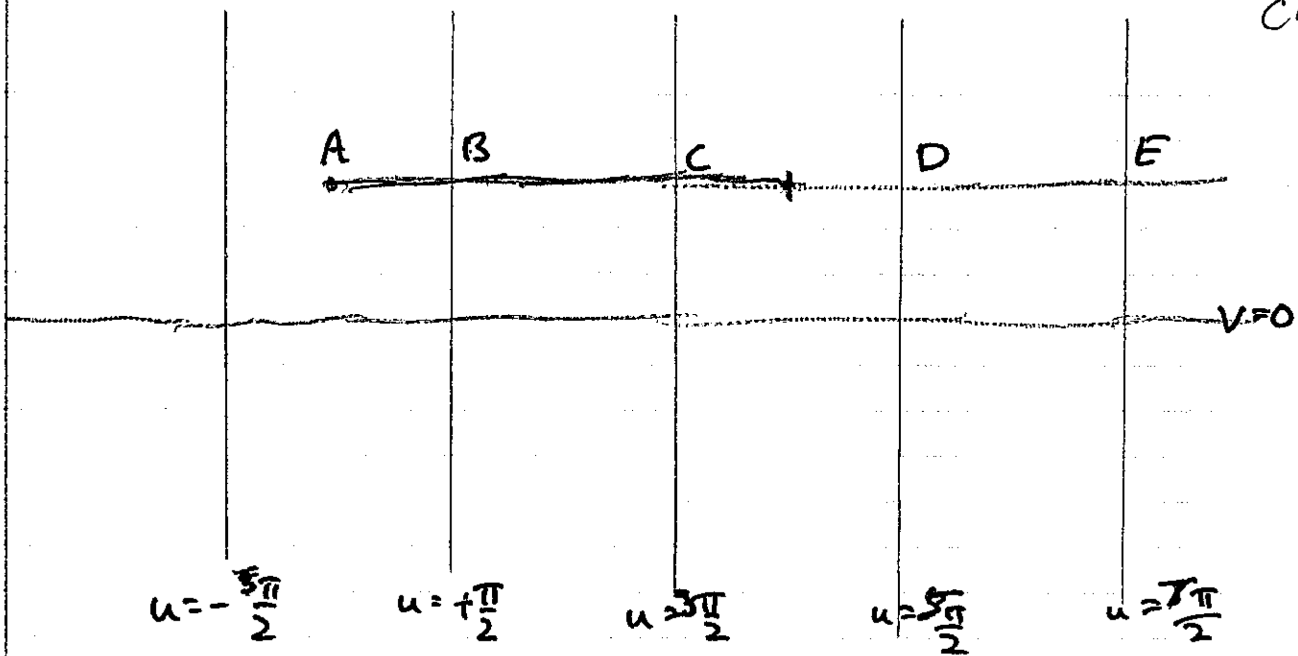
$\cosh v \geqslant 1$ for all v. If $x \neq 0$,
$\cosh v = \dfrac{x}{\sin u}$ $\therefore$ as $u \to 0$, $\cosh v \to \infty$
$\therefore$ $v \to \infty$ or $-\infty$. Within region R,
$v > 0$, so $v \to \infty$ is the only possibility.
For constant x, for $u > 0$, $\cosh v = \dfrac{x}{\sin u}$
so $\cosh v$ decreases as u increases.

For $v = 0$, $\cosh v = 1$ so $x = \sin u$.
Hence the graph of x = constant meets
$v = 0$ only if $-1 \leq x \leq 1$.
If $x > 0$, $v = 0$ then $\sin u > 0$
so $0 < u < \frac{\pi}{2}$ for points in region R.

If u changes to $-u$, x changes to $-x$.
~~Thus~~ If $x > 1$, $u = \frac{\pi}{2}$, $\cosh v = x$ has
a solution. So curves ~~with $x > 1$~~ const. x
with $x > 1$ meet boundary of R on line
$u = \frac{\pi}{2}$.
$x = 1$ if $u = \frac{\pi}{2}$, $v = 0$.



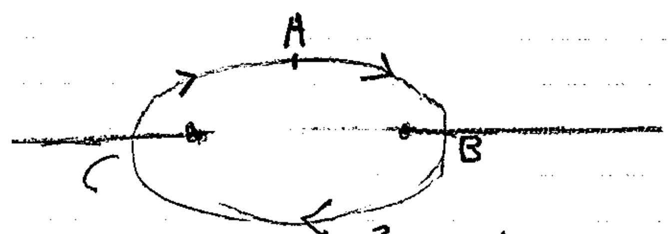
Pass to her
before 11?



A is the point $(0, k)$. ~~If~~ We start at A and move to the right along ABCDE

$$x = \sin u \cosh k \qquad y = \cos u \sinh k.$$

If u increases by 2π, x and y have original values.
If $x = a \sin u$, $y = b \cos u$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
Thus x, y describes an ellipse. Here $a = \cosh k$, $b = \sinh k$. We suppose $k > 0$, so $a > 0$, $b > 0$.
At B, $u = \frac{\pi}{2}$, $x = \cosh k$ so $x > 1$. $y = 0$.
From B to C, $\cos u < 0$, and $\sin u$ steadily decreases from $+1$ to -1. At C, $y = 0$, $x = -\cosh k < -1$.
(x, y) starts at $(0, \sinh k)$, the highest point of the ellipse, and comes there again when $u = 2\pi$, between C and D.



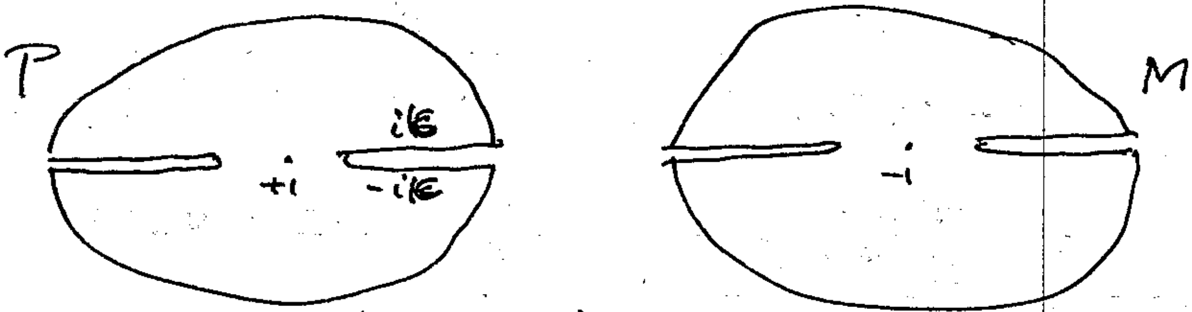
Now $\sin^{-1} z = \int_0^z \frac{dt}{\sqrt{1-t^2}}$

$\sqrt{1-t^2}$ we take real and positive for $-1 < t < 1$ so that, ~~for example,~~ $\sin^{-1} z$ is real and positive for $0 < z < 1$.
Now $\sqrt{1-t^2} = (1-t)^{\frac{1}{2}}(1+t)^{\frac{1}{2}}$ and the analytic continuation of these involves a change of sign for a circuit of $t = 1$ or of $t = -1$. In order to have $\sqrt{1-t^2}$ defined we need cuts that prevent such

FM9 C49

circuits. We suppose cuts made for $1 \le z < +\infty$ and $-\infty < z \le -1$, z being real.

We now have two planes, P and M, each with these cuts. On plane P, $\frac{1}{\sqrt{1-z^2}}$ is defined as having value $+1$ at $z=0$. For the rest of this plane it is defined by analytic continuation.



Consider the values of the function on the edges of the cut.

Suppose we take $(z-1)^{-\frac{1}{2}}$ as being ~~real for~~ +ve for $z>1$, and $(z+1)^{-\frac{1}{2}}$ as being +ve for $z>-1$.

$$\frac{1}{\sqrt{1-z^2}} = \frac{1}{\sqrt{-(z+1)(z-1)}} = \frac{1}{\sqrt{-1}}(z+1)^{-\frac{1}{2}}(z-1)^{-\frac{1}{2}}.$$

On P, we go from $z>1$ to $z<1$ by the path $\curvearrowleft$ Here $z = 1 + re^{i\theta}$ $0 \le \theta \le \pi$.

$z-1 = re^{i\theta}$ $(z-1)^{-\frac{1}{2}} = \frac{1}{\sqrt{r}}e^{-\frac{1}{2}i\theta}$.

This is real positive for $z>1$, since $\theta = 0$.

For $z<1$, $\theta = \pi$ $e^{-\frac{1}{2}i\pi} = \cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2})$
$= -i$.

$(z+1)^{-\frac{1}{2}}$ for $-1<z<1$ is real, +ve

$\therefore$ $(z+1)^{-\frac{1}{2}}(z-1)^{-\frac{1}{2}}$ is $-ik$ where $k>0$.

Hence we need to take $\frac{1}{\sqrt{-1}}$ above as i, to make $\frac{1}{\sqrt{1-z^2}}$ +ve in $(-1,1)$. So $\frac{1}{\sqrt{1-z^2}} = i(z-1)^{-\frac{1}{2}}(z+1)^{-\frac{1}{2}}$ on the plane P, $(z-1)^{-\frac{1}{2}}$ and $(z+1)^{-\frac{1}{2}}$ being the branches defined above. For $z>1$, both $(z-1)^{-\frac{1}{2}}$ and $(z+1)^{\frac{1}{2}}$ are +ve real, so $\frac{1}{\sqrt{1-z^2}}$ is ik ~~above~~ on the upper side of the cut.

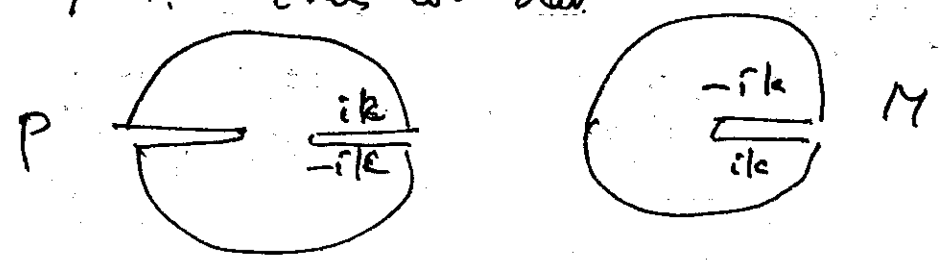
FM 10
C50

On the lower side of the cut $-1 \leq z < +\infty$, $(1+z)^{-\frac{1}{2}}$ is real and positive while on the lower side $\theta = 2\pi$ in $\frac{1}{\sqrt{r}} e^{-\frac{1}{2} i\theta} = (z-1)^{-\frac{1}{2}}$

$\therefore$ $\cancel{(1+z)^{-\frac{1}{2}}}$ $(z-1)^{-\frac{1}{2}} = \frac{1}{\sqrt{r}} e^{-i\pi} = -\frac{1}{\sqrt{r}}$

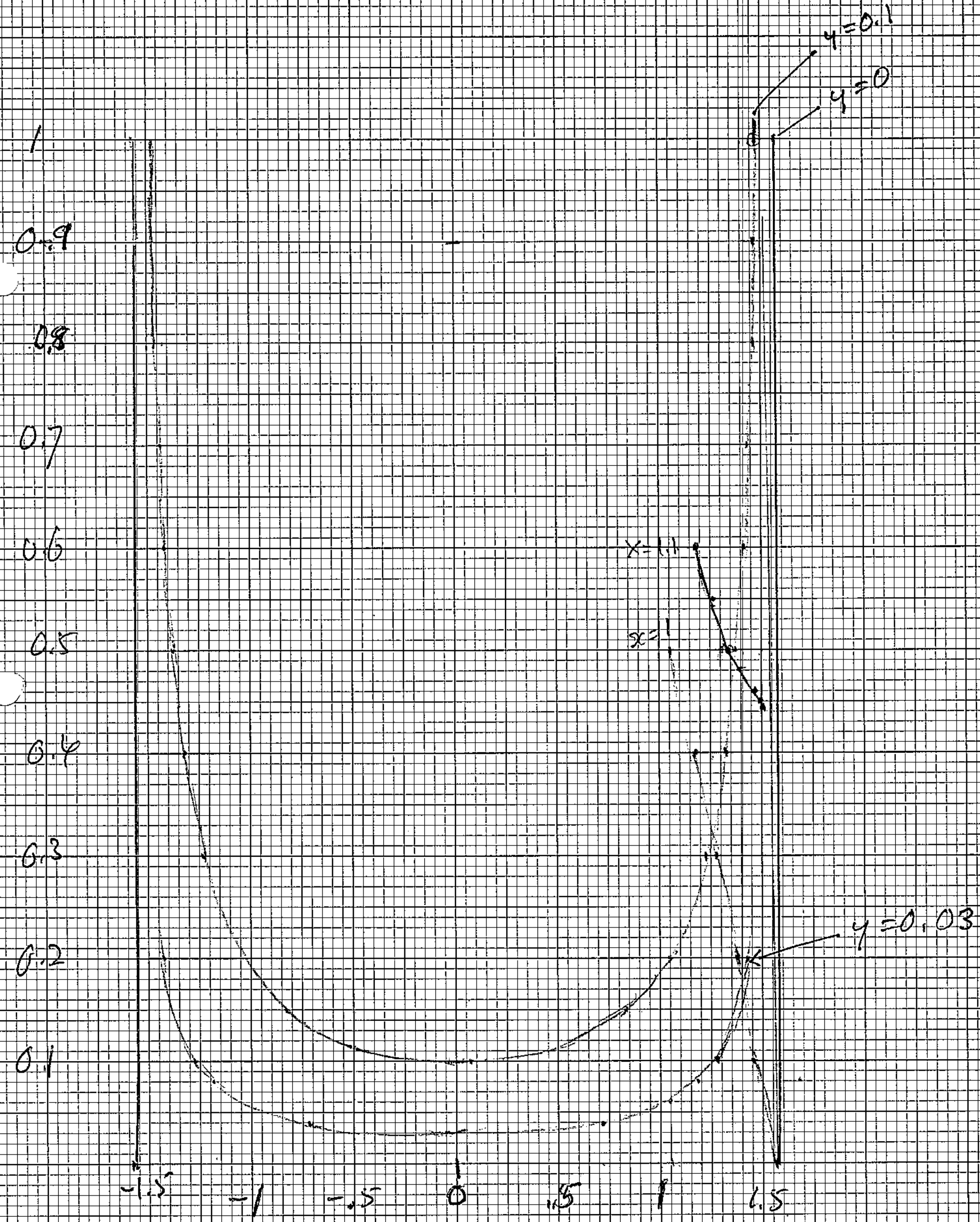
$\therefore$ $i(1+z)^{-\frac{1}{2}}(1-z)^{-\frac{1}{2}} = i(-1)k = -ik$
when k is real +ve.

On M, we get the value at any point by multiplying value on the upper plane by -1. Thus we have

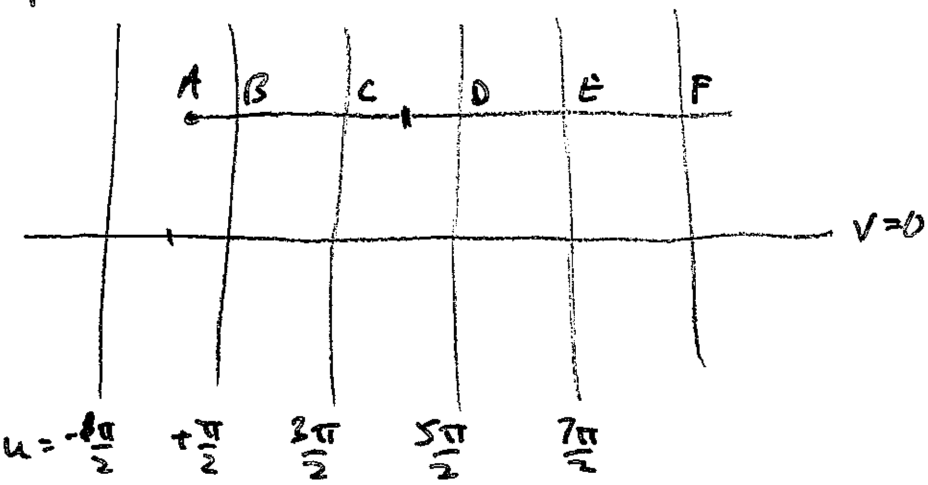


Thus the two planes must be united by a cross over junction.
The same is found at the left hand cut.

C 57



As we move in the w plane, we keep coming to different places where z has the same value. How does such a path appear in the z-plane?



A is the point $(0, k)$ in w plane. We move horizontally to the right from A.

$x = \sin u \cosh k \qquad y = \cos u \sinh k$

If u increases by 2π, x, y come to earlier value.

Consid $x = a \sin u, \quad b = b \cos u. \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

So x, y describes an ellipse.

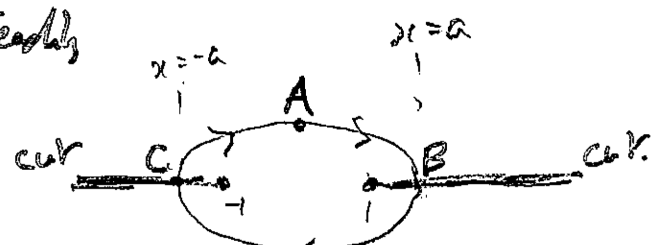
$a = \cosh k, \quad b = \sinh k$. We take $k > 0$,

$a > 1, \ b > 0$.

At B, $u = \frac{\pi}{2}$ $x = \cosh k$, so $x > 1$, $y = 0$.

B to C, $\cos u < 0$ and $\sin u$ steadily decreases from $+1$ to -1, so x from $+a$ to $-a$.

(x, y) returns to A when $u > 2\pi$



$$\sin^{-1} z = \int_0^z \frac{dt}{\sqrt{1-t^2}}$$ We take $\sqrt{1-t^2}$, real, > 0 in $(-1, +1)$.

So $\sin^{-1} z$ is real and positive for $0 < z < 1$.

Now $\sqrt{1-t^2} = (1-t)^{\frac{1}{2}}(1+t)^{\frac{1}{2}}$ and analytic continuation of this involves a change of sign for circuit of $t = 1$ or $t = -1$.

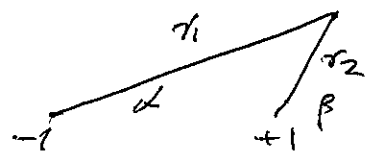
To have $\sqrt{1-t^2}$ precisely defined we need cuts to prevent such circuits. Suppose cuts made from $+1$ to $+\infty$, and -1 to $-\infty$, on real axis

$\int_2$ C53

When we are dealing with $\sin^{-1} z = \int_0^z \frac{dt}{\sqrt{1-t^2}}$,
we obviously want to define the $\surd$
in such a way that it will be real for $-1 < z < +1$.

The natural way to define $\sqrt{z}$ is to take $z = (r, \theta)$
$\sqrt{z} = (\sqrt{r}, \theta/2)$

Similarly, for $\sqrt{z+1}$ and $\sqrt{z-1}$, the natural definition would involve $(\sqrt{r_1}, \alpha/2)$ and $(\sqrt{r_2}, \beta/2)$.



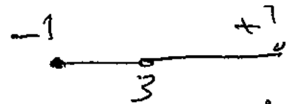
We want $\frac{1}{\sqrt{1-z^2}}$ to have the value $+1$ at $z=0$.

So $\sqrt{1-z^2} = 1$ for $z=0$.

$$\sqrt{1-z^2} = \sqrt{(1-z)(1+z)} = \sqrt{-(z-1)(z+1)}$$
$$= \sqrt{-1}\,\sqrt{z-1}\,\sqrt{z+1}$$

The question is — should $\sqrt{-1}$ be i or $-i$?

If $z=0$, $r_1 = 1$, $\alpha = 0$
so $(\sqrt{r_1}, \alpha/2)$ is $(1, 0)$ (polar)



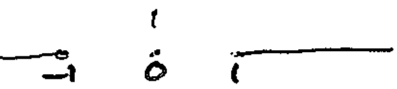
which means 1. So our standard $\sqrt{z+1}$ ~~makes~~ for 1 at 0.

If $z=0$, $r_2 = 1$, $\beta = \pi$, so $\sqrt{z-1}$ is polar $(1, \pi/2)$ which is i. To get 1 at $z=0$ we need to take $\sqrt{(-1)} = -i\,\sqrt{z+1}$.

Thus, with standard $\surd$'s, $\sqrt{1-z^2} = -i\sqrt{z-1}\,\sqrt{z+1}$

$$\frac{1}{\sqrt{1-z^2}} = \frac{i}{\sqrt{z-1}\,\sqrt{z+1}}.$$

We cut $(-\infty, -1)$ and $(1, +\infty)$

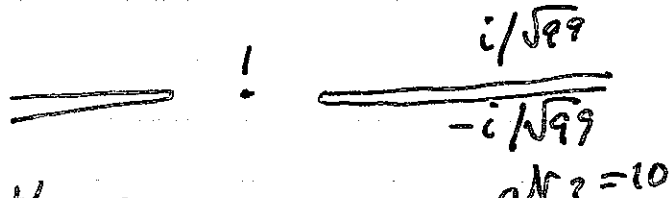


and can continue $\frac{1}{\sqrt{1-z^2}}$ away from 1 at $z=0$ to any part of the cut plane.

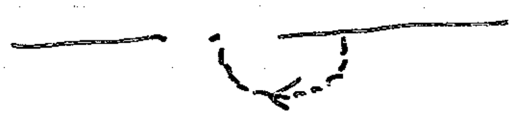
If we now take $z>1$ on the upper side of the cut we have $\alpha = 0$, $\beta = 0$ so $\sqrt{z-1}$ ~~$= \surd$~~ and $\sqrt{z+1}$ are real, so $\frac{1}{\sqrt{1-z^2}} =$ is i, real, +ve

On the lower side of the cut $\alpha = 0$, $\beta = 2\pi$
$\beta/2 = \pi$ and $\sqrt{z-1}$ now means $-\sqrt{z-1}$ in usual sense

Accordingly, on this side of the cut, $\frac{1}{\sqrt{1-z^2}} = -is$
s, real, +ve.



If we cross the cut and come round to the origin



$r_2=1$, $\beta=-\pi$
so second $\sqrt{z-1}$ is polar $(1,-\frac{\pi}{2})$, that is $-i$. $\sqrt{z+1}=1$ as before

Then $\frac{1}{\sqrt{1-z^2}} = \frac{i}{\sqrt{z-1}\,\sqrt{z+1}} = \frac{i}{-i\cdot 1} = -1$

As we would expect, in the other half of the Riemann surface we get -1 at $z=0$, as against $+1$ at $z=0$ in the first cut plane.

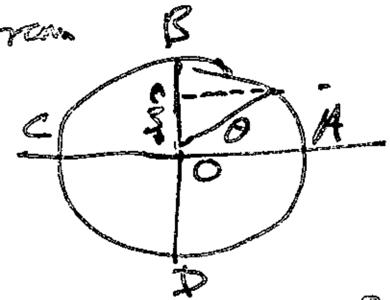
We have negative numbers ~~from~~ for $-1<z<+1$ on this sheet.

In regard to $w=\int_0^z \frac{1}{\sqrt{1-t^2}}\,dt$ it is of course for t that we need these two sheets.

If we interpret $\sin^{-1} z$ to mean "the angle whose ~~sign is~~ sine is z", then $\sin^{-1}\frac{1}{2}$ can mean $\pi/6$, $5\pi/6$, $2\frac{1}{6}\pi$, $2\frac{5}{6}\pi$, $3\frac{1}{6}\pi$, $3\frac{5}{6}\pi$, etc. How do these manage to appear in $\sin^{-1} z = \int_0^z \frac{dt}{\sqrt{1-t^2}}$. This integral implies

$$\frac{d}{dz}\sin^{-1} z = \frac{1}{\sqrt{1-z^2}}.$$

The natural way to visualize $\sin^{-1} z$ is to take θ in the diagram. How does $d\theta/dz$ appear in this?

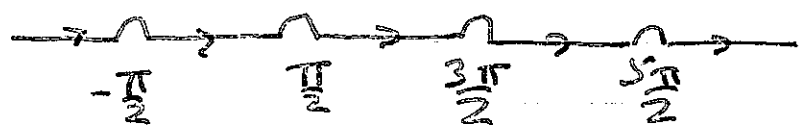


As z rises from 0 to 1 the situation is perfectly straightforward, z is rising and so is θ, $\frac{d\theta}{dz} = \frac{1}{\sqrt{1-z^2}}$. Things change when we pass B. From B to C, θ is increasing but z is decreasing, so it must be that $\frac{d\theta}{dz} = -\frac{1}{\sqrt{1-z^2}}$. Another reversal occurs as the point on the circle passes D: once more we have $d\theta/dz = +\frac{1}{\sqrt{1-z^2}}$.

To get values of $\sin^{-1} z$ larger than $\frac{\pi}{2}$ we have to consider ~~alternate~~ $1/\sqrt{1-z^2}$ periodically changing sign.

Many mistakes are made in elementary calculus courses because this complicated behavior is not understood.

How does this appear in the complex plane? w is to increase steadily from 0. However, we cannot just take w moving along the real axis because in $\int \frac{1}{\sqrt{1-t^2}}\,dt$, the integrand is infinite
for $t=-1$ and $t=+1$ corresponding to $w = -\frac{\pi}{2}$ and $\frac{\pi}{2}$, and these $+\ 2n\pi$.
~~These are imp~~ We have an improper integral, depending on limit $\int_{-1+\varepsilon}^{1-\eta}$ as $\varepsilon \to 0$, $\eta \to 0$. So we make little loops and consider the path for w :-



C57

Let $f(z) = K\,(z-a)^p (z-b)^q (z-c)^r$

If $\frac{dw}{dz} = f(z)$ $\quad dw = f(z)\,dz$.

Now $f(z) = Re^{i\theta}$. Multiplying by $f(z)$ turns quantity multiplied through θ.

Suppose each of powers is chosen so that ( ) positive real gives value positive real.

Then $(x-q)^s$ ($q = a, b$ or c) has argument 0 for $x > q$, $s\pi$ for $x < q$

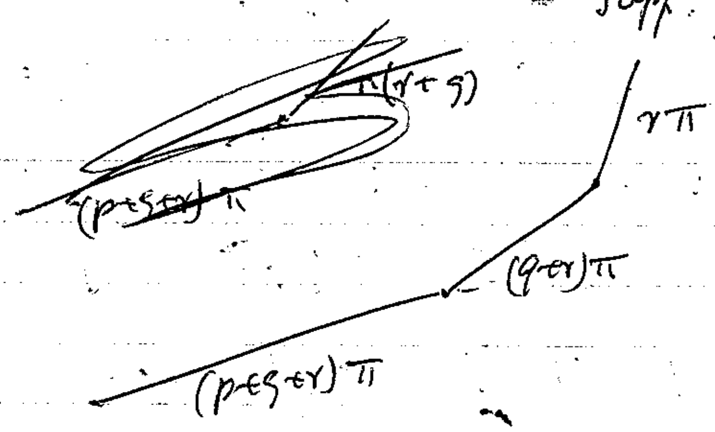
For $z > c$, each contribute 0 to argt.

For $b < z < c$ we have πr

$a < z < b$ $\qquad \pi r + \pi q$

$z < a$ $\qquad \pi r + \pi q + \pi p$

Supp. $p, q, r < 0$.

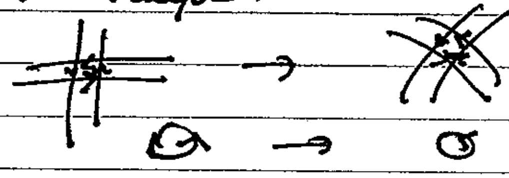


We are concerned with functions, $\mathbb{C} \to \mathbb{C}$, for which $f'(z)$ exists, i.e for $w = f(z)$, $\frac{dw}{dz} = f'(z)$

Consider a particular point z, and variations dz from it. Then $dw = f'(z)\,dz$. Here $f'(z)$ is some particular complex number. Multiplication by it produces a rotation and change of scale, i.e it leaves angles and ratios unchanged. Such a transformation is called <u>conformal</u>. As $dw = f'(z)\,dz$ holds only for differentials, this means that the geometry is preserved only in the limit as we considered neighbourhoods of z and w whose size $\to 0$. This in particular means that the angle between the tangents to two curves, at a place where they cross, is not altered.

Note that multiplication produces a <u>turning</u>: it cannot produce a <u>turning over</u>. Thus, if $z = x + iy$, $w = x - iy$ the function $z \to w$ cannot be analytic.

Always



direction of rotation preserved

Transformation, $w = \frac{az+b}{cz+d}$, maps conformally 1-1 the entire plane z to the entire plane w. It preserves the essential behaviour of functions at each point, and can give a very useful way of putting some situation in an easily understood form.

We suppose, of course, $ad - bc \neq 0$.

Such a transformation can be the combined effect of simpler transformations. For example

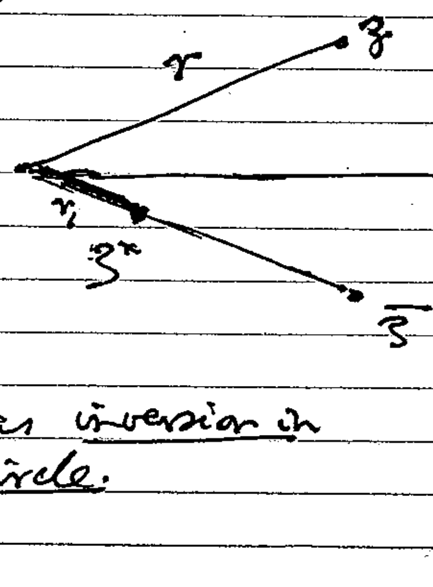
$$\frac{12z+25}{3z+6} = 4 + \frac{1}{3z+6} = 4 + \frac{1}{3(z+2)}$$

Thus $z^* = \dfrac{12z+25}{3z+6}$ can be achieved by the following sequence of steps.

$$z_1 = z + 2 \qquad (1)$$
$$z_2 = 3z_1 \qquad (2)$$
$$z_3 = 1/z_2 \qquad (3)$$
$$z^* = z_3 + 4. \qquad (4)$$

Such a decomposition is always possible.
(1) and (4) are simply translations.
(2) is a change of scale.
(3) is a new kind of transformation.

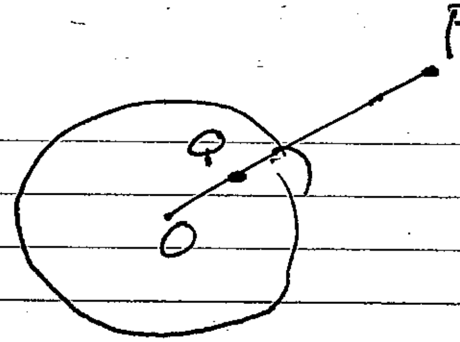
If $z^* = \dfrac{1}{z}$ and z is (r, θ) then z^* is $(\frac{1}{r}, -\theta)$



If $z = x+iy$, $\bar{z} = x - iy$, reflection in the real axis. $\bar{z}$ is at distance r from the origin, z^* at distance $\frac{1}{r}$.

$\bar{z} \to z^*$ is known as <u>inversion in the unit circle.</u>

Given any circle, radius R, centre O, inversion in it sends P to Q where OPQ is straight and $OP.OQ = R^2$.

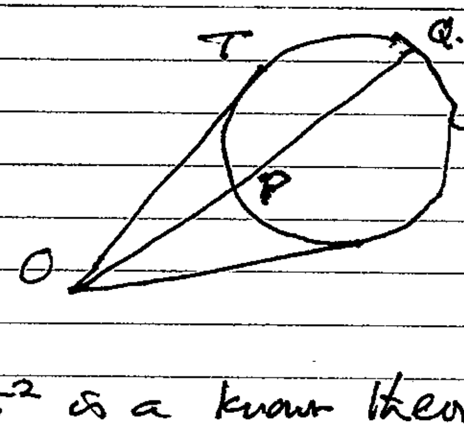


This operation has much in common with reflection. If $P \to Q$, then $Q \to P$. If the radius R becomes very large, it could be mistaken for reflection.

It reverses sense of rotation $\circlearrowleft \to \circlearrowright$.

$z^* = 1/z$ is analytic: it is equivalent to 2 operations, each of which reverses sense of rotation. It could not be equivalent to a single such operation.

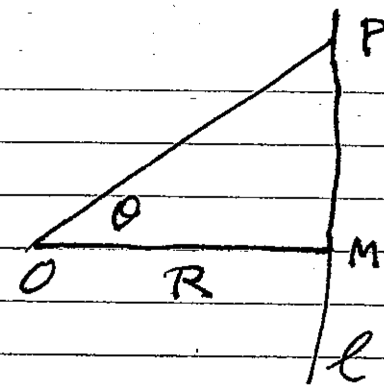
<u>Theorem</u> If we have a circle C, and if OT is a tangent to it, inversion of C in the circle, centre O, radius OT, makes $C \to C$.



<u>Proof</u> $OP.OQ = OT^2$ is a known theorem.

Changing the radius of the circle of inversion only produces a change of scale, so C inverted in any circle, centre O, goes to a circle.

Problem l is a line at perpendicular distance R from a point O. What do we get if we invert l in the circle centre O, radius R?
Suggestion: Find OQ when $\angle POM$ is θ.



$$w = \frac{az+b}{cz+d} \quad \text{where } ad - bc \neq 0.$$ (C22)

Just one value of w corresponds to a given value of z.

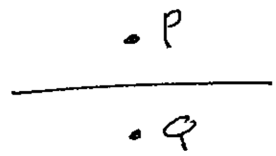
$czw + wd = az + b \quad \therefore \; z = \dfrac{wd - b}{-cw + a}$.

This gives a single value for z when w is known

[Question. What goes wrong with this argument if $ad - bc = 0$?]

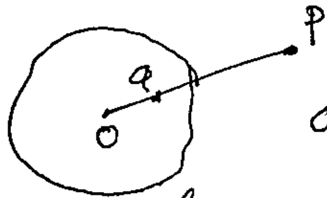
"Kreisverwandschaft."

<u>Inversion</u> Reflection in a line is



$P \to Q$, $Q \to P$

Inversion may be thought of as reflection in a circle.

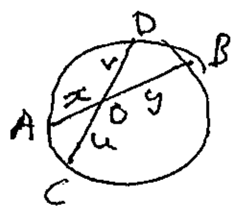


$OP . OQ = R^2$.

Important property of a circle

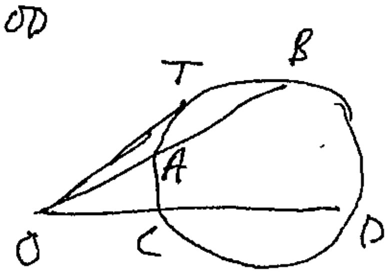
$x . y = u . v$

(lengths)



$OA . OB = OC . OD$

This still holds if O is outside the circle.



For the tangent OT,

$A \to T$, $B \to T$.

So $OA . OB = OC . OD = OT^2$.

If we invert with respect to the circle, centre O, radius OT $\quad A \to B$, $B \to A$

So the circle TBDCA reflects to itself.

If we invert with respect to a circle, centre O, but a different radius, we have similar effect with a change of scale: the circle goes to a different circle.

There is an exception if O lies on the circle BDCA.

1) $f(z) = f(x+iy)$ is analytic in a region if there is a continuous derivative $f'(z)$, independent of direction in which z is changing.

$$df(z) = f'(z)\,dz.$$

Let $f(x+iy) = u+iv$. $\quad f'(z) = p+iq$

$$du + i\,dv = (p+iq)(dx+i\,dy)$$
$$= p\,dx - q\,dy + i(q\,dx + p\,dy)$$

$$du = p\,dx - q\,dy \quad \therefore \frac{\partial u}{\partial x} = p \quad \frac{\partial u}{\partial y} = -q.$$

$$dv = q\,dx + p\,dy \quad \therefore \frac{\partial v}{\partial x} = q \quad \frac{\partial v}{\partial y} = p.$$

Thus $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Cauchy Riemann.

These derivatives must exist and be continuous.

2) Geometrical meaning of Cauchy Riemann equations.

Suppose a point (x,y) starts at $(x_0\, y_0)$ and moves a distance s at angle θ to OX.

Then $x = x_0 + s\cos\theta \quad y = y_0 + s\cdot\sin\theta$

Rate of change of $u(x,y)$ w.r.t. s is then

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta.$$

Suppose v changes as a result of x, y, moving in direction $\theta + \frac{\pi}{2}$. $\cos(\theta+\frac{\pi}{2}) = -\sin\theta$, $\sin(\theta+\frac{\pi}{2}) = \cos\theta$

$$\frac{dv}{ds} = -\frac{\partial v}{\partial x}\sin\theta + \frac{\partial v}{\partial y}\cos\theta$$
$$= \frac{\partial u}{\partial y}\sin\theta + \frac{\partial u}{\partial x}\cos\theta \quad \text{if C-R hold.}$$

Thus rate of change of u in direction θ
= rate of change of v in direction $\theta + \frac{\pi}{2}$.

In particular, if u is constant for $\angle\theta$, v is constant for $\angle\theta + \frac{\pi}{2}$. The curves $u = c$, $v = k$ cross at right angles.

Integration in 2 dimensions.

To define $\int X(x,y)\,dx + Y(x,y)\,dy$ along a given curve, we suppose the curve given by $x = x(t)$, $y = y(t)$ $0 \le t \le 1$.

Then $X(x,y) = X(x(t), y(t))$ a fn of t.

$$dx = \frac{dx}{dt}\,dt = x'(t)\,dt.$$

We define the integral as

$$\int_0^1 X(x(t), y(t))\,x'(t) + Y(x(t), y(t))\,y'(t)\,dt.$$

Suppose for example we require $\int y\,dx + 2x\,dy$ along the line joining $(1,1)$ to $(3,4)$.

$x = 1 + 2t$, $y = 1 + 3t$ takes x, y from $(1,1)$ to $(3,4)$ as t goes from 0 to 1.

$$\int = \int_0^1 (1+3t)2 + (2+4t)3 \,.\, dt$$

$$= \int_0^1 8 + 18t\,dt = \left.8t + 9t^2\right]_0^1 = 17.$$

We would get a different result if we went from $(1,1)$ to $(3,1)$ and then $(3,1)$ to $(3,4)$

In first part $x = 1 + 2t$, $y = 1$ $x' = 2$, $y' = 0$

We have $\int_0^1 2 + 4t\,dt = \left.2t + 2t^2\right]_0^1 = 4.$

In the second part $x = 3$, $y = 1 + 3t$. $x' = 0$ $y' = 3$

$$\int_0^1 6.3\,dt = \left.18t\right]_0^1 = 18. \qquad \int = 22.$$

$\int X\,dx + Y\,dy$ is an expression for work done. This example corresponds to a non conservative system. Work could be obtained from it by following an appropriate loop.

The result will be independent of the path if $X\frac{dx}{dt} + Y\frac{dy}{dt} \equiv \frac{d}{dt} f(x,y)$ for some $f(x,y)$ For $\int_0^1 \frac{d}{dt} f(x,y)\,dt = f(x_1,y_1) - f(x_0,y_0)$.

Now $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

This case arises if there exists $f(x,y)$:-

$$X = \frac{\partial f}{\partial x} \qquad Y = \frac{\partial f}{\partial y}.$$

We suppose X and Y have continuous derivatives.

Then $\frac{\partial}{\partial y} X = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f = \frac{\partial}{\partial x} Y$

$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}$ is the usual condition for $X\,dx + Y\,dy$ to be an exact differential.

[Psaggio. Appendix A. $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ if f is continuous, and has continuous first and second derivatives.]

If $\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}$ we can show that there is an f with the required property. $\int X\,dx + Y\,dy$ is then independent of the path between given end points, provided $\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}$ holds throughout the region. Stokes's theorem.

$$\oint X\,dx + Y\,dy = \iint \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\,.\,dx\,dy.$$

If $f(x+iy)$ satisfies the Cauchy-Riemann conditions, $\int_a^b f(z)\,dz$ does not depend on the path connecting given a, b.

Proof. Let $f(x+iy) = u + iv$.

$$\int (u+iv)(dx+i\,dy) = \int u\,dx - v\,dy + i\int v\,dx + u\,dy.$$

$u\,dx - v\,dy$ is exact if $\frac{\partial}{\partial y}(u) = \frac{\partial}{\partial x}(-v)$

i.e. $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. This is a C-R condition.

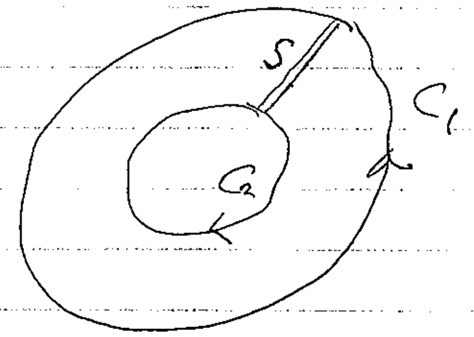
$v\,dx + u\,dy$ is exact if $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$, the other C-R condition.

So we have

Cauchy's Theorem.

If $f(z)$ is analytic in some region, $\oint f(z)\,dz = 0$ for any smooth closed curve in that region, and for any curve consisting of a finite number of smooth curves.

Corollary If C_1 and C_2 are two closed curves, C_1 inside C_2, and a region in which $f(z)$ is analytic contains C_1 and C_2.



then $\oint_{C_1} = \oint_{C_2}$

Proof Consider region bounded by C_2, C_1 and strip S.

This is bounded by C_1, S, C_2, S. The two parts on S cancel and we have $\int_{C_1 - C_2} f(z)\,dz = 0$.

Residue Theorem. $f(z)$ is analytic on a region containing curve C. A point $\underline{a}$ is inside C.

Consider $\oint_C \frac{f(z)\,dz}{z-a}$. By previous theorem

this equals ~~val~~ value for a small circle, centre $\underline{a}$.

On this circle $f(z) \doteqdot f(a)$. In the limit we have $f(a)\oint \frac{dz}{z-a}$, integral around small circle.

On the small circle $z = a + re^{i\theta}$ where r is radius of circle, $0 \le \theta \le 2\pi$.

$$dz = re^{i\theta} i\,d\theta.$$

$$\oint \frac{dz}{z-a} = \int_0^{2\pi} \frac{re^{i\theta} i\,d\theta}{re^{i\theta}} = \int_0^{2\pi} i\,d\theta = 2\pi i$$

$$\therefore \quad 2\pi i\, f(a) = \oint_C \frac{f(z)\,dz}{z-a}$$

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)\,dz}{z-a}$$

Given $f(z)$ on the boundary, this gives an explicit solution for values of $f(z)$ inside the curve.

C28

$I = \int x^{-3/2}(1-x)^{-3/2}\,dx$ let $y^2 = x - x^2$

If $y = tx$ $t^2x^2 = x - x^2$

$x = \dfrac{1}{1+t^2}$

$1-x = \dfrac{t^2}{1+t^2}$

$dx = \dfrac{-2t}{(1+t^2)^2}$

$t^2 = \dfrac{x}{1-x}$

$t = \sqrt{\dfrac{x}{1-x}}$

$$I = \int \left[\frac{t^2}{(1+t^2)^2}\right]^{-3/2} \frac{-2t}{(1+t^2)^2}\,dt$$

$$= \int \frac{(1+t^2)^3}{t^3}\,\frac{-2t}{(1+t^2)^2}\,dt$$

$$= -2\int \frac{1+t^2}{t^2}\,dt$$

$$= -2\left[t - \frac{1}{t}\right]$$

$$= 2\left[\sqrt{\frac{1-x}{x}} - \sqrt{\frac{x}{1-x}}\right]$$

$$= \frac{2}{\sqrt{x(1-x)}}\left[1-x-x\right]$$

$$= \frac{2(1-2x)}{\sqrt{x(1-x)}}$$

Check $\dfrac{d}{dx}(2-4x)\,x^{-1/2}(1-x)^{-\frac{1}{2}}$

$= \cancel{-4x^{\frac{1}{2}}}$ $\ln I = \ln 2 + \ln(1-2x) - \frac{1}{2}\ln x - \frac{1}{2}\ln(1-x)$

$$\frac{I'}{I} = \frac{-2}{1-2x} - \frac{1}{2x} + \frac{1}{2(1-x)}$$

$$= \frac{-4x(1-x) - (1-x)(1-2x) + x(1-2x)}{2x(1-x)(1-2x)}$$

Numer $=$ $-4x + 4x^2$
$-1 + 3x - 2x^2$ $= 1.$
$+x - 2x^2$

$$I' = \frac{I}{2x(1-x)(1-2x)} = \frac{\cancel{2}(\cancel{1-2x})}{\sqrt{x(1-x)}}\cdot\frac{1}{\cancel{2}x(1-x)(\cancel{1-2x})}$$

$$= \frac{1}{\sqrt{x^3(1-x)^3}}$$

$$w(z) = \frac{2(1-2z)}{\sqrt{z(1-z)}}$$

If z is real between 0 and 1

w is real and goes from $+\infty$ to $-\infty$.

z real, > 1. z is pure imaginary.

z real < 0 z is pure imaginary

<u>Transformations $\mathbb{C} \to \mathbb{C}$</u>

$$w = f(z) = \prod_{n=1}^{N} (z - a_n)^{\alpha_n}$$

We need to specify the branch of $(z - a_n)^{\alpha_n}$.
We suppose, ~~that~~ for $z = a_n + r$, $(r \in \mathbb{R})$
we take the real value of r^{α_n}. (α_n is real, so we can always have such a value.)

Let $\omega = z - a_n$. We consider ω traversing a semicircle, centre a_n, radius r.

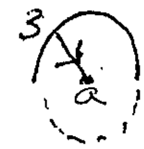
Thus $\omega = r e^{i\theta}$, θ from 0 to π.
Then ω^{α_n} goes from r to r^{α_n} to
$$(r e^{i\pi})^{\alpha_n} = r^{\alpha_n} e^{i\pi\alpha_n}$$
Thus the value of $(z - a_n)^{\alpha_n}$ at $a_n + r$
is $e^{-i\pi\alpha_n}$ times its value at $a_n - r$.
<u>This is independent of r</u>

~~If we suppose~~ When $z > a_N$, all the factors are real. Hence $\arg f(z) = 0$ for $z > a_N$.

Any z between a_{N-1} and a_N has $(z - a_N)^{\alpha_N}$ equal to $e^{i\pi\alpha_N}$ times value ~~of~~ for some z to the right of a_N.
Hence $(z - a_N)^{\alpha_N} = \rho e^{i\pi\alpha_N}$, $\rho \in \mathbb{R}$.
Similarly, for $a_{N-2} < z < a_{N-1}$,
$$f(z) = \rho e^{i\pi\alpha_{N-1}} \cdot e^{i\pi\alpha_N}$$
By induction $\arg f(z)$, for $a_{N-k-1} < z < a_{N-k}$ is $\sum_{s=N-k}^{N} i\pi\alpha_s$.

If $f(z)$ also has factors of the form $(a-z)^\alpha$ it will be convenient to have a real value for $z < a$. Let $\zeta = a - z$

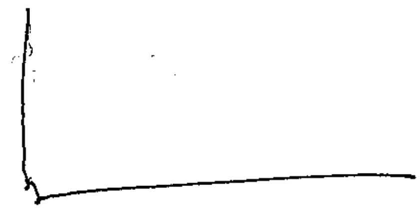
If z describes the semicircle, the vector $a-z$, drawn with a as beginning, will describe dotted circle. Thus $\zeta = re^{i\theta}$



with θ going from 0 to $-\pi$.
Thus $(a-z)^\alpha = \zeta^\alpha = r^\alpha e^{i\alpha\theta}$ and has the final value $r^\alpha e^{-i\alpha\pi}$.

Hence, for factor $(z-a)^\alpha$, the value to the left is $e^{i\pi\alpha}$ times that to the right.
For $(a-z)^\alpha$ the value to the right is $e^{-i\pi\alpha}$ times that to the left.

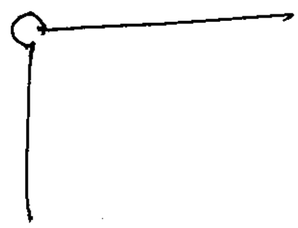
$w = \sqrt{z}$



w

$\alpha = \frac{1}{2}$ $e^{i\pi\alpha} = e^{\frac{1}{2}i\pi} = +i$

$\frac{dw}{dz} = \sqrt{z}$ $w = \frac{2}{3} z^{3/2}$



C 58

The following page seem to be of interest because it ~~~~ ~~the~~ appears to have the program of a prodigy that Warwick was working with.

The only Paul Cook I could locate was in the US and took his PhD in 1975 – obviously not the right one.

[signature]

Ofc. Its raised question. (1)

C59

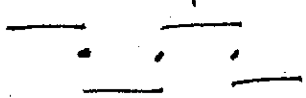
Paul Cook

Started here, September 1985 Age 13

Started co-ord. geometry. Quadratics by completing square. $(a+b)^2$, $(a-b)^2$. Elem algebra.
(Carpet problem)
Nov. 30. Paul started calculus. Diffn of polynomial.

(1986 – I was out of action, surgery)

Jan 1987. How Newton found series for $\sin^{-1}x$.
Differential equation for circuit including resistance and capacity. Paul intrigued by gamma function, $n! = \int_0^\infty x^n e^{-x}\,dx$.

February 1987. Theory of vibrating piano string.
$\partial^2 f/\partial x^2 = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$. Variables separated.
Remark by musician Rameau that he could hear many harmonics simultaneously led to Fourier series $\sum c_n \sin nx$. Very much interested in fact that graph  could be obtained by such a series.
Elementary work on Fourier series, much as Piaggio Diff Eqns Chap. 4.

March 1987 Diffn Inverse functions. ~~Series for π~~ Paul interested in series for π.
Gibbs phenomenon for parapet function We looked at the sum ~~of~~ $y = \sin x + \frac{1}{3}\sin 3x + \dots \frac{1}{179}\sin 179x$ both by calculus and numerically by computer.

April 1987. Envelope of $y = mx + (a/m)$. Reflection in a circular mirror. Integration by parts

Theme ← $\boxed{\text{Elementary algebra}}$

May Wallis on $\int_0^1 (1-x^2)^n\,dx$.
June Theory of beat note. Euler's Beta function as generalization of Wallis.
Matrix for rotation. $\sin(A+B)$

c60

Summary
March to June 1989. Analysis. ~~[illegible]~~. Paradoxes of show ~~conditionally cgt series and non unif space~~ need for rigour. $\varepsilon - \delta$. Absolute and uniform space. Various topics from Tims A ——— .

Sept 88 — More Elements of projective geometry. Cross ratio. Harmonic range. Jorachimsthal method. Polar. Homogeneous co-ordinates.

October. ~~Galois group of a finite field~~. Some discussion of groups, finite fields. Error free coding.

1988 ~~April 1988~~ Feb – May Partial differentiation. $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$ for current in uniform sheet. Counter example to

$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$. In 3 dimensions,

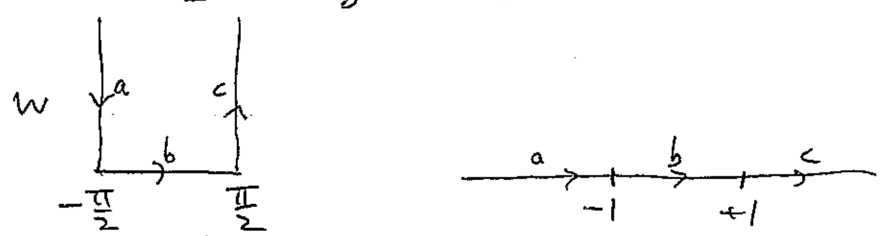
$\frac{1}{r}$ satisfies $\nabla^2 V = 0$.

The mapping $w = \sin^{-1} z$. $\quad \sin w = \dfrac{e^{iw} - e^{-iw}}{2i} \quad \sin it = \dfrac{e^{-t} - e^{t}}{2i} = i \sinh t$

If $\quad -\frac{\pi}{2} \le w \le \frac{\pi}{2}, \quad -1 \le z \le +1$, both in reals.

If $\quad w = \frac{\pi}{2} + it \quad z = \sin\left(\frac{\pi}{2} + it\right) = \cos it = \cosh t$
with values from 1 to $+\infty$.
$w = -\frac{\pi}{2} + it \quad z = \sin\left(-\frac{\pi}{2} + it\right) = -\cos it = -\cosh t$.

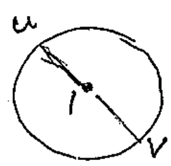
Hence



In other problems of mapping upper half plane to polygon, we use an integral. We wish to show from the integral that c, in w plane, is upwards.

$$w = \int_0^z \frac{du}{\sqrt{1-u^2}} \qquad |z| \le 1, \text{ mapping of } b \text{ straightforward}$$

Path of z



$1 - u = v - 1$

Path of $1-u$

Path of $\dfrac{1}{1-u}$

since $\arg \frac{1}{z} = -\arg z$.

$\dfrac{1}{\sqrt{1-u}}$

so $\arg \dfrac{1}{\sqrt{1-u}} = +\dfrac{\pi}{2}$ when $u = 1+$.

$$\arg \frac{1}{\sqrt{1-u^2}} = \arg \frac{1}{\sqrt{1+u}} + \arg \frac{1}{\sqrt{1-u}}$$

$\arg \dfrac{1}{\sqrt{1+u}}$ is 0 both for $u = 1 - \varepsilon$ and $u = 1 + \varepsilon$.

Hence $\dfrac{1}{\sqrt{1-u}} = +ik$ for $u = 1+$.

The argument holds for any $u > 1$.

Integral of $\dfrac{1}{\sqrt{1-u^2}}$ converges, so we do not have any contribution from small semicircle over 1.

If $f(z)$ is regular at $z = z_0$ and not identically zero in a neighbourhood of z_0, then z_0 cannot be a limit point for zeros of $f(z)$.

Proof $f(z)$ has a Taylor series in a domain including z_0. If this is $c_0 + c_1 z + \ldots$ with $c_0 \neq 0$, by continuity there is a neighbourhood of z_0 in which $f(z)$ differs little from c_0, hence $f(z) \neq 0$ in this neighbourhood and so zeros cannot get near to z_0.

If $c_0 = 0$ and first nonzero coefficient is c_n,
$f(z) = c_n(z - z_0)^n + c_{n+1}(z - z_0)^{n+1} + \ldots$ so
$g(z) = f(z)/(z - z_0)^n$ is regular in the domain in question, and by the argument above, $g(z) \neq 0$ in some neighbourhood. In this neighbourhood, $f(z) = 0$ only for $z = z_0$.

If neither of above apply, all c_i are zero, and $f(z) \equiv 0$ in the domain. Q. E. D. (Hurwitz Courant pp 306-307.)

Analytic Continuation

If $\varphi(z)$ is defined in a domain G' containing the domain G, and $\varphi(z) = f(z)$ in G, then $\varphi(z)$ is the continuation of $f(z)$ in G'.

If such exists, it is unique. If φ_1, φ_2 two continuations, $\varphi_1(z) - \varphi_2(z)$ is regular in G' and certainly zero in G. Let G^x be the set of points for which $\varphi_1 - \varphi_2 = 0$. If there is a place where $\varphi_1 - \varphi_2 \neq 0$, join it to a point in G^x by a suitable arc. There will be a point P on this arc that is the limit point of points in G^x, and also of points not in G^x. Since $\varphi_1 - \varphi_2$ is regular in G', there is a power series for $\varphi_1 - \varphi_2$ around P. As P is limit point of zeros, this power series must have all coefficients zero. Hence P is not the limit of points where $\varphi_1 - \varphi_2 \neq 0$. Contradiction.

(Slightly adapted from p. 369 Hurwitz Courant)

SMOOTH APPROXIMATION OF CURVE.

Hurwitz Courant. III. 1. §2. p261 – 263 only.

Continuous curve. $\{(x,y) : x = \varphi(t), y = \psi(t)$ continuous fns, $t_1 \le t \le t_2\}$.

Smooth arc of a curve. At all points, including ends, a tangent exists and its direction varies continuously. "Piecewise smooth curves". Curves assumed to be this; word "path" also used.

Analytically, C defined by $x = \varphi(t)$, $y = \psi(t)$. φ, ψ continuous, φ' and ψ' piecewise continuous, not zero simultaneously.

Open curve $t_1 < t < t_2$. Closed $\varphi(t_1) = \varphi(t_2)$, $\psi(t_1) = \psi(t_2)$. Can be seen as periodic fns.

Simple, = free from double points.

A piecewise ~~continuous~~ smooth curve C can be approximated by $C_1, C_2 \ldots$ This means, $x = \varphi_n(t)$, $y = \psi_n(t)$ $t_1 \le t \le t_2$ with $\varphi_n(t) \to \varphi(t)$, $\psi_n(t) \to \psi(t)$

There is "smooth approximation" if in addition tangent to C_n differs little from tangent to C, except when the point on C is a corner or sharp point (?). Suppose C has discontinuous tangent direction for value $T_1, T_2 \ldots$ we shut T_ν in by an interval $|t - T_\nu| < \varepsilon$ with arb^y small ε and require that for all other t $\varphi_n(t) \xrightarrow{\text{unif}} \varphi(t)$, $\psi_n(t) \xrightarrow{\text{unif}} \psi(t)$ and at the same time $\varphi_n'(t) \xrightarrow{\text{unif}} \varphi'(t)$, $\psi_n'(t) \xrightarrow{\text{unif}} \psi'(t)$. It is understood that at the corners of C_n, this holds both for right and left derivatives. (C_n may have more corners than C, even $\to \infty$ — example polygon approx to circle)

We can smoothly approximate any piecewise smooth curve by polygons.

p271. If $C_1, C_2 \ldots C_n \ldots$ smoothly approximate C, and $J_n = \int_{C_n} a\,dx + b\,dy$, $J = \int_C a\,dx + b\,dy$ then $J_n \to J$. $\underline{a}$, $\underline{b}$ are continuous real functions of x, y.

Courant Hurwitz.

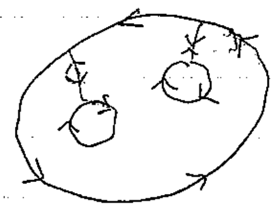
III.2 p 275–. §1. $\zeta = f(z)$ is regular in domain G if $d\zeta/dz$ exists and is continuous in this domain. Cauchy Riemann $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. These derivatives <u>continuous</u>. §2. If $f'(z_0) \neq 0$ the inverse function $\zeta \to z$ is defined and is regular in some neighbourhood of z_0. §3 Definite integral. $\int_C f(z)dz = \int_C (u\,dx - v\,dy) + i\int_C v\,dx + u\,dy$. If $|f(z)| \leq M$ and C has length L, $|\int_C f(z)dz| \leq ML$. §4 Indefinite integral. $f(z)$ regular in domain. C from z_0 to Z. $F(z) = \int_{z_0 (C)}^{Z} f(z)dz$ is independent of path, and is analytic.

$F'(z) = f(z)$. Independent of path by Green's Th^m type of argument.

$F(z) = U + iV$ Then $\partial U/\partial x = u$ $\partial U/\partial y = -v$ $\partial V/\partial x = v$ $\partial V/\partial y = u$

$\to$ Cauchy Riemann for U, V, so F regular.

§5. Multiply connected domains



Integral round total boundary is 0, if fn. regular in domain. Integral round outer curve = sum of integrals around inner. Considering only one inner curve, integral around this unchanged if replaced by similar circuit of singularity. $\frac{1}{2\pi i}\oint f(z)dz$ = residue (def.). Curve containing a finite no. of singularities. $\int_C = 2\pi i \sum$ residues.

§6. Def. $\log z = \int_1^z \frac{dt}{t}$. Principal value, cut $(-\infty, 0)$. $+2n\pi i$ for n circuits of 0. $\log z = \log\rho + i\theta$ by path. $w = \log z$. Given w, ρ determined and θ with $2n\pi i$ uncertainty, so position of z is fixed. Hence $z(w)$ is uniform. Analytic by inverse, §2 above. $z^\alpha = e^{\alpha \log z}$ (def). Many valued.

§7. Cauchy Integral formula. $f(z)$ regular. ~~[illegible]~~ $f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}dt$

Still valid if C is boundary of region of regularity, and f is continuous when region closed by including C.

If C is simple, closed, piecewise smooth curve, and $\varphi(z)$ continuous on C, then $f(z) = \frac{1}{2\pi i}\int_C \frac{\varphi(t)dt}{t-z}$ is analytic inside C, and has derivatives of every order $f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{\varphi(t)dt}{(t-z)^{n+1}}$.

Proof. $\frac{f(z+h) - f(z)}{h} - \frac{1}{2\pi i}\int_C \frac{\varphi(t)dt}{(t-z)^2} = \frac{h}{2\pi i}\int_C \frac{\varphi(t)\,dt}{(t-z-h)(t-z)^2}$

$h \to 0$. $\int_C$ bounded, as interior point has non-zero minimum distance from C.

Mapping, upper half of z to w plane $\quad w = \int_0^z \frac{dt}{\sqrt{1-t^2}}$

$z=0, w=0$. $\quad 0<z<1$, w real, increasing, towards $\pi/2$.

Near $z=1$,
$$\frac{dw}{dz} = \frac{1}{\sqrt{1-z^2}} = \frac{1}{\sqrt{1-z}}\,\frac{1}{\sqrt{1+z}}$$
$$= \frac{1}{\sqrt{z-1}} \cdot \sum_0^\infty c_r (z-1)^r$$
$$= \sum_0^\infty c_r (z-1)^{r-\frac{1}{2}}$$
$$\therefore\ w = \text{constant} + \sum_0^\infty \frac{c_r (z-1)^{r+\frac{1}{2}}}{r+\frac{1}{2}}$$

$z=1, w=\frac{\pi}{2}$
$$w - \frac{\pi}{2} = \sum_0^\infty \frac{c_r (z-1)^{r+\frac{1}{2}}}{r+\frac{1}{2}}$$
$$= \sum_0^\infty \frac{c_r u^{2r+1}}{r+\frac{1}{2}} \quad \text{where } u^2 = (z-1)$$

Thus w is regular fn of u near $u=0$, i.e. $z=1$, so mapped conformally.

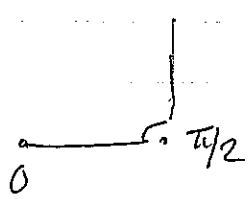
$u = \sqrt{z-1}$ so as z does 

u does .

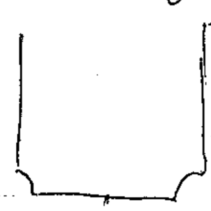
~~For~~ $t>1$ $\quad \frac{1}{\sqrt{1-t^2}}$ is pure imaginary. By continuity, it is to be ki, $k>0$.

Thus

maps to



Similarly for $z<0$



III 2 §8 Conformal mapping.

$w = f(z)$, regular. $f'(z) \neq 0$. Then mapping conformal.

III 3 §1. Arithmetic mean. Maximum principle. Schwarz' Lemma

$f(z) = \frac{1}{2\pi i}\oint \frac{f(t)\,dt}{t-z}$ Let $t = z + \rho e^{i\theta}$ $dt = i\rho e^{i\theta}d\theta$

$$f(z) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z+\rho e^{i\theta})\, i\rho e^{i\theta}d\theta}{\rho e^{i\theta}} = \frac{1}{2\pi}\int_0^{2\pi} f(z+\rho e^{i\theta})d\theta$$

= average on circle. ∴ No interior point can make $|f(z)|$ a maximum.

If $f(z) \neq 0$ in region, min $|f(z)|$ must occur on boundary. Proof: $1/f(z)$ regular. Consider its maximum. Schwarz Lemma omitted in notes here.

§2 Estimates. Liouville Theorem.

C a curve within domain of regularity, length L. $M = \max |f(z)|$ on C. z inside C, and at least δ distant from it. Then $|f(z)| \leq ML/(2\pi\delta)$.

Also $|f^{(n)}(z)| \leq n!\, ML/(2\pi\delta^{n+1})$.

If C circle, radius ρ, $|f^{(n)}(z)| \leq n!M/\rho^n$. Put $n=1$, $|f'(z)| \leq M/\rho$. If $f(z)$ regular in whole plane and bounded, $f'(z)=0$, so $f(z)$ is constant. (Liouville).

§3. Uniform convergence.

§4 Taylor and Laurent series.

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(t)}{t-z}dt \qquad \frac{1}{t-z} = \frac{1}{t-z_0}\,\frac{1}{1-\frac{z-z_0}{t-z_0}} \text{ Expand.}$$

Theorem $f(z)$ regular at z_0, and $\not\equiv 0$ in a neighbourhood of z_0, then z_0 is not a limit point of zeros of $f(z)$.

If $f(z_0) \neq 0$, $f(z) = c_0 + c_1(z-z_0) + $ etc By continuity $f(z) \neq 0$ in neighbourhood of z_0. If $f(z_0)=0$, let $f(z) = (z-z_0)^n[c_n + c_{n+1}(z-z_0) + \dots]$ Apply same argument to [ ]. Used later to prove uniqueness of continuation.

§5. Application of Integral & Residue Theorems

$$\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}\text{ etc.} \qquad \frac{1}{2\pi i}\oint \frac{f'(z)}{f(z)}dz = N - P.$$

§6. Principle of accumulation for analytic functions. Uniformly bounded in domain G.

§7 a. Relation to potential theory.

For each particle

$$m\ddot{x} = X_E + X_I \quad m\ddot{y} = Y_E + Y_I \quad m\ddot{z} = Z_E + Z_I$$

Where $X_E Y_E Z_E$ is external force on m, $X_I Y_I Z_I$ internal.

Internal forces do no work in an admissible displacement $\delta x\ \delta y\ \delta z$ of each particle

$$\therefore \sum m\ddot{x}\delta x + m\ddot{y}\delta y + m\ddot{z}\delta z = \sum X_E\delta x + Y_E\delta y + Z_E\delta z$$

$$= \sum Q_r \delta q_r.$$

$\delta x, \delta y, \delta z$ is due to a change $\delta q_1 \ldots \delta q_n$ in $q_1 \ldots q_n$

$$\therefore \delta x = \sum \frac{\partial x}{\partial q_r}\delta q_r = \sum \frac{\partial \dot{x}}{\partial \dot{q}_r}\delta q_r$$

$$\therefore Q_r = \sum m\ddot{x}\frac{\partial \dot{x}}{\partial \dot{q}_r} + m\ddot{y}\frac{\partial \dot{y}}{\partial \dot{q}_r} + m\ddot{z}\frac{\partial \dot{z}}{\partial \dot{q}_r}$$

$$T = \sum \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2 + \tfrac{1}{2}m\dot{z}^2$$

$$\therefore \frac{\partial T}{\partial \dot{q}_r} = m\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_r} + m\dot{y}\frac{\partial \dot{y}}{\partial \dot{q}_r} + m\dot{z}\frac{\partial \dot{z}}{\partial \dot{q}_r}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) = \sum m\ddot{x}\frac{\partial \dot{x}}{\partial \dot{q}_r} + \text{etc} + \sum m\dot{x}\frac{d}{dt}\left(\frac{\partial \dot{x}}{\partial \dot{q}_r}\right) + \text{etc.}$$

$$\frac{d}{dt}\left(\frac{\partial \dot{x}}{\partial \dot{q}_r}\right) = \frac{d}{dt}\left(\frac{\partial x}{\partial q_r}\right) = \frac{\partial \dot{x}}{\partial q_r} \qquad \text{by } *$$

$$\therefore \text{Second part above} = \sum m\dot{x}\frac{\partial \dot{x}}{\partial q_r} + m\dot{y}\frac{\partial \dot{y}}{\partial q_r} + m\dot{z}\frac{\partial \dot{z}}{\partial q_r}$$

$$= \frac{\partial}{\partial q_r}\left[\sum\left(\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2 + \tfrac{1}{2}m\dot{z}^2\right)\right]$$

$$= \frac{\partial T}{\partial q_r}$$

$$\therefore Q_r = \sum m\ddot{x}\frac{\partial \dot{x}}{\partial \dot{q}_r} + m\ddot{y}\frac{\partial \dot{y}}{\partial \dot{q}_r} + m\ddot{z}\frac{\partial \dot{z}}{\partial \dot{q}_r} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r}$$

If $Q_r = -\dfrac{\partial V}{\partial q_r}$ V being independent of $\dot{q}_r$

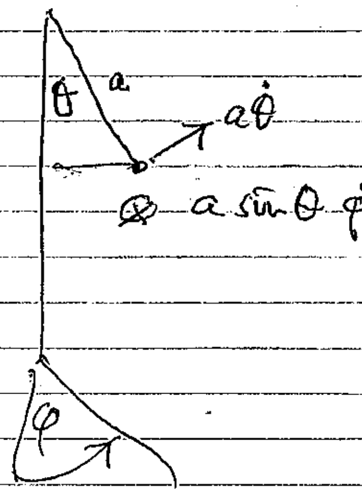
$$LHS = \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_r}\right) - \frac{\partial V}{\partial q_r}$$

$L = T - V$ gives $\quad \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_r}\right) - \dfrac{\partial L}{\partial q_r} = 0$

i.e. $\int L\,dt$ is stationary.

Useful example to illustrate $\frac{\partial \dot{x}}{\partial \dot{q}} = \frac{\partial x}{\partial q}$ etc C68

$T = \frac{1}{2} m a^2 [\dot{\theta}^2 + \sin^2\theta \, \dot{\varphi}^2]$



$x = a \sin\theta \cos\varphi$

$y = a \sin\theta \sin\varphi$

$z = a \cos\theta$

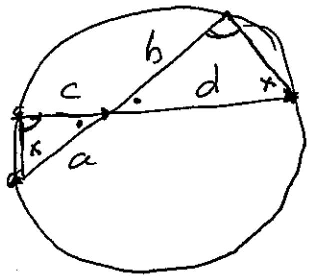
$\dot{x} = a \cos\theta \cos\varphi \, \dot{\theta} - a \sin\theta \sin\varphi \, \dot{\varphi}$

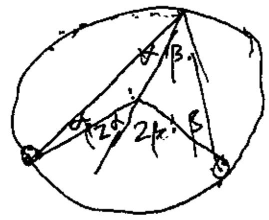
$\frac{\partial \dot{x}}{\partial \theta} = -a \sin\theta \cos\varphi \, \dot{\theta} - a \cos\theta \sin\varphi \, \dot{\varphi}$

$\frac{\partial x}{\partial \theta} = a \cos\theta \cos\varphi$

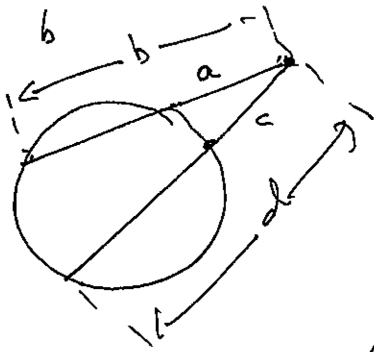
$\frac{d}{dt}\left(\frac{\partial x}{\partial \theta}\right) = -a \sin\theta \cos\varphi \, \dot{\theta} - a \cos\theta \sin\varphi \, \dot{\varphi}$

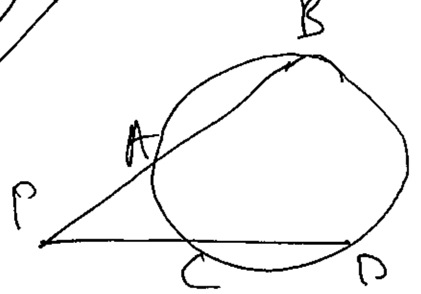
$\frac{\partial \dot{x}}{\partial \dot{\theta}} = a \cos\theta \cos\varphi .$





$\frac{a}{c} = \frac{d}{b}$ $\therefore$ $a \cdot b = c \cdot d$.

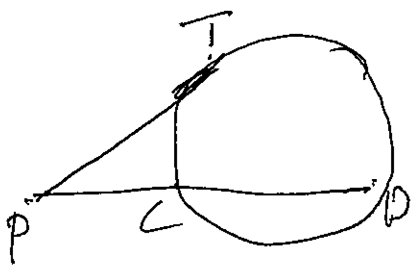


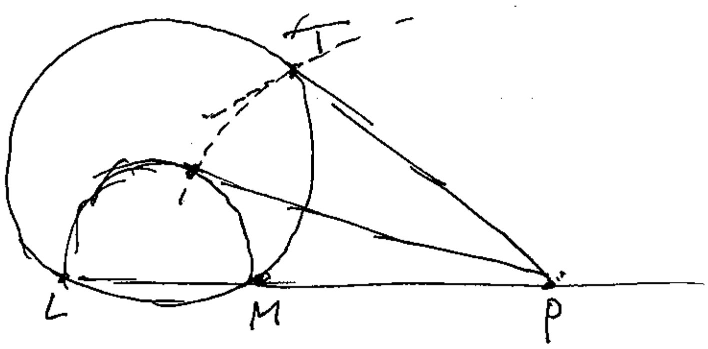


$PA \cdot PB$

$= PC \cdot PD$

$PC \cdot PD = PT^2$





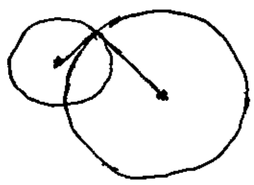
Circle through L, M. Tangent PT

$PT^2 = PL \cdot PM$

MacRobert, p6, q.13

Prove that $\left|\dfrac{z-1}{z+1}\right| =$ constant and amp$\left(\dfrac{z-1}{z+1}\right) =$ constant are orthogonal circles.

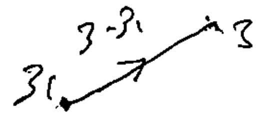
("amp" is short for "amplitude", ie θ for the point; "argument" is used more now.)



Orthogonal circles. The tangents at intersection go through centres

$$\left|\frac{z-1}{z+1}\right| = \sqrt{a}.$$

$$|z-1| = \sqrt{a}\,|z+1|$$

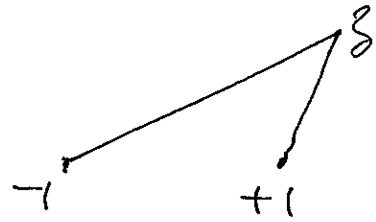


What is the meaning of $|z-z_1|$?

It is distance of z from z_1

So $|z-1|$ is distance of z from 1

$|z+1|$ — — — z -1



Let $z = x+iy$ $\quad |z-1|^2 = |x-1+iy|^2 = (x-1)^2+y^2$

$$|z+1|^2 = |x+1+iy|^2 = (x+1)^2+y^2$$

So $(x-1)^2+y^2 = a\{(x+1)^2+y^2\}$

$$\begin{aligned}0 &= ax^2 + 2ax + a + ay^2 \\ &\quad -x^2 + 2x - 1 - y^2 \\ &= (a-1)x^2 + (a-1)y^2 + 2(a+1)x + a - 1\end{aligned}$$

$$\begin{aligned}0 &= x^2+y^2 + \frac{2(a+1)}{a-1}x + 1 \\ &= \left(x + \frac{a+1}{a-1}\right)^2 + y^2 + 1 - \left(\frac{a+1}{a-1}\right)^2\end{aligned}$$

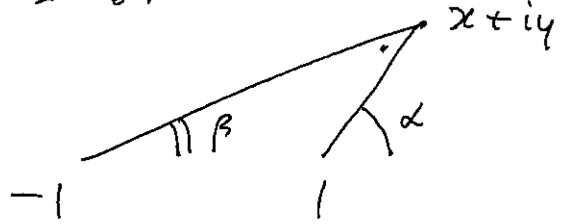
Centre is $\left(-\dfrac{a+1}{a-1}, 0\right)$, radius squared $\left(\dfrac{a+1}{a-1}\right)^2 - 1$

p2
C70

Q.3 $r_1^2 = \left(\frac{a+1}{a-1}\right)^2 - 1 = \frac{a^2+2a+1}{a^2-2a+1} - 1$

$$= \frac{4a}{a^2-2a+1} = \frac{4a}{(a-1)^2}$$

$$\arg \frac{z-1}{z+1} = 6.$$



$$\arg \frac{z-1}{z+1} = \alpha - \beta$$

$$\tan \alpha = \frac{y}{x-1} \qquad \tan\beta = \frac{y}{x+1}$$

$$\tan(\alpha-\beta) = \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}$$

$$= \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta + \sin\alpha\sin\beta}$$

÷ above & below by $\cos\alpha\cos\beta$

$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$= \frac{\frac{y}{x-1} - \frac{y}{x+1}}{1 + \frac{y}{x-1}\cdot\frac{y}{x+1}}$$

× above & below by $(x-1)(x+1)$

$$= \frac{y(x+1) - y(x-1)}{x^2-1+y^2}$$

$(x-p)^2+(y-q)^2=r^2$

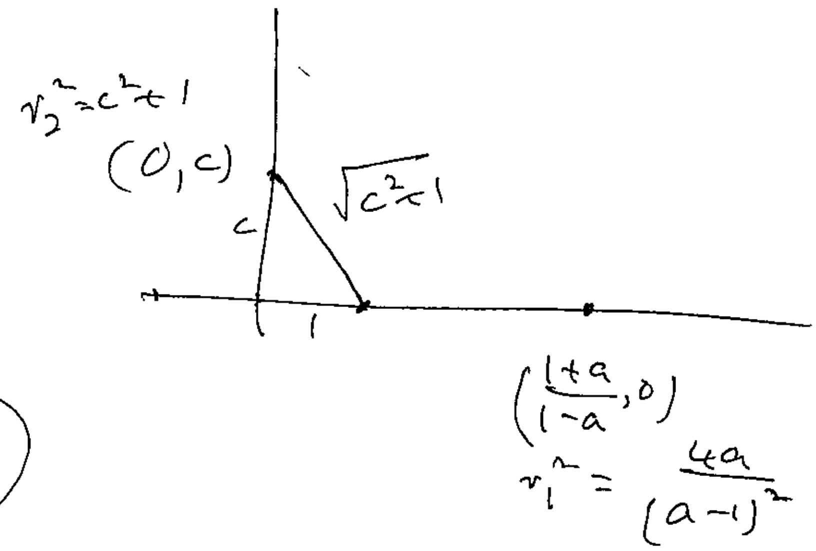
$$= \frac{2y}{x^2+y^2-1} = \text{constant } \frac{1}{c}$$

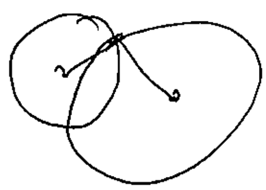
$$x^2+y^2-1 = 2cy$$

$$x^2+y^2-2cy-1=0$$

$$x^2+(y-c)^2 = c^2+1$$

Centre $(0, c)$ $\quad r_2^2 = c^2+1$





For orthogonal circles r_1, r_2 distance between centres for rt $\angle d$ $\triangle$.

$$r_1^2 + r_2^2 = (\text{distance between centres})^2$$

To prove $\frac{4a}{(a-1)^2} + c^2 + 1 = c^2 + \left(\frac{1+a}{1-a}\right)^2$

This will be so if

$$\frac{4a}{(a-1)^2} + 1 = \left(\frac{1+a}{a-1}\right)^2$$

i.e. if $\quad 4a + (a-1)^2 = (a+1)^2$

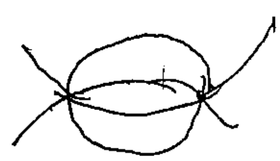
$$4a + a^2 - 2a + 1 = (a+1)^2$$

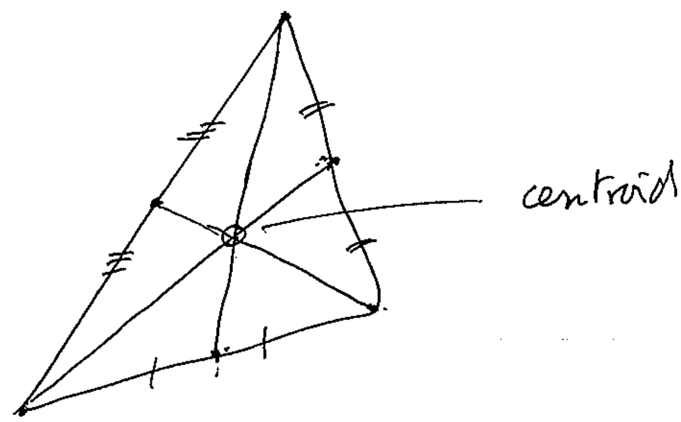
which it does.

Reverse order for formal proof

$4a \leq (a+1)^2$ since $(a+1)^2 - 4a = a^2 - 2a + 1$
$= (a-1)^2 \geqslant 0.$

All the circles for any constant pass through $(-1,0)$ and $+1, 0$





3 Prove that modulus of the quotient of two conjugate numbers is unity.

$x+iy$ and $x-iy$ are conjugate.

To prove $\left|\dfrac{x+iy}{x-iy}\right| = 1$.

We know $|z_1 z_2| = |z_1|\,|z_2|$ multiplication.

? $\left|\dfrac{z_1}{z_2}\right| = |z_1|/|z_2|$

$$\frac{z_1}{z_2} = w \iff z_1 = w z_2.$$

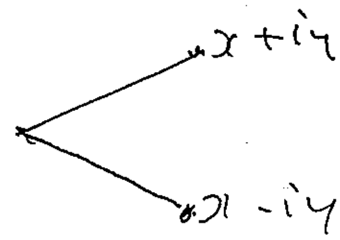
$$|z_1| = |w|\,|z_2|$$

$$\therefore |w| = \frac{|z_1|}{|z_2|}$$

$$|x+iy| = \sqrt{x^2+y^2}$$

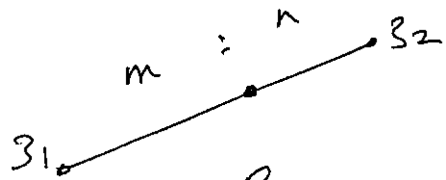
$$|x-iy| = \sqrt{x^2+y^2}$$

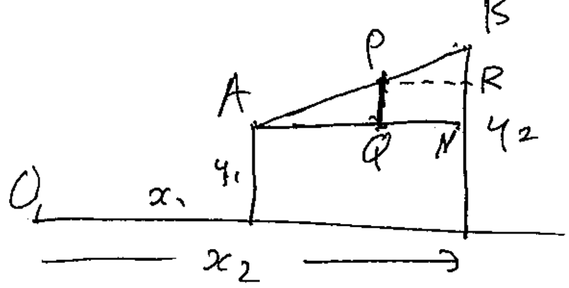
$$\therefore \frac{|x+iy|}{|x-iy|} = 1$$



Q.1. p 6. MacRobert.

Divide in ratio m:n





P is to divide AB in ratio m:n

$$\frac{m}{n} = \frac{AP}{PB} = \frac{AQ}{QN}$$

If P ~~is~~ is (x, y)

$$AQ = x - x_1$$
$$QN = x_2 - x$$

$$\therefore \quad \frac{x - x_1}{m} = \frac{x_2 - x}{n}$$

$\times (mn)$

$$nx - nx_1 = mx_2 - mx$$
$$(m+n)x = mx_2 + nx_1$$
$$\therefore \quad x = \frac{mx_2 + nx_1}{m+n}$$

$$\frac{NR}{RB} = \frac{AP}{PB} = \frac{m}{n} \qquad y = \frac{my_2 + ny_1}{m+n}$$

P represents $x + iy = \dfrac{m(x_2 + iy_2) + n(x_1 + iy_1)}{m+n}$

$$= \frac{mz_2 + nz_1}{m+n}$$

$\mathbb{C} \to \mathbb{C}$ $w = \sin z$

$$\sin iy = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = \frac{e^{-y} - e^{y}}{2i} = i\left(\frac{e^{y} - e^{-y}}{2}\right) = i \sinh y.$$

Real part $= 0$.

$z = x + iy$ $\dfrac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$

$$= \frac{1}{2i}\left\{e^{-y}e^{ix} - e^{y}e^{-ix}\right\}$$

~~Real part~~ $= \dfrac{1}{2i}\left[e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)\right]$

Real part $= \frac{1}{2}(e^{-y}\sin x + e^{y}\sin x)$
$= \sin x \cosh y.$

$k = \sin x \cosh y.$

ASN $= \sin^{-1}$

We want a collection (x, y, k).
For $k > 1$, $y \geq \cosh^{-1} k$. Take in turn $y_0 = 1, 2, 3 \ldots$
and $k = \cosh y_0$. $x = \sin^{-1}(k/\cosh y)$ $k = \cosh y_0$.
For $y \geq y_0$,
$y_0 = 0$, $k = 1$. $x = \sin^{-1}(1/\cosh y)$. This gives
a succession of x, y for $k = 1$.
$y_0 = m$ $k = \cosh m$ $x = \sin^{-1}(k/\cosh y)$ for $y \geq m$.
$\cosh 4.8 = 121.510$ Take 0.4 as unit ~~separate~~ for y. 4.8 to k
Thus $y = 0, 0.4, 0.8, 1.2 \ldots$.
table. Plotted at $0, 1 \ldots 12$.

We plot x, y. The value of $\cosh y$ is not plotted.
$\xi = \dfrac{2}{\pi}\sin^{-1}(k/\cosh y)$ gives ~~cosh~~ x as fraction of $\frac{\pi}{2}$.

Let $C(N) = \cosh(0.4 \times n)$

```
~~DEF FNC(N)~~ PROC COSH(N)
  X = 0.4 * N
  Y = EXP(X) + EXP(-X)
  C(N) = Y/2   gives cosh(n 0.4)
END PROC
```

The program is to find a number of points on the graph $k = \sin x \cosh y$. C75

k will be in turn $0, 0.4, 0.8 \ldots$ 4.8

so $K_m = \cosh(0.4 * M)$.

Having m, we take $n = m, m+1 \ldots$ 12

and find $x = \sin^{-1}(K/\cosh y_n)$

where $y_n = 0.4 * n$.

PROCCOSH(Y) finds ~~cosh y_n~~ cosh Y

```
PROC GRAPH
 M=0
 REPEAT
  Y = 0.4 * M
  K = FNCOSH(Y)   P. "K=";K
  N = M
  REPEAT   Y = 0.4×N
   X = 2/π ASN(K/FNCOSH(Y))
   P."X=";X;"  N=";N
   N = N+1
  UNTIL N>12
  P."---"
  M = M+1
 UNTIL M>12
ENDPROC.
```

We are going to relax in turn

~~$y = 0,\ 0.4,\ 0.8 \ldots$~~ say ~~$y = M \times 0.4$~~

$k = 1,\ \cosh 0.4,\ \cosh 0.8, \ldots\ldots$

and also $y = 0.4 \times 0.8$ ~~M~~ for $N \geq M$

when $k = \cosh M \times 0.4$

So choose M. $k = \cosh M(0.4)$.

Then $N = M, M+1 \ldots\ldots$ 12

find $X(M,N) = \dfrac{2}{\pi} \sin^{-1}\left(\dfrac{k_m}{\cosh y_n}\right)$

$k = 0.2,\ 0.4,\ 0.6,\ 0.8$

$.2 = \sin x \cosh y$

$y = 0$

$y = 0.4$

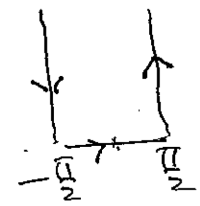
```
DEF PROCLESS
    K=0.2
    REPEAT . P"K=";K
     N=0 :
    REPEAT  Y=0.4*N
   X= (2/PI) ASN (K/FN COSH(Y))
    P. "X=";X;"  N=";N
      N=N+1
      UNTIL N>12
     P"---"
      K=K+0.2
     UNTIL K>0.8
     ENDPROC
```

A study of $\mathbb{C} \to \mathbb{C}$: $w = \sin z$.

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = \frac{1}{2i}(e^{ix-y} - e^{-ix+y})$$
$$= \frac{1}{2i}\left[e^{-y}(\cos x + i \sin x) - e^{y}(\cos x - i\sin x)\right]$$
$$= \tfrac{1}{2}(e^{-y} + e^{y})\sin x + i\tfrac{1}{2}(e^{y} - e^{-y})i \cos x$$
$$= \cosh y \sin x + i \sinh y \cos x.$$

$u = \cosh y \sin x \qquad v = \sinh y \cos x$

$v = 0$ if $y = 0$; $x = -\frac{\pi}{2}$; $x = \frac{\pi}{2}$



We are interested in the behaviour of w for z in the strip enclosed by these three lines.

$u = 0 \Rightarrow \sin x = 0$ so $x = 0$ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, y arbitrary.

Consider z moving round the boundary in the direction shown by the arrows.

When $x = -\frac{\pi}{2}$, $u = -\cosh y$. $\cosh t$ is 1 for $t = 0$ and rises steadily to $+\infty$ for $t = +\infty$.

Thus u starts at $-\infty$ and grows steadily to -1 at $z = -\pi/2$.

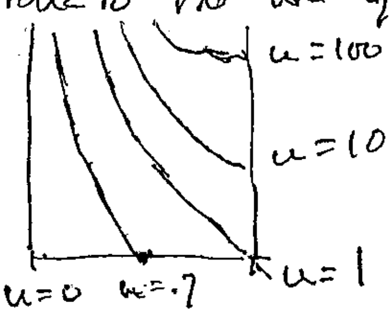
Between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, z is real and $w = \sin z$ increases from -1 to $+1$.

When $z = \pi/2$, $\sin x = 1$ and $u = \cosh y$, so u rises from 1 to $+\infty$ as z moves along the right-hand boundary.

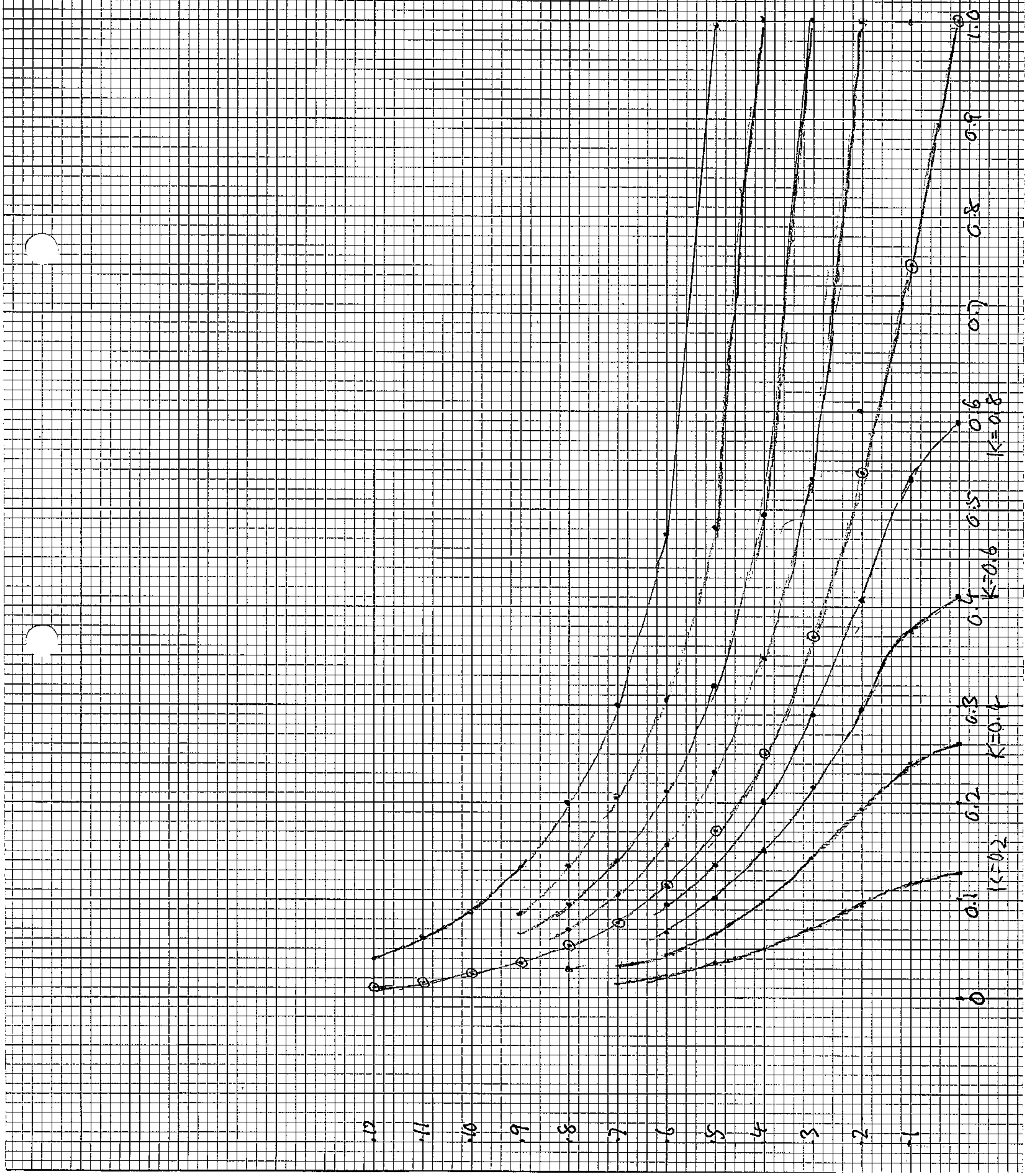
Thus as z moves around $\sqcup$, w is real and goes steadily from $-\infty$ to $+\infty$, taking each value once.

<u>The lines u = constant.</u>

$u = \cosh y \sin x$. Suppose $0 < x < \pi/2$, then x decreasing implies y increasing. If $x \to 0$, $\sin x \to 0$ so $\cosh y \to \infty$ so $y \to \infty$. The curves u = const are thus all asymptotic to the line $y = 0$.



If $(x,y) \to (-x,y)$ then $u \to -u$, so the curves in the region $-\pi/2 < x < 0$ are the reflection in the y axis of those to the right and the values of u are the negatives of the curve they reflect.


0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
K=0.2
K=0.4
K=0.6
K=0.8
.1
.2
.3
.4
.5
.6
.7
.8
.9
.10
.11
.12

x, y metres $K = \sin \frac{\pi}{2} x \cosh y$. $y = 0.4 \times N$. C 80

K=1
X=1 N=0
X=0.751883096 N=1
X=0.537683651 N=2
X=0.372488828 N=3
X=0.253652831 N=4
X=0.171273631 N=5
X=0.11519038 N=6
X=0.077330551 N=7
X=0.0518713337 N=8
X=0.03478099 N=9
X=0.0233175886 N=10
X=0.0156312093 N=11
X=0.0104782028 N=12

K=1.08107237
X=1 N=1
X=0.599243252 N=2
X=0.407330196 N=3
X=0.275541824 N=4
X=0.185549525 N=5
X=0.124645598 N=6
X=0.0836348538 N=7
X=0.0560871675 N=8
X=0.0376039277 N=9
X=0.0252089523 N=10
X=0.0168987551 N=11
X=0.0113277819 N=12

K=1.33743495
X=1 N=2
X=0.52906968 N=3
X=0.347313321 N=4
X=0.231374154 N=5
X=0.154739556 N=6
X=0.103627451 N=7
X=0.0694353839 N=8
X=0.0465356049 N=9
X=0.0311912631 N=10
X=0.0209073831 N=11
X=0.0140144138 N=12

K=1.81065557
X=1 N=3
X=0.495861245 N=4
X=0.319653238 N=5
X=0.211291719 N=6
X=0.140820489 N=7
X=0.0941601695 N=8
X=0.0630480813 N=9
X=0.0422416717 N=10
X=0.0283092213 N=11
X=0.0189743688 N=12

K=2.57746447
X=1 N=4
X=0.480479354 N=5
X=0.307048457 N=6
X=0.202209978 N=7
X=0.134548063 N=8
X=0.0899006754 N=9
X=0.0601763912 N=10
X=0.0403117724 N=11
X=0.0270140824 N=12

K=3.76219569
X=1 N=5
X=0.473462786 N=6
X=0.301345731 N=7
X=0.1981166 N=8
X=0.131725735 N=9
X=0.0879855566 N=10
X=0.0588856925 N=11
X=0.0394445024 N=12

K=5.55694717
X=1 N=6
X=0.470287673 N=7
X=0.298775131 N=8
X=0.196274677 N=9
X=0.130456759 N=10
X=0.0871247877 N=11
X=0.0583056683 N=12

K=8.25272841
X=1 N=7
X=0.468856379 N=8
X=0.297618401 N=9
X=0.195446505 N=10
X=0.129886404 N=11
X=0.086737968 N=12

K=12.2866462
X=1 N=8
X=0.468212314 N=9
X=0.297098306 N=10
X=0.195074273 N=11
X=0.129630093 N=12

K=18.3127791
X=1 N=9
X=0.467922726 N=10
X=0.296864543 N=11
X=0.194906996 N=12

K=27.3082328

C81

```
K=0.2
X=0.128188434  N=0
X=0.118457992  N=1
X=0.0955585432  N=2
X=0.0704630399  N=3
X=0.0494486247  N=4
X=0.0338589514  N=5
X=0.022917522  N=6
X=0.0154296148  N=7
X=0.0103632493  N=8
X=6.95287619E-3  N=9
X=4.66251674E-3  N=10
X=3.12594031E-3  N=11
X=2.09554973E-3  N=12
-----
K=0.4
X=0.261979761  N=0
X=0.241286679  N=1
X=0.193359469  N=2
X=0.141808327  N=3
X=0.0991987755  N=4
X=0.0678141567  N=5
X=0.0458648105  N=6
X=0.0308683027  N=7
X=0.0207292461  N=8
X=0.0139065819  N=9
X=9.32528359E-3  N=10
X=6.25195599E-3  N=11
X=4.19112217E-3  N=12
-----
K=0.6
X=0.409665529  N=0
X=0.374566919  N=1
X=0.296168621  N=2
X=0.215022809  N=3
X=0.149568942  N=4
X=0.101964361  N=5
X=0.0688719821  N=6
X=0.0463251847  N=7
X=0.0311007443  N=8
X=0.0208619475  N=9
X=0.0139885508  N=10
X=9.37812244E-3  N=11
X=6.28674002E-3  N=12
-----
K=0.8
X=0.590334471  N=0
X=0.530354704  N=1
X=0.408202511  N=2
X=0.291340355  N=3
X=0.200914293  N=4
X=0.136413473  N=5
X=0.0919698641  N=6
X=0.061809479  N=7
X=0.0414805112  N=8
X=0.0278198053  N=9
X=0.0186525688  N=10
X=0.012504515  N=11
X=8.382426E-3  N=12
-----
```

Gamma Function

Macrobert p109 Defines by $\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_1^\infty \{(1+\frac{z}{n})e^{-z/n}\}$

Where γ is Euler's constant. RHS is an integral fn, with zeros $z = 0, -1, -2, \ldots$ $\Gamma(z)$ is regular except for poles at $0, -1, -2 \ldots$ It has no zeros in finite plane.

p 141 Gauss' definition $\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)(z+2)\ldots(z+n)}$

Then $\frac{1}{\Gamma(z)} = \lim_{n\to\infty}\{z(1+\frac{z}{1})(1+\frac{z}{2})\ldots(1+\frac{z}{n})e^{-z\ln n}\}$

$= \lim_{n\to\infty}\{z(1+\frac{z}{1})e^{-z}.(1+\frac{z}{2})e^{-\frac{z}{2}}\ldots(1+\frac{z}{n})e^{-\frac{z}{n}}e^{z(1+\frac{1}{2}+\ldots\frac{1}{n}-\ln n)}\}$

$= e^{\gamma z} z \prod_1^\infty (1+\frac{z}{n})e^{-z/n}\}$ in agreement with p109.

The following properties are easily deduced from Gauss def.

(i) $\Gamma(z+1) = z\Gamma(z)$

(ii) $\Gamma(m+1) = m!$ if $m \in \mathbb{N}$

(iii) $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$.

(iv) residue at $z=-m$ is $(-1)^m/m!$ $m = 0,1,2\ldots$.

Information on $\psi(z) = \frac{d}{dz}\ln\Gamma(z+1)$.

Euler's Definition $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt$ $R(z) > 0$.

Proof of equivalence, p142. (See p. ②.)

Contour integral p143

Asymptotic expansion, p146 from $\frac{d^2}{dz^2}\ln\Gamma(z) = \sum_{n=0}^\infty \frac{1}{(z+n)^2}$

Comes easily from p109 def. Differentiate twice formally & then check integration by uniform convergence.

Equivalence of definitions.

Remember $\Gamma(z)$ gives $(n-1)!$ when $z = n$.

Euler's definition is $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$.

Gauss $\lim_{n\to\infty} \dfrac{n!\, n^z}{z(z+1)\cdots(z+n)}$.

We have $\int_0^1 y^{z-1}(1-y)^n dy = \dfrac{n}{z}\int_0^1 y^z (1-y)^{n-1} dy$

by partial integration $= \dfrac{n(n-1)\cdots 1}{z(z+1)(\cdot)\cdots(z+n-1)} \int_0^1 y^{z+n-1} dy$

$$= \frac{n!}{z(z+1)\cdots(z+n)}$$

Let $y = u/n$

$$\int_0^1 y^{z-1}(1-y)^n dy = \int_0^n \frac{u^{z-1}}{n^{z-1}}\left(1-\frac{u}{n}\right)^n \frac{du}{n} = \frac{1}{n^z}\int_0^n u^{z-1}\left(1-\frac{u}{n}\right)^n du$$

$$\therefore \frac{n!\, n^z}{z(z+1)\cdots(z+n)} = \int_0^n u^{z-1}\left(1-\frac{u}{n}\right)^n du$$

When $n\to\infty$, two changes occur. (i) $\int_0^n$ becomes $\int_0^\infty$; (ii) $\left(1-\frac{u}{n}\right)^n \to e^{-u}$. We have to show carefully that it is justified to carry these out formally in the integral expression. MacRobert does this in some detail, pp 142-3.

$\Pi(z) = z!$

$\Pi(z) = z\,\Pi(z-1)$

If z near 0, $\Pi(z-1) = \dfrac{\Pi(z)}{z} \sim \dfrac{\Pi(0)}{z} = \dfrac{1}{z}$

So, for small t, $\Pi(-1+t) \sim \dfrac{1}{t}$

$t = z+1$, t small when z ~~near~~ -1 $\Pi(z) \sim \dfrac{1}{z+1}$

z near -2. $z = -2+t$

$$\Pi(-2+t) = \frac{\Pi(-1+t)}{-1+t}$$

$$\sim \frac{1}{-1+t}\cdot\frac{1}{t} \sim \frac{-1}{t}$$

Near $z=-2$ $\Pi(z) \sim \dfrac{-1}{z+2}$

z near -3. $z = -3+t$

$$\Pi(-3+t) = \frac{\Pi(-2+t)}{-2+t}$$

$$= \frac{1}{-2+t}\cdot\frac{1}{-1+t}\,\Pi(-1+t)$$

$$\sim \frac{1}{-2+t}\cdot\frac{1}{-1+t}\,\frac{1}{t} \sim \frac{1/2!}{t}$$

$$\Pi(z) \sim \frac{1/2!}{z+3}$$

So we have

-5	-4	-3	-2	-1	0	1	2	3
$\dfrac{+1/4!}{z+5}$	$\dfrac{-1/3!}{z+4}$	$\dfrac{+1/2!}{z+3}$	$\dfrac{-1/1!}{z+2}$	$\dfrac{1}{z+1}$	1	1	$2!$	$3!$

$$\frac{-1/3!}{z+4} \quad \frac{1/2!}{z+3} \quad \frac{-1/1!}{z+2} \quad \frac{1}{z+1} \qquad \Gamma(z)$$

$$\frac{-1/3!}{z+4} \quad \frac{+1/2!}{z+3} \quad \frac{-1/1!}{z+2} \quad \frac{1}{z+1} \quad 1 \quad 1! \quad 2! \quad 3! \quad 4!$$

$$-3! \quad -2! \quad -1! \quad -1 \quad -\frac{1}{z} \quad \frac{+1/1!}{z+1} \quad \frac{-1/2!}{z+2} \quad \frac{+1/3!}{z+3} \quad \frac{-1/4!}{z+4}$$

Product $\frac{+1}{z+4} \quad \frac{-1}{z+3} \quad \frac{+1}{z+2} \quad \frac{1}{z+1} \quad \frac{1}{z}$

$$\frac{1}{\Gamma(z)} = e^{\gamma z} \prod_{1}^{\infty} \left(1+\frac{z}{n}\right)e^{-z/n}$$

$$-\ln\Gamma(z) = \gamma z + \sum_{n=1}^{\infty}\left[\ln\left(1+\frac{z}{n}\right) - z/n\right]$$

Formal diff fn

$$-\frac{d}{dz}\ln\Gamma(z) = \gamma + \sum_{n=1}^{\infty}\left[\frac{1/n}{1+z/n} - \frac{1}{n}\right]$$

$$= \gamma + \sum_{n=1}^{\infty}\left[\frac{1}{z+n} - \frac{1}{n}\right]$$

$$= \gamma + \sum_{n=1}^{\infty}\frac{-z}{n(z+n)}$$

This is useful : Integration leads back to prev

$$\therefore \frac{d}{dz}\ln\Gamma(z) = -\gamma + \sum_{n=1}^{\infty}\frac{z}{n(z+n)}$$

$$= -\gamma + \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right)$$

Formal diff again

$$\frac{d^2}{dz^2}\ln\Gamma(z) = \sum_{n=1}^{\infty}$$

$$I = \frac{1}{2\pi i}\oint f(\zeta)\left[\frac{1}{\zeta - z} - \frac{1}{\zeta} - \frac{z}{\zeta^2} \cdots - \frac{z^{m-1}}{\zeta^m}\right] d\zeta$$

<u>First part</u> $\frac{1}{2\pi i}\oint \frac{f(\zeta)\,d\zeta}{\zeta - z}$

We replace curve by circles around the separate singularities, namely ① $\zeta = z$
② $\zeta = a_1, a_2 \ldots$ singularities of $f(\zeta)$

① $\frac{1}{2\pi i}\oint \frac{f(\zeta)}{\zeta - z}\,d\zeta$ around z gives $f(z)$

② Consider a singularity, $\underline{a}$, of $f(\zeta)$ and suppose (for example)

$$f(\zeta) = \frac{\alpha}{(\zeta-a)^3} + \frac{\beta}{(\zeta-a)^2} + \frac{\gamma}{z-a} + \text{regular part}$$

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta-a)-(z-a)} = -\left[\frac{1}{z-a} + \frac{\zeta-a}{(z-a)^2} + \frac{(\zeta-a)^2}{(z-a)^3} + \cdots\right]$$

Pick out term in $\frac{1}{\zeta-a}$ (the only one that contributes anything) in the product of the above. It is

$$-\frac{1}{\zeta-a}\left[\frac{\alpha}{(z-a)^3} + \frac{\beta}{(z-a)^2} + \frac{\gamma}{(z-a)}\right]$$

① $\frac{1}{2\pi i}\oint \frac{f(\zeta)\,d\zeta}{\zeta - z} = -\left[\frac{\alpha}{(z-a)^3} + \frac{\beta}{(z-a)^2} + \frac{\gamma}{z-a}\right]$ *

This is what we get if we replace ζ by z in principal part of $-f(\zeta)$ at $\underline{a}$, denoted by $-g\left(\frac{1}{z-a}\right)$

<u>Second part</u> $-\frac{1}{2\pi i}\oint f(\zeta)\left[\frac{1}{\zeta} + \frac{z}{\zeta^2} + \cdots \frac{z^{m-1}}{\zeta^m}\right] d\zeta$

~~also~~ $\oint \frac{f(\zeta)}{\zeta^r}\,d\zeta$ is independent of z, and we

The integral is taken around $\underline{a}$, a polynomial of degree $m-1$ denote it by $h(z)$ for $\underline{a}$ (or less, if some integral is zero.)

Thus $I = \sum_K \left[-g_K\left(\frac{1}{z-a_K}\right) + h_K(z)\right] + f(z)$

where $h_K(z)$ is the polynomial arising from the integral around a_K. $f(z)$ from ① above, [ ] from ②.

Now $\frac{1}{\zeta} + \frac{z}{\zeta^2} + \cdots \frac{z^{m-1}}{\zeta^m} = \frac{1 - z^m/\zeta^m}{\zeta - z}$ by GP formula.

$$I = \frac{1}{2\pi i}\oint f(\zeta)\left[\frac{1}{\zeta-z} - \frac{1 - z^m/\zeta^m}{\zeta - z}\right) d\zeta$$

$$= \frac{1}{2\pi i}\oint \frac{f(\zeta)\, z^m/\zeta^m}{\zeta - z}$$

$$= \frac{1}{2\pi i}\oint \frac{z^m f(\zeta)}{\zeta^m(\zeta - z)}\,d\zeta$$

Let $I_n = \int_0^\infty x^n e^{-x}\,dx$

Integrate by parts. $\int u\,dv = uv - \int v\,du$

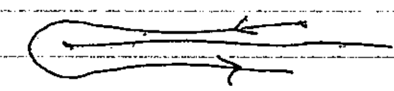
Take $v = -e^{-x}$, $u = x^n$ $\int x^n e^{-x} dx = -x^n e^{-x} + \int e^{-x} . n x^{n-1} dx$

If $\int_0^\infty$, $\left[-x^n e^{-x}\right]_0^\infty = 0$ provided $\left[x^n e^{-x}\right]_{x=0} = 0$. This is so if $n > 0$.

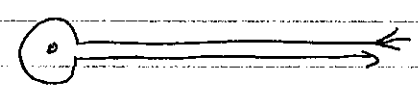
Then $I_n = n I_{n-1}$.

$I_0 = \int_0^\infty e^{-x} dx = \left[-e^{-x}\right]_0^\infty = 1.$

This agrees with $I_n = n!$ and for $n > 0$ we define $n!$ as I_n. (Gauss $\Pi(n)$; Euler, $\Gamma(n+1)$)

Contour Integral C the path 

C may be deformed to



Consider $\int_C z^n e^{-z} dz$, $n > 0$. Suppose the small circle has radius r. If $n > 0$, $z^n e^{-z}$ is finite on the circle, and length of arc is $2\pi r$.

$\therefore$ Integral around circle $\to 0$ as $r \to 0$.

We take $\arg z = 0$ ~~at~~ on upper line.

$z^n = e^{n \log z}$ When z goes round circle $\log z$ increases by $2\pi i$, so values of z^n below line are $e^{2\pi i n}$ times those above.

In the limit, integral $= (e^{2\pi i n} - 1)\int_0^\infty z^n e^{-z} dz$

$$\therefore \quad n! = \frac{1}{e^{2\pi i n} - 1} \int_C z^n e^{-z} dz$$

Let $I_n = \int_0^\infty x^n e^{-x}\,dx$.

This integral always converges at ∞, and converges at $x=0$ provided $n>-1$. Thus I_n is defined for real n with $n>-1$.

Suppose $N>0$. Apply $\int$ by parts to I_N.
$\int u\,dv = uv - \int v\,du$. Take $v = -e^{-x}$, $u = x^N$
$$\int_0^\infty x^N e^{-x}\,dx = \left[-x^N e^{-x}\right]_0^\infty - \int_0^\infty -e^{-x}\cdot N x^{N-1}\,dx$$
Provided $N>0$, $[\ ]=0$. Hence $I_N = N I_{N-1}$, for $N>0$.
As $N>0$, $N-1>-1$, so I_{N-1} is defined.
Now consider $n = p+iq$. $x^n = e^{n\log x} = e^{p\log x + iq\log x}$
$= e^{p\log x} e^{iq\log x} = x^p e^{iq\log x}$ $\therefore |x^n| = x^p$.
$\therefore |x^n e^{-x}| = x^p e^{-x}$ for $0<x<\infty$.
Hence $\int_0^\infty x^n e^{-x}\,dx$ converges provided $p>-1$, i.e. $\mathcal{R}(n)>-1$. Thus I_n is defined for $\mathcal{R}(n)>-1$, and the equation $I_n = I_{n+1}/(n+1)$ holds for $n>-1$. [This is $I_N = N I_{N-1}$, $N>0$ with $n=N-1$.]

For $\mathcal{R}(n)>-1$, we define $n! = I_n$.

Analytic continuation preserves an equation such as $f(z) = \frac{1}{z+1} f(z+1)$. If $-2<\mathcal{R}(z)$, $-1<\mathcal{R}(z+1)$ so $(z+1)!$ is defined. Hence for $-2<\mathcal{R}(z)$ we can define $z!$ as $\frac{1}{z+1} f(z+1)$. Similarly for $-3<\mathcal{R}(z)$, $(z+1)!$ is defined $\therefore$ $z! = \frac{1}{z+1} f(z+1) = \frac{1}{(z+1)(z+2)} f(z+2)$
In this way we can extend the definition of the function indefinitely.

Now $z!$ is finite for $R(z) > 0$.

If $-1 < R(z)$, $z! = \frac{1}{z+1}(z+1)!$ so we can get a pole only as $z \to -1$. In the same way, $z!$ is regular for $R(z) > -2$, ~~and~~ apart from the pole at $z = -1$ already mentioned, and we see there is a pole at $z = -2$.

The only singularities of $z!$ are poles at $-1, -2, -3, \ldots$

$$(-1+t)! = \frac{1}{t} \cdot t!$$

$0! = 1$ Hence, near $t = 0$

$$(-1+t)! \sim \frac{1}{t}$$

Putting $z = t+1$ $z! \sim \frac{1}{t+1}$ near -1.

Graph of $x!$ for real x

$$(-2+t)! = \frac{(-1+t)!}{t-1} = \frac{t!}{(t-1)t}$$

If $t \to 0$, this $\sim \frac{-1}{t}$ so $z! \sim \frac{-1}{z+2}$

$$(-3+t)! = \frac{1}{-2+t}(-2+t)! \sim \frac{1}{\cancel{2t}\,2t}$$

$$z! \sim \frac{1}{2(z+3)} \text{ near } z = -3$$

$$(-4+t)! = \frac{1}{-3+t}(-3+t)! \sim \frac{-1}{3.2t}$$

$$z! \sim \frac{-1}{3.2(z+4)} \text{ near } z = -4.$$

So we have
$$z! \sim \frac{(-1)^{k+1}}{(k-1)!\,(z+k)} \text{ near } z = -k$$

An approach to the Γ function from the start.

$n!$ is defined only for $n \in \mathbb{N}$. It is natural to try to interpolate it. $n! = n.(n-1)!$

If we write $\Gamma(n) = (n-1)!$, $n! = \Gamma(n+1)$ and we want to have $\Gamma(n+1) = n\,\Gamma(n)$.

Suppose then there exists a function that satisfies

(1) .... $G(z+1) = z\,G(z)$, analytic in some domain.

Then $\ln G(z+1) = \ln z + \ln G(z)$

Differentiate $\frac{d}{dz}\ln G(z+1) = \frac{1}{z} + \frac{d}{dz}\ln G(z)$

Differentiate again $\frac{d^2}{dz^2}\ln G(z+1) = -\frac{1}{z^2} + \frac{d^2}{dz^2}\ln G(z)$

$$\frac{d^2}{dz^2}\ln G(z+n+1) = -\frac{1}{(z+n)^2} + \frac{d^2}{dz^2}\ln G(z+n)$$

Combining these $\frac{d^2}{dz^2}\ln(z+n+1) = -\sum_{k=0}^{n}\frac{1}{(z+k)^2} + \frac{d^2}{dz^2}\ln\Gamma(z)$. ... (2)

The steps can be reversed, if suitable constants of integration are used.

Let $f(z) = \sum_{k=0}^{\infty}\frac{1}{(z+k)^2}$. Series represents a function analytic everywhere except for poles at $0, -1, -2, -3 \dots$

$$f(z+n+1) = \sum_{k=0}^{\infty}\frac{1}{(z+n+1+k)^2} = \sum_{n+1}^{\infty}\frac{1}{(z+l)^2}$$

$$f(z+n+1) - f(z) = \sum_{k=0}^{n}\frac{1}{(z+k)^2}$$

This suggests we define $\Gamma(z)$ by

$$\frac{d^2}{dz^2}\ln\Gamma(z) = f(z) = \sum_{r=0}^{\infty}\frac{1}{(z+r)^2}$$

Take $\int_1^z \frac{d^2}{dt^2}\ln\Gamma(t).dt = \int_1^z \sum_0^{\infty}\left(\frac{1}{t+r}\right)^2.dt$

$$\left.\frac{d}{dt}\ln\Gamma(t)\right]_1^z = \sum_0^{\infty}\left[-\frac{1}{t+n}\right]_1^z = \sum_0^{\infty}\left(-\frac{1}{z+n} + \frac{1}{n+1}\right)$$

$$\frac{d}{dz}\ln\Gamma(z) - \frac{\Gamma'(1)}{\Gamma(1)} = \left(-\frac{1}{z}+1\right) + \left(-\frac{1}{z+1}+\frac{1}{2}\right) + \left(-\frac{1}{z+2}+\frac{1}{3}\right) \cdot$$

$$= -\frac{1}{z} + \left(1 - \frac{1}{z+1}\right) + \left(\frac{1}{2} - \frac{1}{z+2}\right) + \dots$$

$$\frac{d^2}{dz^2}\ln\Gamma(z) = \sum_{n=0}^{\infty}\frac{1}{(z+n)^2}$$

We assume also $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(1)=1$.

$\int_1^z \cdots dz$ gives $\frac{d}{dz}\Gamma(z) - \frac{\Gamma'(1)}{\Gamma(1)} = \sum_{n=0}^{\infty}\left[\frac{1}{n+1} - \frac{1}{z+n}\right]$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{z+n-1}\right)$$

$$= \left(-\frac{1}{z} + 1\right) + \left(-\frac{1}{z+1} + \frac{1}{2}\right) + \left(-\frac{1}{z+2} + \frac{1}{3}\right)\cdots$$

$$= -\frac{1}{z} + \left(1 - \frac{1}{z+1}\right) + \left(\frac{1}{2} - \frac{1}{z+2}\right) + \cdots$$

$$= -\frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{z+n}\right)$$

As $\Gamma(1)=1$

$$\frac{d}{dz}\Gamma(z) = \Gamma'(1) - \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{z+n}\right)$$

$\cancel{\int_0^z dz}$ As $\Gamma(z+1) = z\Gamma(z)$ $\ln\Gamma(z+1) = \ln z + \ln\Gamma(z)$

$$\therefore \frac{d}{dz}\ln\Gamma(z+1) = \frac{1}{z} + \frac{d}{dz}\ln\Gamma(z)$$

$$\therefore \frac{d}{dz}\ln\Gamma(z+1) = \Gamma'(1) + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{z+n}\right)$$

$\int_0^z$ gives $\ln\Gamma(z+1) - \ln\Gamma(1) = \Gamma'(1)z + \sum_{n=1}^{\infty}\left(\frac{z}{n} - \ln(z+n) + \ln n\right)$

So $\ln\Gamma(z+1) = \Gamma'(1)z + \sum_{n=1}^{\infty}\left[\frac{z}{n} - \ln\left(1+\frac{z}{n}\right)\right]$

as $\Gamma(1)=1$

Put $z=1$. As $\Gamma(z+1) = z\Gamma(z)$, $\Gamma(2) = 1.\Gamma(1) = 1$.

$$\therefore 0 = \Gamma'(1) + \sum_{n=1}^{\infty}\left[\frac{1}{n} - \ln\left(1+\frac{1}{n}\right)\right]$$

$$\sum_{n=1}^{N}\ln\left(1+\frac{1}{n}\right) = \sum_{n=1}^{N}\ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty}\left[\ln(n+1) - \ln n\right]$$

$$= \ln(N+1) - \ln 1 = \ln(N+1)$$

$$\therefore \sum_{n=1}^{N}\left(\frac{1}{n} - \ln\left(1+\frac{1}{n}\right)\right) = -\ln(N+1) + \sum_{n=1}^{N}\frac{1}{n}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n} - \ln n\right) = \gamma$$

$\therefore -\Gamma'(1) = \gamma$. $\therefore \Gamma(z+1) = e^{-\gamma z}\prod_{1}^{\infty}\frac{e^{z/n}}{1+z/n}$ $\Gamma(z) = \frac{e^{-\gamma z}}{z}\prod\frac{e^{z/n}}{1+\frac{z}{n}}$

pp 110-111 Mittag-Leffler deals with the series

$$F_0(z) + \sum_{n=1}^{\infty} \{F_n(z) - h_n(z)\} \qquad (1)$$

with $\qquad |F_n(z) - h_n(z)| < \varepsilon_n \qquad n \geq 1 \qquad (2)$

where $\quad \sum \varepsilon_n$ is a convergent series of positive numbers (3)

In any finite domain, the tail of the series converges absolutely (since (2) involves absolute values) and uniformly (since it is dominated by a fixed series of constants). We have to refer to the tail because the earlier terms have poles inside the domain.
Absolute convergence allows terms to be rearranged.
Uniform convergence means that term-by-term integration is legitimate.

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=-\infty}^{+\infty}{}' \left(\frac{1}{z-n} + \frac{1}{n}\right)$$

Taking n and $-n$ together we have $\frac{1}{z-n} + \frac{1}{n} + \frac{1}{z+n} - \frac{1}{n}$
$= \frac{1}{z-n} + \frac{1}{z+n}$

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n}\right)$$

Integrate. $\frac{d}{dz}\log \sin \pi z = \cancel{\frac{1}{\sin \pi z} \cdot \pi \cos \pi z} = \pi \cot \pi z$

$$\therefore \quad \log \sin \pi z = \cancel{C + \log z + \sum_{n=1}^{\infty} \left(\log(z-n) + \log(z+n)\right)}$$

$$\therefore \quad \sin \pi z = \cancel{K z \prod_{n=1}^{\infty} [(z-n)(z+n)]}$$

$$\pi \cot \pi z - \frac{1}{z} = \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n}\right)$$

$$\int_0^Z \left(\pi \cot \pi z - \frac{1}{z}\right) dz = \sum_{n=1}^{\infty} \Big[\log(z-n) + \log(z+n)\Big]_0^Z$$

$$= \sum_{n=1}^{\infty} \left[\log(Z-n) - \log(-n) + \log(Z+n) - \log n\right]$$

$$= \sum_{n=1}^{\infty} \left\{\log\left(1 - \frac{Z}{n}\right) + \log\left(1 + \frac{Z}{n}\right)\right\}$$

$$\int_0^Z \left(\pi \cot \pi z - \frac{1}{z}\right) dz = \Big[\log \sin \pi z - \log z\Big]_0^Z$$

$$\log \sin \pi z - \log z = \log \frac{\sin \pi z}{z}$$

As $z \to 0$, $\frac{\sin \pi z}{z} \to \pi$

Hence LHS $= \log \sin \pi Z - \log Z - \log \pi$

$$\log \sin \pi z = \log \pi + \log z + \sum_{n=1}^{\infty}\left\{\log\left(1-\frac{z}{n}\right) + \log\left(1+\frac{z}{n}\right)\right\}$$

$$\therefore \sin \pi z = \pi z \prod \left(1-\frac{z}{n}\right)\left(1+\frac{z}{n}\right)$$

$$\sin \pi z = \pi z \prod \left(1 - \frac{z^2}{n^2}\right)$$

There is a general theorem about the possibility of factorizing a function.

In general $\Pi() = \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \dots \frac{1}{k_n}\left(\frac{z}{a_n}\right)^{k_n}}$

and function equals $e^{H(z)}\,\Pi$ where $H(z)$ is some integral function. This exponential is necessary. For instance, e^z has no zeros in finite plane. For it, $\Pi = 1$.

If $f(z)$ (is an integral function and) has no finite zeros, $f(z) = e^{H(z)}$ where $H(z)$ is integral.

$\log f(z)$ is regular except where $f(z)$ is 0 or ∞. $\log f(z) \neq \infty$ since $f(z)$ is regular everywhere. $\log f(z) \neq -\infty$ since $f(z)$ is never 0. So $\log f(z)$ is entire fn. $H(z)$

$$\therefore f(z) = e^{H(z)}$$

Book shows $\log f(z)$ regular by considering $f'(z)/f(z)$

Continuity at $z = a$ $\quad f(z) \to f(a)$ as $z \to a$, $f(a)$ finite.

§ 13. <u>Uniform Continuity.</u> For any $\varepsilon > 0$, $\exists \eta$:—

$$|z-a|<\eta \Rightarrow |f(z)-f(a)|<\varepsilon .$$

<u>Theorem</u> If $f(z)$ is continuous in a region, it is uniformly continuous in that region.

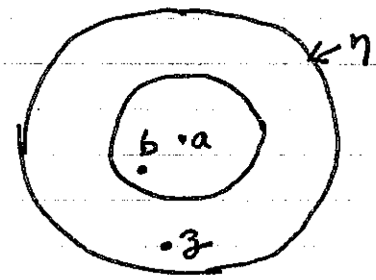
<u>Lemma</u> If $|f(z)-f(a)|<\varepsilon$ for $|z-a|<\eta$ and $|b-a|<\frac{1}{2}\eta$, then $|z-b|<\frac{1}{2}\eta \Rightarrow |f(z)-f(b)|<2\varepsilon$.

<u>Proof</u> ~~$z-b = z-b + a-b$~~

$\because z-a = z-b + b-a.$

$\therefore |z-a| \le |z-b| + |b-a|$

$\quad < \frac{1}{2}\eta + \frac{1}{2}\eta = \eta.$



Also $|b-a| < \frac{1}{2}\eta < \eta$.

$\therefore |f(z)-f(a)| < \varepsilon$

$\quad |f(b)-f(a)| < \varepsilon$

$\therefore |f(z)-f(b)| < 2\varepsilon$.

<u>Proof of Theorem</u> If $f(z)$ is uniformly continuous in a closed region A and also in a closed region B with a boundary common to A, then it is uniformly continuous in $A \cup B$. Hence, if we divide a rectangle

into 2 parts and function is u-cont in each it is u-cont in the rect-gle. If we know it is not u-cont in the rectangle, it must be not u-cont. in one of the parts. Thus we can obtain a region of non-continuity of arbitrarily small size, by repeated bisection. Let a be a point interior to every such rectangle. $f(z)$ is continuous at $z=a$, so $\exists\, \eta$:– $|z-a|<\eta \Rightarrow |f(z)-f(a)| < \varepsilon/2$. By the lemma above, for any point b with $|b-a| < \frac{1}{2}\eta$, $|z-b| < \frac{1}{2}\eta \Rightarrow |f(z)-f(b)| < \varepsilon$. Thus the circle centre a, radius $\frac{1}{2}\eta$ is a region of uniform continuity. But by repeated bisection we can obtain a region of non-uniform continuity lying entirely inside this circle. Contradiction, if theorem is false.

C95

McRobert.

p37 §21 Conformal representation.
§22 Singular points.

$\sin\theta$

p64. Ex. 8. Finding definite integral.
? p66 Ex. 10.
p67 No of poles & zeros §31
p68 Liouville's Theorem. §32
p69 §33 Fundamental Theorem of algebra.
p88 Classification of Uniform Fns

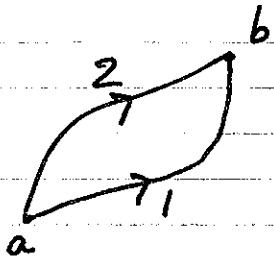
Myself — work through §13. §24, §34
Differentiation under ∫ sign.

Hurwitz Convent. p88. / p108 Meromorphic

Micromath article.
Problem from Sunday Times

Cauchy's Theorem. f analytic in a region R. C a simple closed curve inside R. Then $\oint f(z)\,dz = 0$, the integral being around C.

1) Integral independent of path.



Within R, we may find $\int_a^b f(z)\,dz$ along path 1 or path 2. These are equal, for circuit of loop is path 1 and then path 2 reversed. $\therefore \int_{a\,(1)}^{b} - \int_{a\,(2)}^{b} = 0$.

2) If $F(z) = \int_a^z f(t)\,dt$, then dF/dz exists and $F'(z) = f(z)$.

$$F(z+h) - F(z) = \int_a^{z+h} - \int_a^z = \int_z^{z+h} f(t)\,dt.$$

f is continuous, so for small h, $f(t)$ is very close to the constant value, $f(z)$. (Constant for fixed z, i.e. independent of t.)

$$\therefore \int_z^{z+h} f(t)\,dt \sim f(z)\int_z^{z+h} dt$$

$$= f(z)\,[t]_z^{z+h} = f(z)\,h$$

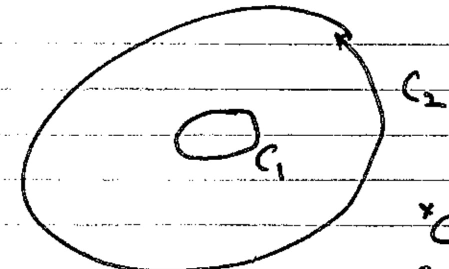
$$\therefore f(z) = \frac{F(z+h) - F(z)}{h}$$

Rather loose argument. Formal treatment, see MacRobert p. 52. Above shows essential idea.

3) $F(z)$ is analytic function. For it has a definite $F'(z)$.

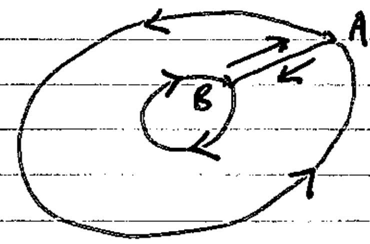
4) If $f(z)$ is analytic in the region between C_1 and C_2.*

Then $\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$.



*C_1 and C_2 are supposed to lie inside a region in which $f(z)$ is

Proof Draw a line to join C_1 to C_2 and take integral round path shown



$\int_{BA}$ cancels $\int_{AB}$ so

integral around this path $= \oint_{C_2} - \oint_{C_1} = 0$

$\therefore \oint_{C_2} = \oint_{C_1}$ Q.E.D. This generalizes easily to several curves inside.

5) If a curve C surrounds a point a,

$\oint_C \frac{dz}{z-a} = 2\pi i$. For, by (4), the integral equals that for a circle around a. Let $z-a = re^{i\theta}$

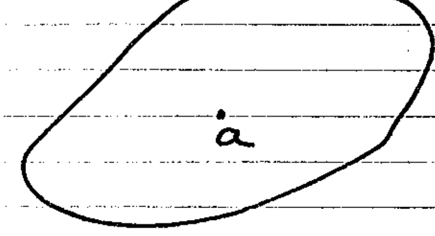
$dz = re^{i\theta} i\,d\theta$ $\oint \frac{dz}{z-a} = \int_0^{2\pi} \frac{re^{i\theta} i\,d\theta}{re^{i\theta}} = 2\pi i$.

6) Cauchy's Residue Theorem.

Definition. The residue of $f(z)$ at a is $\frac{1}{2\pi i}\oint f(z)dz$ taken round a small circle, enclosing a but not enclosing any other singularity of $f(z)$.

Theorem. a inside a curve C, all within R. Then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{z-a}.$$



Proof Enclose a in a small circle. $f(z)/(z-a)$ is analytic in region between this circle and C, so $\oint_C = \oint_{\text{small circle}}$.

On the small circle $f(z)$ is 'practically' $f(a)$

so $\oint_{\text{circle}} = f(a)\oint \frac{dz}{z-a} = 2\pi i f(a)$ Q.E.D.

GOTO p 4

7) Notes on residues.

(i) If $n \in \mathbb{Z}$, $n \neq 1$, residue of $(z-a)^{-n}$ at a is 0.

For $\int (z-a)^{-n} dz = \dfrac{(z-a)^{-n+1}}{-n+1}$ and this returns to its original value when a circle around $\underline{a}$ is described.

(ii) The residue of $(z-a)^{-1}$ at a is 1.

For $\dfrac{1}{2\pi i}\oint \dfrac{dz}{z-a} = \dfrac{1}{2\pi i}(2\pi i) = 1.$

(iii) If $f(z) = \dfrac{A_1}{z-a} + \dfrac{A_2}{(z-a)^2} + \ldots + \dfrac{A_n}{(z-a)^n} + \varphi(z)$

where $\varphi(z)$ is analytic at a, residue of $f(z)$ at a is A_1.

For $\oint \varphi(z)dz$ around curve close to a is 0.— we have no ~~other~~ singularity within it. Rest follows from (i) and (ii).

8) Theorem. If $\lim\limits_{z \to a} (z-a)f(z)$ is A, then the residue of $f(z)$ at $\underline{a}$ is A.

Proof. $F(z) = (z-a)f(z) = A + \varepsilon(z)$

where $\varepsilon(z) \to 0$ as $z \to a$.

$$\oint f(z) = \oint \frac{A}{z-a} + \oint \frac{\varepsilon(z)\,dz}{z-a}.$$

$$\therefore \oint f(z) - 2\pi i A = \oint \frac{\varepsilon(z)\,dz}{z-a}$$

If circle of radius ρ, $z - a = \rho e^{i\theta}$ $\dfrac{dz}{z-a} = i\,d\theta$.

RHS has modulus $\leq$ max $\varepsilon(z)$. length of path

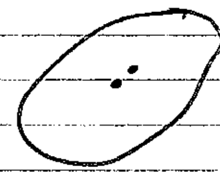
$= 2\pi$ max $\varepsilon(z)$.

$\therefore$ RHS can be made as small as we like by taking ρ small. But LHS is independent of ρ, once circle is small enough to exclude other singularities of $f(z)$. $\therefore \oint f(z) = 2\pi i A$ Q.E.D.

This theorem might be guessed from 7(iii)

9) Formula for $f'(z)$.

We have $f(a) = \frac{1}{2\pi i}\oint \frac{f(z)\,dz}{z-a}$



$$f(a+h) = \oint \frac{f(z)\,dz}{z-a-h}$$

$$\frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i}\oint \left\{ \frac{1}{z-a-h} - \frac{1}{z-a} \right\} \frac{f(z)\,dz}{h}$$

$$= \frac{1}{2\pi i}\oint \frac{h}{(z-a)(z-a-h)} \frac{f(z)\,dz}{h}$$

$$= \frac{1}{2\pi i}\oint \frac{f(z)\,dz}{(z-a)(z-a-h)}$$

~~Now~~ $f'(a)$ = limit of this. <u>We are entitled to take limit of integral = integral of limit.</u>

<u>Proof of this</u> Putting $z-a = q$, $\frac{1}{(z-a)(z-a-h)} = \frac{1}{q(q-h)} \to \frac{1}{q^2}$

$$\frac{1}{q(q-h)} - \frac{1}{q^2} = \frac{qh}{q^2(q-h)} = \frac{h}{q(q-h)}$$

Now a is inside curve, so $|z-a| > m$ where m is the least distance from z to the curve.

For small h, $|h| < \frac{1}{2}|q|$ so $|q-h| > \frac{1}{2}|q|$

Hence $|q(q-h)| > \frac{1}{2}|q|^2 > \frac{1}{2}m^2$, and $\frac{1}{(z-a)(z-a-h)}$ differs from its limiting value by $< \frac{2}{m^2}h$

$|f(z)| < M$, so $\frac{f(z)}{(z-a)(z-a-h)}$ differs from its limit by $< \frac{2M}{m^2}h$. Hence $\to$ uniformly.

We assume length of the curve is finite.

~~It follows~~

$$f'(a) = \frac{1}{2\pi i}\oint \frac{f(z)\,dz}{(z-a)^2}.$$

Note that $\frac{1}{(z-a)^2} = \frac{d}{da}\frac{1}{z-a}$. Here differentiation under integral sign is justified.

9) continued.

Applying the same process to $f'(a)$ we find

$$f''(a) = \frac{1}{2\pi i}\oint \frac{2f(z)\,dz}{(z-a)^3}$$

and eventually

$$f^{(n)}(a) = \frac{1}{2\pi i}\oint \frac{n!\,f(z)\,dz}{(z-a)^{n+1}}$$

$$= \frac{n!}{2\pi i}\oint \frac{f(z)\,dz}{(z-a)^{n+1}}$$

Thus, if f' exists, $\left(\frac{d}{dx}\right)^n f$ exists, for all n.

Entirely unlike situation for real variables. Our condition picks out extremely smooth functions.

Taylor, Laurent and residues

Note Deal with unbounded.

finite no

∞ pts

If $\overline{\lim} = l$
only finite no $> l + \alpha \quad \alpha > 0$.

$f(z) = c_0 + c_1 z + c_2 z^2 + \dots$

Consider the ~~set~~ sequence $|c_0|, \sqrt[2]{|c_2|}, \sqrt[3]{|c_3|} \quad \sqrt[n]{c_n}$
(real, non neg. nos.)

Let $l = \overline{\lim} \sqrt[n]{c_n}$

Then radius of convergence $r = 1/l$.

Proof (1) Suppose $|z| > r$.

$$\overline{\lim} \sqrt[n]{|c_n z^n|} = \overline{\lim} \sqrt[n]{|c_n|}|z| > lr = 1.$$

Suppose $lr = 1 + 2k, \; k > 0$

There are ∞ elements in $(lr - k, lr + k)$

i.e. there are ∞ terms :-

$$\sqrt[n]{|c_n z^n|} > 1 + k > 1$$

$\therefore |c_n z^n| > 1 \quad \therefore$ Series cannot converge.

(2) Suppose $|z| < r$ Then $\overline{\lim} \sqrt[n]{|c_n z^n|} = |z| \overline{\lim} \sqrt[n]{c_n} = l|z| < lr \leq 1$

Let $l|z| = 1 - 2k$ There are only a finite number of terms :- $|z| |\sqrt[n]{c_n}| > 1 - k = q$ say.

For all the terms except this finite set, $|z| \sqrt[n]{|c_n|} \leq q$

$\therefore |c_n z^n| \leq q^n$

Thus there is a stage after which the term $c_n z^n$ are smaller in norm than those of the gt GP $\sum q^n$.

Hurwitz Courant p 49

C/02

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(t)\,dt}{t-z}$$

$$= \frac{1}{2\pi i}\oint \left[\frac{1}{t} + \frac{z}{t^2} + \frac{z^2}{t^3} + \quad \frac{z^n}{t^{n+1}} + \dots\right] f(t)\,dt$$

$$= \frac{1}{2\pi i}\left\{ \oint \frac{f(t)}{t}dt + z\oint \frac{f(t)dt}{t^2} + z^2\oint \frac{f(t)dt}{t^3} + \dots \right.$$
$$\left. + z^n \oint \frac{f(t)dt}{t^{n+1}} + \dots\dots \right\}$$

This step has been shown to be justified by uniform convergence of the series being integrated

Theorem If for all finite values, $|f(t)| \leq M$, a fixed number, then f is a constant, f being an integral function

Proof $\left|\oint \frac{f(t)dt}{t^{n+1}}\right| \leq \frac{ML}{R^{n+1}}$

where the curve is a circle of radius R and L is its length As $L = 2\pi R$

$$\left|\frac{1}{2\pi i}\oint \frac{f(t)dt}{t^{n+1}}\right| \leq \frac{2\pi MR}{R^{n+1}} = \frac{2\pi M}{R^n}$$

If $n \geq 1$, as this holds for all R, it follows that the coefficient of z^n is 0.

$\therefore \quad f(z) = c_0.$

Definition of essential singularity

s a singular point. If there is a neighbourhood of s in which $1/f(z)$ can be represented by a power series $\sum c_r(z-s)^r$ (except at the point s itself) then s is called a non-essential singularity or pole. Otherwise it is essential.

For a pole $\frac{1}{f(z)} = c_k(z-s)^k + \dots = (z-s)^k P(z-s)$
where $c_k \neq 0$, so $P(0) \neq 0$.

Then $f(z) = \frac{1}{(z-s)^k P(z-s)} = \frac{1}{(z-s)^k} P_1(z-s)$

$= \frac{1}{(z-s)^k}\left[a_0 + a_1(z-s) + a_2(z-s)^2 + \dots\right]$

k = order of the pole.

$$f(z) = \frac{a_0}{(z-s)^k} + \dots \frac{a_{k-1}}{z-s} + a_k + a_{k+1}(z-s) + \dots$$

$$g\left(\frac{1}{z-s}\right) = \frac{a_0}{(z-s)^k} + \dots \frac{a_{k-1}}{z-s}$$

called principal part or meromorphic part of f at s.

A pole is always an isolated singularity.

For, we can

If we have a sequence of numbers, c/04
$c_1\ c_2\ c_3\ c_4 \dots\dots$, L is said to be a limit of the sequence if, for any $\varepsilon > 0$, an infinite number of terms c_n lie in $(L-\varepsilon\ L+\varepsilon)$.

The upper limit, $\overline{\lim}\,(c_1\ c_2 \dots)$ is the largest limit of the sequence.

Theorem. A bounded sequence has at least one limit point.

Proof. Let J_0 be a finite interval containing all the points of the sequence. Divide J_0 into halves. If the lower half contains only a finite number of points, let J_1 = upper half. If ~~J_0~~ the lower half contains ∞ points take J_1 = lower half (even if the upper half has ∞ points). So continue, getting $J_0\ J_1\ J_2 \dots$ each contained in the previous and half as long.
They define a real number ξ.
This ξ is a limit point.

Theorems in Hurwitz Courant.

II. I, 2, §1. If $P(z)$ cgt for $z = z_0$ then $P(z)$ cgt <u>absolutely and uniformly</u> for z with $|z| < |z_0|$. Hence, in a circle.

§2. Let $P(z) = \sum c_n z^n$ and $l = \overline{\lim} \{ \sqrt[n]{c_n} \}$. Then radius of convergence $r = 1/l$.

§3. Operations on power series. $+, -$ as expected.

Multiplication $P_1(z) = \sum c_n^{(1)} z^n$, $P_2(z) = \sum c_n^{(2)} z^n$. Then the formal product $\sum d_n z^n$ with $d_n = \sum_{i=0}^{n} c_i^{(1)} c_{n-i}^{(2)}$ is absolutely convergent in circle for which both series converge, and gives $P_1(z) P_2(z)$.

Formal division for $1/P(z)$ is valid in some circle provided $P(z)$ cgt. in some circle, and $c_0 \neq 0$.

Thus all rational operations, $+, -, \times, \div$ can be carried out formally on power series if these converge appropriately.

§4 If $P(z)$ does not have all coefficients zero, the zeros of $P(z)$ cannot have 0 as a limit point. i.e. there is a circle around 0 in which $P(z) \neq 0$ unless perhaps for $z = 0$.

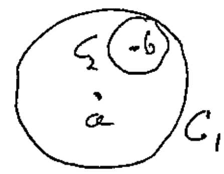
Corollary. If $P_1(z) = P_2(z)$ for a set of points with 0 as a limit point, the series having a common circle of convergence, then $\forall_n$ $c_n^{(1)} = c_n^{(2)}$.

§5. $P(z|a)$, a power series in $z - a$, cgt for $|z-a| < r$, dgt for $|z-a| > r$.

$P(z|\infty) = \sum c_n z^{-n}$ cgt for $\left|\frac{1}{z}\right| < r$, dgt for $\left|\frac{1}{z}\right| > r$

i.e. convergent outside circle $|z| = \frac{1}{r}$, dgt inside.

§6 $P(z|a)$ cgt inside circle C_1. We form $P_1(z|b)$ formally. It converges at least in circle C_2.



§7 $P'(z|a)$ has same radius of convergence as $P(z|a)$ and can be found by term-by-term differentiation

§8. Direct continuation. $P(z|a)$ and $P_1(z|a_1)$ have S, a region of convergence, in common. If they are equal for $b_1, b_2 \ldots b_n \ldots$ which $\to b$ in S, they are equal everywhere in S.

P.T.O.

It may happen that no circle of convergence reaches z_0. Then $f(z_0)$ is not defined.

The set of points that can be reached is called "domain of regularity" of $f(z)$.

It can happen that two distinct power series in the system have the same c_0. Then $f(z_0)$ is regarded as having as many values as there are distinct $P(z|z_0)$.

§3 Single valued branches of an analytic function.

Σ a set of points in complex plane or on number sphere. A fixed point $\underline{a}$ can stand in one of 3 relations to Σ.

(1) all points in some neighborhood of $\underline{a}$ are in Σ

(2) no point in some circle around a belongs to Σ

(3) any circle around $\underline{a}$ includes a point in Σ and a point not in Σ.

(1) $\underline{a}$ is in interior of Σ: it is an inner point.

(2) $\underline{a}$ is external to Σ.

(3) $\underline{a}$ is a boundary point of Σ.

A point $\underline{a}$ of Σ is called isolated if some circle around $\underline{a}$ contains no other point of Σ.

A continuous curve is a set of points (x, y) :-

$$x = \varphi(t), \quad y = \psi(t) \quad \text{for} \quad t_0 \le t \le t_1$$

φ and ψ being continuous (real) functions.

A curve is called closed if $\varphi(t_0) = \varphi(t_1), \psi(t_0) = \psi(t_1)$

It is simple if no other pair of values, t_2, t_3 give same point.

Domain. A set of points is called a domain if (1) all its points are inner points, (2) any two points can be joined by a continuous curve lying entirely in the domain.

Jordan's Theorem. A simple closed curve divides the plane into exactly 2 domains.

I §3. If $1 - c_1 z - c_2 z^2 - \ldots$ has radius of convergence $r > 0$, then $1/(1 - c_1 z - c_2 z^2 - \ldots)$ equals a power series with positive radius of cgnce.
Proof $\overline{\lim} \sqrt[n]{|c_n|} = l$ is finite $\therefore$ Only a finite no. of $\sqrt[n]{|c_n|} > l + \varepsilon$. These have finite upper bound. $\therefore \sqrt[n]{|c_n|} < g$ for all n. From formal expansion it is proved that power series cgt for $|z| < \frac{1}{2g}$, and may of course converge for larger values.

§5 A point is regular if function is given by a power series, cgt in some circle ($r > 0$) with the point as centre. Otherwise it is a singularity.

The singularities of $1/p(z)$, where $p(z)$ is a polynomial, are the roots of $p(z)$ only. For if $p(z_0) \neq 0$, we can express polynomial as $\sum_{i=0}^{n} c_i z^i$, with $c_0 \neq 0$. The radius of convergence is ∞ for $p(z|z_0)$ hence $1/p(z)$ has radius of cgnce > 0 when expanded about z_0.

If $p(z_0) = 0$, $1/p(z_0) = \infty$ and z_0 must be a singularity.

The proof that there must be a singularity on the circle of convergence is fairly complicated. If every point is regular we can put a circle around it to extend fn. These circles overlap, so extensions agree - we get same $f(z)$ whichever circle near z we use to extend. From continuity and closure we show radii of these circles have a nonzero minimum.

As we know result from IV.2 §7 we need not go into this on Saturday a.m.

If $f(z) = \sum_0^\infty a_n z^n$ cgt for all finite z, $f(z)$ called integral fn. It has no singularity for finite z. This $\Rightarrow$ fn is integral, for radius of cgnce goes to nearest singularity.

A bounded integral fn is constant. Liouville's Theorem

For $|a_n| \leq \dfrac{M}{r^n}$ for any r

If $n \geq 1$, it follows $|a_n| = 0$.

Cauchy: Partial Fractions.

$f(z)$ meromorphic with ∞ poles: $a_0=0$; $a_1, a_2, \ldots$.

$g_n\left(\frac{1}{z-a_n}\right)$ meromorphic part for a_n, $n \geq 1$.

$g_0\left(\frac{1}{z-a_0}\right)$ is $g_0\left(\frac{1}{z}\right)$ if 0 is a pole; otherwise 0.

C is a simple closed rectifiable curve, not through any pole. It has $a_0, a_1 \ldots a_r$ inside it. For any $m \in \mathbb{N}$,

$$I = \frac{1}{2\pi i}\int_C f(\zeta)\left[\frac{1}{\zeta-z} - \frac{1}{\zeta} - \frac{z}{\zeta^2} \cdots - \frac{z^{m-1}}{\zeta^m}\right]d\zeta.$$

$$= \frac{1}{2\pi i}\int_C \frac{z^m f(\zeta)\,d\zeta}{\zeta^m(\zeta-z)}.$$

It can be calculated as sum of residues. z is a fixed point inside C, distinct from $a_1 \ldots a_r$. The infinities of the integrand in I are z, $a_0=0$, $a_1 \ldots a_r$.

~~It~~ ~~Also~~ $\zeta = z$ in integrand, apart from $\zeta - z$ ~~term~~ factor, is $z^m f(z)/z^m = f(z)$ which is finite, by choice of z.

$\therefore$ Residue $= f(z)$

Near $\zeta = a_k$ we have $f(\zeta) = g_k\left(\frac{1}{\zeta-a_k}\right) + \mathcal{P}(\zeta - a_k)$

$\mathcal{P}$ being a power series.

$$\frac{1}{\zeta-z} = \frac{1}{\zeta-a_k-(z-a_k)} = -\left[\frac{1}{z-a_k} + \frac{\zeta-a_k}{(z-a_k)^2} + \frac{(\zeta-a_k)^2}{(z-a_k)^2} + \cdots\right]$$

Let $g_k\left(\frac{1}{\zeta-a_k}\right) = \sum_{s=1}^{\ell} \frac{\alpha_s}{(\zeta-a_k)^s}$

$$\frac{f(\zeta)}{\zeta-z} = \left[\sum_{s=1}^{\ell}\frac{\alpha_s}{(\zeta-a_k)^s} + \mathcal{P}(\zeta-a_k)\right]\left[\sum_{t=0}^{\infty}\frac{(\zeta-a_k)^t}{(z-a_k)^{t+1}}\right]$$

Coeff of $\frac{1}{\zeta-a_k} = \sum_{t=0}^{\ell-1}\frac{\alpha_{t+1}}{(z-a_k)^{t+1}} = \sum_{t=1}^{\ell}\frac{\alpha_t}{(z-a_k)^t} = g_k\left(\frac{1}{z-a_k}\right)$

I also includes

$$-\frac{1}{2\pi i}\int_C f(\zeta)\left[\frac{1}{\zeta} + \frac{z}{\zeta^2} + \cdots + \frac{z^{m-1}}{\zeta^m}\right]d\zeta$$

the terms that follow $\frac{1}{\zeta-z}$

Let $h_k(z)$ denote coefficient of $\frac{1}{\zeta - a_k}$ in the expansion of above expression in rising powers of $\zeta - a_k$. Then $h_k(z)$ is a polynomial of degree $m-1$ at most in z, and the contribution of residue at a_k to ~~this part~~ I is

$$-\left\{ g_k\left(\frac{1}{z-a_k}\right) - h_k(z) \right\}.$$

$$\therefore I = f(z) - \sum_{k=0}^{r} \left\{ g_k\left(\frac{1}{z-a_k}\right) - h_k(z) \right\}$$

This may be shown as $\frac{1}{2\pi i}\int \frac{z^m}{\zeta^m}\frac{f(\zeta)}{\zeta - z}\,d\zeta$

$$\therefore f(z) = \sum_{k=0}^{r} \left\{ g_k\left(\frac{1}{z-a_k}\right) - h_k(z) \right\} + \frac{1}{2\pi i}\int \frac{z^m f(\zeta)}{\zeta^m(\zeta - z)}\,d\zeta$$

<u>This holds for any m</u> : we have used only an algebraic identity.

Meromorphic function with prescribed principal parts

$r_1 < |a_1| < r_2 < |a_2| < \quad r_n < |a_n|$

$F_1(z)$ is regular in $|z| \leq r_1$

Let $P_1(z)$ be polynomial formed from the series for $F_1(z)$:- $|F_1(z) - P_1(z)| < \varepsilon_1$ for $|z| \leq r_1$

$|F_2(z) - P_2(z)| < \varepsilon_2 \qquad |z| \leq r_2$

$\sum \varepsilon_r$ is cgt. ($P_1(z), P_2(z)$ etc may have different degrees.)

$\{F_1(z) - P_1(z)\} + \{F_2(z) - P_2(z)\} + \dots$

G any domain. Let k be the least number :- ~~$G \in \{z : |z| \leq r_k\}$~~ G is in circle radius r with $r < r_k$.

Then G is contained in every circle $\{z : |z| < r_s\}$ with $s \geq k$.

$\therefore$ For $s \geq k$ $\quad |F_s(z) - P_s(z)| < \varepsilon_s$.

Thus $\phi(z) = \sum_{s=k}^{\infty} \{F_s(z) - P_s(z)\}$ is $\{$u-cgt, abs cgt$\}$ in G.

By Weierstrass Theorem, this sum can be replaced by a single series, as the radius of convergence of each $F_s(z)$ is $|a_s|$, and all of these are larger than r, and $\phi(z)$ is u-cgt on circle of radius r. The series is found by naive addition of the individual series.

Now let

$F(z) = F_0(z) + \{F_1(z) - P_1(z)\} + \dots + \{F_{k-1}(z) - P_{k-1}(z)\} + \phi(z)$.

$\phi(z)$ is regular in G. We have in front of $\phi(z)$ a finite sum, which can have poles only at $a_1, a_2 \dots a_{k-1}$ and the meromorphic parts at these poles are given by $F_s(z)$ $s = 0 \dots k-1$.

$F_0(z)$ corresponds to pole at origin, if there is one

I.4. §2 dealt with $e^{iz} = \cos z + i \sin z$ and showed $\cos z$ and $\sin z$ so defined agreed with elem. trig.

§3. Logarithm. Bittam d p. 76. $l(z)$ principal logarithm: real for real +ve z.

If $w = \ln z$ $\quad z = e^w = e^{u+iv} = e^u(\cos v + i \sin v)$
$x = e^u \cos v, \quad y = e^u \sin v.$

$|z| = e^u$ If $z = re^{i\theta}$ $\quad |z| = r \quad u = \ln r.$

For real +ve z, $y = \Im|z| = 0$, so $v = s\pi \quad s \in \mathbb{Z}$
Also $z = e^u \cos v$ and $z > 0 \therefore \cos v = +1.$

The principal logarithm is defined as that which has $v = 0$ for real positive z.

v varies continuously.

If $z = \rho e^{i\theta} = \rho e^{u+iv}$

$e^u = \rho$ and $e^{i\theta} = e^{iv}$

$\therefore e^{i(v-\theta)} = 1 \quad v - \theta = 2n\pi i$

Principal log has $n = 0$. $\therefore v = \theta$.

Thus $\ln z$ approach $\ln \rho + i\pi$ if z approaches $-\rho$ from above: $\ln \rho - i\pi$ from below.

The general logarithm is $l(z) + 2n\pi i = f_n(z)$

If z approaches $-\rho$ from above, this approaches

$l(\rho) + (2n+1)\pi i$

This agrees with $f_{n+1}(z)$ when the same point is approached from below

§4. The general power.

Chap ~~VI~~. §1 Meromorphic. §2 Deals with fn with finite number of poles

See Macrobert p. 105. essentially same as §5?

$$\frac{1}{a-z} = \frac{1/a}{1-z/a} = \frac{1}{a}\left[1 + \frac{z}{a} + \frac{z^2}{a^2} + \dots\right.$$

Let $S^{(a)} = 1 + \frac{z}{a} + \frac{z^2}{a^2} + \dots \frac{z^{n-1}}{a^{n-1}}$

$$\left(1-\frac{z}{a}\right)S^{(a)} = 1 - \frac{z^n}{a^n}$$

$$\therefore S^{(a)} = \frac{1}{1-\frac{z}{a}} - \frac{z^n}{a^{n-1}(a-z)}$$

$$= \frac{a}{a-z} -$$

$$\therefore \frac{a}{z-a} + S(a) = \frac{-z^n}{a^{n-1}(a-z)}$$

Suppose $f(z)$ has simple poles $|a_1|<|a_2|<|a_3| \dots$ such that $\sum_{r=1}^{\infty} \frac{1}{|a_r^{n+1}|}$ is convergent.

Let C be circle, radius r, $R < |a_{p+1}|$

If z is inside C, for $r = p+1, p+2, p+3 \dots$

$|z - a_r| \geq k$ where $k = |a_{p+1} - R|$
the shortest distance from a_{p+1} to circle C.

$$\left|\frac{z}{a_r} - 1\right| \geq \frac{k}{|a_r|} \geq \frac{k}{|a_{p+1}|} = \frac{|a_{p+1} - R|}{|a_{p+1}|} = \mu \text{ say}$$

Let $w_r(z) = \frac{1}{z-a_r} + \frac{1}{a_r} S(a_r) = -\frac{z^n}{a_r^n(a_r - z)}$

$$= -\frac{z^n}{a_r^{n+1}\left(1-\frac{z}{a_r}\right)}$$

$$|w_r(z)| < \frac{R^n}{a_r^{n+1}\mu}. \quad \sum w_r z = \frac{R^n}{\mu}\sum\frac{1}{a_r^{n+1}}$$

$\therefore \sum w_r(z)$ is convergent absolutely inside C.

This is true for any R.

A more general theorem. To get poles $\frac{c_r}{z-a_r}$ we require $\sum \frac{|c_r|}{|a_r^{n+1}|}$ cgt.

Weierstrass Sum Theorem.

A circle K lies <u>inside</u> the circle of convergence of $P_r(z)$ for each r. Also $\sum_1^\infty P_r(z)$ is <u>u-cgt on the circumference of K</u>

Then $\sum_1^\infty P_r(z)$ is convergent for any point <u>inside K</u> and is given by the power series obtained by summing coefficients in the natural way.

i.e. If $P_n(z) = \sum_{k=0}^\infty c_k^{(n)} z^k$ and $c_k = \sum_{n=1}^\infty c_k^{(n)}$

then $P(z) = \sum_0^\infty c_k z^k$ equals $\sum_{n=1}^\infty P_n(z)$ <u>inside K</u>.

From uniform convergence of $\sum P_r(z)$ on K we have that, for any $\varepsilon > 0$ $\exists N :- n > N :- |P_{n+1}(z) + P_{n+2}(z) + \dots P_{n+m}(z)| < \varepsilon$ for every m.

This is finite sum.

Coefficient of z^k in $P_{n+1}(z) \dots P_{n+m}(z)$ is

$$\frac{1}{2\pi i} \oint_K \frac{P_{n+1}(z) + \dots + P_{n+m}(z)}{z^{k+1}} dz$$

and is $c_k^{(n+1)} + \dots + c_k^{(n+m)}$, which cannot exceed

$$\frac{\varepsilon \cdot 2\pi\rho}{2\pi \cdot \rho^{k+1}} = \frac{\varepsilon}{\rho^k}.$$

Hence $c_k = c_k^{(1)} + c_k^{(2)} + \dots$ is cgt. and if

$c_k = \sum_{s=1}^n c_k^{(s)} + r_k^{(n)}$, by result above

$$r_k^{(n)} \leq \frac{\varepsilon}{\rho^k}$$

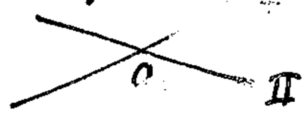
$$\therefore\ f(z) = \sum_{k}\left[g_k\left(\frac{1}{z-a_k}\right) - h_k(z)\right] + \frac{1}{2\pi i}\oint \frac{z^m f(\zeta)}{\zeta^m(\zeta - z)}\, d\zeta.$$

C115

DETERMINANTS.

§1.
$$ax + by = 0 \quad (\mathrm{I})$$
$$cx + dy = 0 \quad (\mathrm{II})$$

In general these represent 2 lines crossing at the origin. The only solution is $x=0,\ y=0$



If we wish to find x, y formally we may take

$d(\mathrm{I}) - a(\mathrm{II})$ $\quad x(ad-bc)=0. \quad (\mathrm{III})$

$-c(\mathrm{I}) + a(\mathrm{II})$ $\quad y(ad-bc)=0. \quad (\mathrm{IV})$

If $ad-bc \neq 0$ we have $x=0,\ y=0.$

If $ad-bc=0$, (III) and (IV) do not prove $x=0$, $y=0$.

Both (I) and (II) may represent the same line, when $ad-bc=0$. In that case (x,y) can be any point on that line.

A possible case is $a=0,\ b=0,\ c=0,\ d=0.$ In this case, any point of the plane is a solution.

We define $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

Note $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}.$

§2.
$$ax + by + cz = 0 \quad (\mathrm{I})$$
$$dx + ey + fz = 0. \quad (\mathrm{II})$$

$-d(\mathrm{I}) + a(\mathrm{II})$ $\quad y(ae-bd) + z(af-cd) = 0. \quad (\mathrm{III})$

$f(\mathrm{I}) - c(\mathrm{II})$ $\quad x(af-cd) + y(bf-ce) = 0 \quad (\mathrm{IV})$

$e(\mathrm{I}) - b(\mathrm{II})$ $\quad x(ae-bd) + z(ce-bf) = 0 \quad (\mathrm{V})$

If
$$x = \rho(bf-ce)$$
$$y = -\rho(af-cd)$$
$$z = \rho(ae-bd)$$
equations III, IV, V are satisfied, and it can be checked that (I) and (II) are satisfied.

$$\frac{x}{\begin{vmatrix} b & c \\ e & f \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a & c \\ d & f \end{vmatrix}} = \frac{z}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} \quad (\mathrm{VI})$$

DL

2 equations in 3 unknowns always have a solution other than 0, 0, 0.

This is reasonable. (I) and (II) represent 2 planes through the origin. These must have a line of intersection.

Algebraically there are various possibilities.

1) a, b, c, d, e, f are all zero. In this case any x, y, z is a solution.

2) It may be that they are not all zero, but
$\begin{vmatrix} b & c \\ e & f \end{vmatrix} = 0 \quad \begin{vmatrix} a & c \\ d & f \end{vmatrix} = 0 \quad \begin{vmatrix} a & b \\ d & e \end{vmatrix} = 0$.

~~If a and d are both zero (x, 0, 0) is a solution for any x.~~

Then equations III, IV, V show that
$-d(I) + a(II) = 0, \quad f(I) - c(II) = 0,$
$e(I) - b(II) = 0$.

If $a \neq 0$, $(II) = \frac{d}{a}(I)$. Any solution of (I) will be a solution of (II). We can choose any y, z we like and solve for x.

Similarly if any other number in a, b, c, d, e, f is not zero.

3) If $\begin{vmatrix} b & c \\ e & f \end{vmatrix}, \begin{vmatrix} a & c \\ d & f \end{vmatrix}, \begin{vmatrix} a & b \\ d & e \end{vmatrix}$ are not all zero, (VI) gives a solution other than 0, 0, 0.

~~DE~~ D3

§3.
$$ax + by + cz = 0 \quad (\mathrm{I})$$
$$dx + ey + fz = 0 \quad (\mathrm{II})$$
$$gx + hy + iz = 0 \quad (\mathrm{III})$$

If $\begin{vmatrix} e & f \\ h & i \end{vmatrix}, -\begin{vmatrix} d & f \\ g & i \end{vmatrix}, \begin{vmatrix} d & e \\ g & h \end{vmatrix}$ are not all zero, these give a solution of (II), (III) other than 000.

This will satisfy (I) if
$$a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = 0. \quad (\mathrm{IV})$$

We define $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ as the LHS of (IV).

This will be zero if $\begin{vmatrix} e & f \\ h & i \end{vmatrix}, \begin{vmatrix} d & f \\ g & i \end{vmatrix}, \begin{vmatrix} d & e \\ g & h \end{vmatrix}$ are all zero.

In this case (II) and (III) can be replaced by one of them. That equation, together with (I), will have a solution other than 000.

Either way, there is a solution of (I), (II) (III) not 000, if IV holds.

Determinants

$px + qy = 0$, where p and q are not both zero, has the general solution (σ arbitrary)

$$x = +\sigma q, \quad y = -\sigma p \qquad (1)$$

Consider the equations
$$ax + by = 0 \qquad (2)$$
$$cx + dy = 0 \qquad (3)$$
Where c and d are not both zero.

(3) has solution $x = +\sigma d$, $y = -\sigma c$ (4)

This will satisfy (2) as well if
$$a(\sigma d) + b(-\sigma c) = 0$$
i.e.
$$\sigma(ad - bc) = 0 \qquad (5)$$

If $ad - bc \neq 0$, $\sigma = 0$ and the only solution of (2), (3) is $x = 0$, $y = 0$.

If $ad - bc = 0$, (5) is satisfied by any σ. So (4) gives solutions for (2), (3).

We have excluded the possibility $c = 0, d = 0$. When this is so, there are nonzero solutions of (2), (3) and $ad - bc = 0$. So we have

If $ad - bc \neq 0$, the only solution of (2)(3) is $x = 0, y = 0$

If $ad - bc = 0$, there are ∞ non-zero solutions.

We write $ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Transpose. We can see $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

This is a general property of determinants, $\det M = \det M^T$. This is due to the fact that when the columns are linearly dependent, so are the rows.

(Vectors u, v, w are linearly dependent if $c_1 u + c_2 v + c_3 w = 0$, not all c_1, c_2, c_3 zero. Similarly for any number)

Rows dependent $\Rightarrow$ columns dependent. We have $c = ka$, $d = kb$ (or a, b in terms of c, d).

Equations are then
$$ax + by = 0 \qquad (i)$$
$$k(ax + by) = 0 \qquad (ii)$$

Any solution of (i) is a solution of (ii), so $x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix} = 0$.

A similar argument is available in any number of dimensions.

$$ax + by + cz = 0 \qquad (6)$$
$$dx + ey + fz = 0 \qquad (7)$$

If we take $s(6) + t(7)$ coefficient of z is $sc + ft$. This will be 0 if $s = f$, $t = -c$. The new equation is then

$$x(af - cd) + y(bf - ce) = 0$$

with solution

$$x = \sigma(bf - ce) = \sigma \begin{vmatrix} b & c \\ e & f \end{vmatrix}$$

$$y = -\sigma(af - cd) = -\sigma \begin{vmatrix} a & c \\ d & f \end{vmatrix}$$

Similarly $-d(6) + a(7)$ gives

$$y(ae - bd) + z(af - cd) = 0.$$

With $y = -\sigma(af - cd)$ this will be satisfied if

$$-\sigma(ae - bd) + z = 0$$
$$z = \sigma(ae - bd)$$

Thus solution is

$$\frac{x}{\begin{vmatrix} b & c \\ e & f \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a & c \\ d & f \end{vmatrix}} = \frac{z}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} \qquad (8)$$

Note: the determinant under x corresponds to the letters remaining if we strike out the coefficients of x. Similarly for the others.

$$\begin{matrix} \cancel{a} & b & c \\ \cancel{d} & e & f \end{matrix}$$

Note signs $+ - +$.

There is an exceptional case. If $d = ka$, $e = kb$, $f = kc$, when we eliminate z we get $0x + 0y + 0z = 0$. In this case, the second equation gives no extra information. Any solution of (6) satisfies (7). The three determinants in (8) are all zero.

3×3 determinant

This is the condition for x, y, z, not all zero to satisfy

$$ax + by + cz = 0 \qquad (i)$$
$$dx + ey + fz = 0 \qquad (ii)$$
$$gx + hy + kz = 0 \qquad (iii)$$

The general solution of (ii) and (iii) is

$$x = \sigma \begin{vmatrix} e & f \\ h & k \end{vmatrix}$$

$$y = -\sigma \begin{vmatrix} d & f \\ g & k \end{vmatrix}$$

$$z = \sigma \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

apart from the exceptional case when all of these are zero (which means that (ii) and (iii) are essentially the same equation).

Substituting in (i) we have

$$\sigma \left\{ a \begin{vmatrix} e & f \\ h & k \end{vmatrix} - b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \right\} = 0$$

If $\{ \} \neq 0$, $\sigma = 0$ and $0,0,0$ is the only solution

Hence $\{ \} = 0$ is the condition for a solution other than $0,0,0$ to exist.

We define

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = a \begin{vmatrix} e & f \\ h & k \end{vmatrix} - b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

Note that aek has coefficient $+1$.

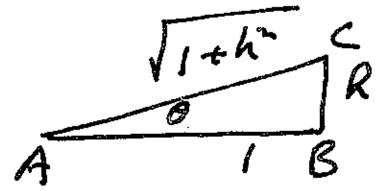
The meaning of the determinant is fixed by the properties (i) condition for solution other than 000,
(ii) linear expression in each row
(iii) $+aek$ occurs in expression.

Note $\begin{vmatrix} e & f \\ h & k \end{vmatrix}$ comes if we delete row and column containing a.

Small oscillations of masses on a string.

"Small" is important. We make many approximations that are reasonable only for small displacements.

(i) When B moves to C the distance 1 is replaced by



$\sqrt{1+h^2} \doteqdot 1 + \frac{1}{2}h^2$. If $h = .001$, $h^2 = .000001$.

Thus we may take AC as still being of length 1.

(ii) The tension of an elastic string increases when the length increases. However, (i) shows that the increase of length is negligible. So the change of tension is negligible.

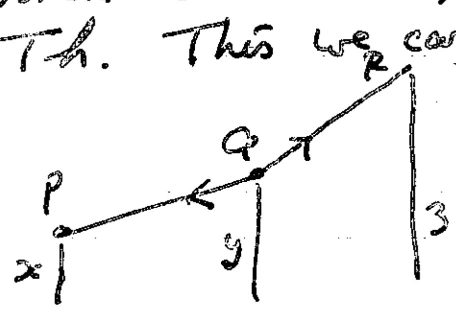
(iii) The tension acting in the direction AC has component $T\cos\theta$ along AB. Effectively $\cos\theta = 1$, so the horizontal component may be taken as T.

The horizontal forces cancel.

We can imagine the masses moving in straight lines ⊥ horizontal.

The transverse component is $T\sin\theta$, which we take as Th. This we cannot neglect.

If x, y, z are the displacements of P, Q, R



Transverse force on Q

$$= T(z-y) - T(y-x)$$

$$= T(z-2y+x).$$

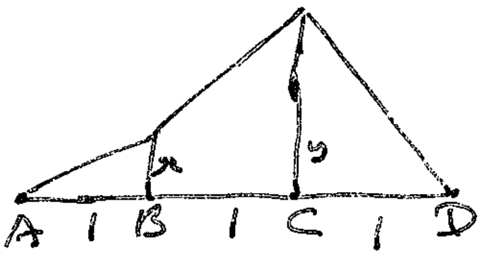
This force $= m\ddot{y}$

0.2
D8

Two masses.

$m\ddot{x} = (y - 2x)T$

$m\ddot{y} = (-2y + x)T$



$m\ddot{x} = (-2x + y)T$
$m\ddot{y} = (x - 2y)T$

For SHM $\ddot{x} = -\omega^2 x$ $\quad \ddot{y} = -\omega^2 y$

Let $\lambda = -m\omega^2/T$.

$$\left.\begin{aligned} \lambda x &= -2x + y \\ \lambda y &= x - 2y \end{aligned}\right\}$$

$$0 = -(\lambda+2)x + y$$
$$0 = x - (\lambda+2)y$$

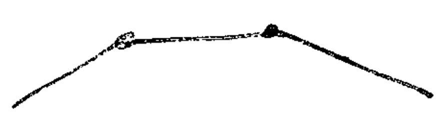
$$0 = \begin{vmatrix} -(\lambda+2) & 1 \\ 1 & -(\lambda+2) \end{vmatrix}$$

$= (\lambda+2)^2 - 1 \qquad \lambda + 2 = 1$ or -1.

$\underline{\lambda = -1 \text{ or } -3.}$

(a) $\lambda = -1$ $\quad \left.\begin{aligned} 0 &= -x + y \\ 0 &= x - y \end{aligned}\right\}$

So $x = y$.



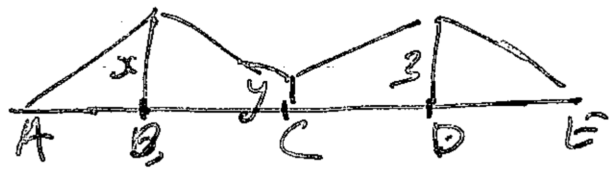
(b) $\lambda = -3$ $\quad \begin{aligned} 0 &= x + y \\ 0 &= x + y \end{aligned}$

$y = -x$

Q. 3

D9

Three masses.



$$m\ddot{x} = T(-2x + y)$$
$$m\ddot{y} = T(x - 2y + z)$$
$$m\ddot{z} = T(\quad y - 2z)$$

This leads to
$$\lambda x = -2x + y$$
$$\lambda y = x - 2y + z$$
$$\lambda z = \quad y - 2z$$

$$0 = -(\lambda+2)x + y$$
$$0 = x - (\lambda+2)y + z$$
$$0 = y - (\lambda+2)z$$

$$0 = \begin{vmatrix} -(\lambda+2) & 1 & 0 \\ 1 & -(\lambda+2) & 1 \\ 0 & 1 & -(\lambda+2) \end{vmatrix}$$

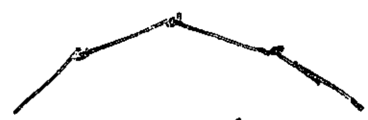
$$= -(\lambda+2)^3 + 2(\lambda+2)$$

$\lambda+2 = 0$ or $(\lambda+2)^2 = 2$ $\quad \lambda+2 = \pm\sqrt{2}$

$$\lambda = -2+\sqrt{2}\,; \quad \lambda = -2\,; \quad \lambda = -2-\sqrt{2}.$$

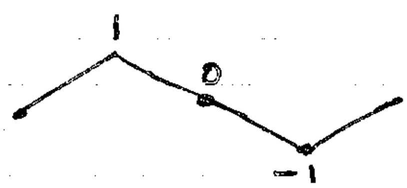
(a) $\lambda = -2+\sqrt{2}$ $\quad \lambda+2 = \sqrt{2}$
$$0 = -x\sqrt{2} + y$$
$$0 = x - y\sqrt{2} + z$$
$$0 = y - z\sqrt{2}.$$

If $y = 1$, $x = \frac{1}{\sqrt{2}}$, $z = \frac{1}{\sqrt{2}}$



Note $x = \sin 45^\circ$ $\quad y = \sin 90^\circ$ $\quad z = \sin 135^\circ$

(b) $\lambda = -2$ $\quad \lambda+2 = 0$ $\quad 0 = y$
$$0 = x + z$$
$$0 = y$$



Note $\quad x = \sin 2(45^\circ)$
$y = \sin 2(90^\circ)$
$z = \sin 2(135^\circ)$

Q.4
D 10

(c) $\lambda = -2 - \sqrt{2}$ $\quad \lambda + 2 = -\sqrt{2}$

$$0 = x\sqrt{2} + y$$
$$0 = x + y\sqrt{2} + z$$
$$0 = y + z\sqrt{2}.$$

If $y = -1$, $x = \frac{1}{\sqrt{2}}$, $z = \frac{1}{\sqrt{2}}$ from 1st and 3rd eqns

$$x + y\sqrt{2} + z = \frac{1}{\sqrt{2}} - \sqrt{2} + \frac{1}{\sqrt{2}}$$
$$= \frac{2}{\sqrt{2}} - \sqrt{2} = 0.$$

Note $\quad x = \sin 3(45^\circ)$

$y = \sin 3(90^\circ)$

$z = \sin 3(135^\circ)$

D 11

General dynamics.

The position of a single particle is determined by its coordinates x, y, z: how these change is determined by Newton's Law. If the force on it is X, Y, Z then $m\ddot{x} = X$; $m\ddot{y} = Y$; $m\ddot{z} = Z$.

In general dynamics we are concerned with systems of particles, subject to certain constraints: for example, in a rigid body the distance between any two points must remain constant. This must be enforced in some way, perhaps by rods joining the particles. We can imagine a 2-dimensional rigid body as, for example

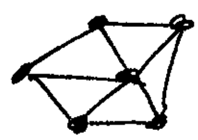


Figure 1

An admissible displacement is one that does not violate the constraints.

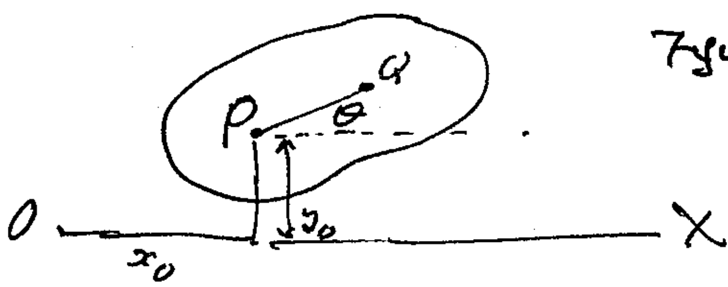


Figure 2

For example, if the position of a lamina is specified by the co-ordinates $x_0\ y_0$ of a point P in it, and by θ, the angle that PQ makes with OX, then any change in $x_0\ y_0\ \theta$ is an admissible displacement, for it does not imply any change in the distances between points of the lamina. x_0, y_0, θ are called generalized co-ordinates, and in the general theory would be represented by q_1, q_2, q_3.

We may suppose that the lamina is light but that forces act in a specified way at particular points of it. We can find the work done if changes $\delta x_0, \delta y_0, \delta\theta$ occur.

Example A light rod of length a experiences a horizontal force 10 at one end and a vertical force 20 at the other end as shown.

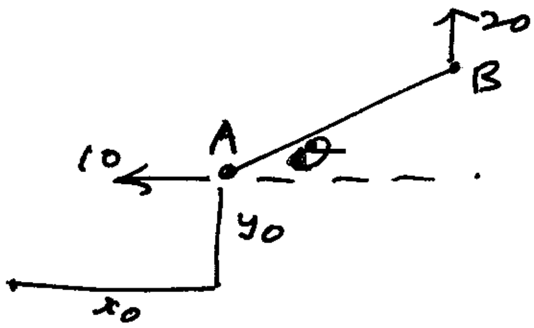


Figure 3.

If $x_0\ y_0\ \theta$ change by $\delta x_0\ \delta y_0\ \delta\theta$, the point A moves δx_0 to the right. Work done by the force 10 is $-10\,\delta x_0$.

The height of B is $y_0 + a\sin\theta$. B will move $\delta y_0 + a\cos\theta\,\delta\theta$, and work is $20\,\delta y_0 + 20a\cos\theta\,\delta\theta$

Total work done =

$$-10\,\delta x_0 + 20\,\delta y_0 + 20a\cos\theta\,\delta\theta.$$

The coefficients of $\delta x_0, \delta y_0, \delta\theta$ define the <u>generalized force</u> acting on the system. In the general notation $Q_1 = -10$, $Q_2 = 20$, $Q_3 = 20a\cos\theta$,

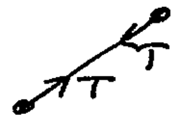
by the external forces

and the work done in the displacement $\delta q_1, \delta q_2, \delta q_3$ is $Q_1\,\delta q_1 + Q_2\,\delta q_2 + Q_3\,\delta q_3$.

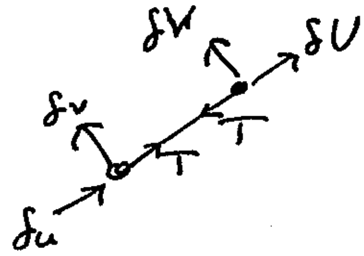
The difficulty in dealing with a system such as that in Figure 1 is that an unknown tension or thrust exists in each joining rod. If we are concerned with the motion of the body as a whole, and not bothered about internal stresses, we want to arrive at equations

 $\mathcal{D}/13$

that do not involve these internal forces, but are sufficient in number to determine the behaviour of the generalized co-ordinate.



<u>Theorem</u> If the distance between the ends of a rod does not change, the total work done by the internal tensions (or thrusts) is zero.

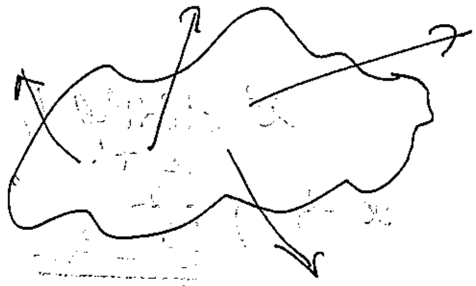


Proof Let the displacements of the end points along the rod and ⊥ rod be as shown. The displacements δv and δV are perpendicular to the directions of the tensions, so do not contribute to the work done.

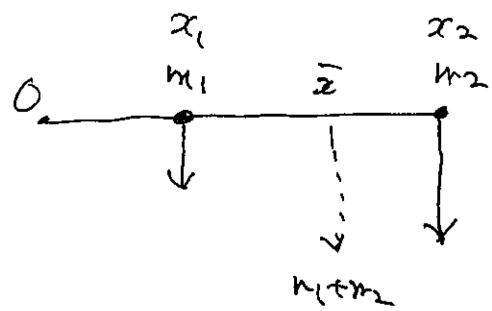
The work done is thus $T\delta U - T\delta u = T(\delta U - \delta u)$
This will be zero if $\delta U = \delta u$, i.e. if the distance between the ends does not change.

~~If we imagine~~ On page 2 the expression $Q_1\delta q_1 + Q_2\delta q_2 + Q_3\delta q_3$ was found for the work done by the external forces: since the total work done by the tensions is zero, this expression in fact gives the work done in the displacement by <u>all</u> forces.

If we have a rigid body with various forces acting on it, these will be equivalent to a single force if we can find a force such that it is the vector sum of all the forces acting, and has the same moment about a fixed point.



Example



Weight m_1 at x_1, m_2 at x_2

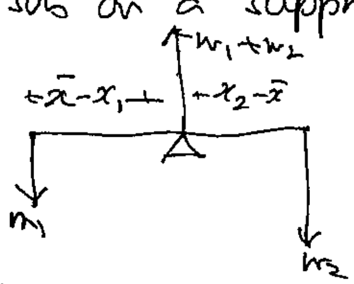
Vector sum is $m_1+m_2 \downarrow$

Suppose it acts at $\bar{x}$.

Moments about origin. $m_1x_1 + m_2x_2 = (m_1+m_2)\bar{x}$

$$\therefore \quad \bar{x} = \frac{m_1x_1 + m_2x_2}{m_1+m_2} \qquad (1)$$

If the beam is light, and sits on a support at $\bar{x}$



Turning effect about $\triangle$ is

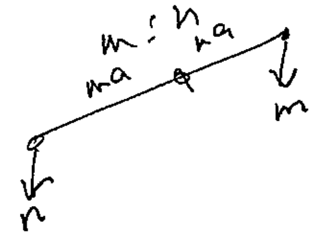
$m_2(x_2-\bar{x}) - m_1(\bar{x}-x_1)$

$= m_2x_2 - m_2\bar{x} - m_1\bar{x} + m_1x_1$

$= (m_2x_2 + m_1x_1) - (m_1+m_2)\bar{x}$

$= 0$ by (1)

$\bar{x}$ is C.G. (centre of gravity) of m_1 at x_1, m_2 at x_2



PRINCIPLES OF DYNAMICS

Dynamics

Force = mass × acceleration.

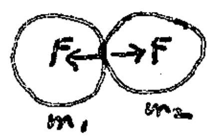
Action and reaction are equal and opposite.

Momentum = mv acceleration = dv/dt

Force $F = m\frac{dv}{dt} = \frac{d}{dt}(mv)$ force = time rate of change of momentum.

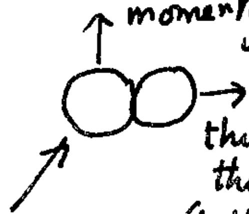
This can be used to find pressure of hose jet (calculate the rate at which momentum disappears)

Conservation of momentum in collision.



$$-F = \frac{d}{dt} m_1 v_1$$
$$F = \frac{d}{dt} m_2 v_2$$
$$\therefore 0 = \frac{d}{dt}(m_1 v_1 + m_2 v_2) \text{ during contact.}$$
$$\therefore m_1 v_1 + m_2 v_2 \text{ unchanged.}$$

Billiard balls



Energy $F = m\frac{dv}{dt}$

$$= m\frac{dv}{dx}\frac{dx}{dt} = m\frac{dv}{dx}v = \frac{d}{dx}\left(\tfrac{1}{2}mv^2\right)$$

Force = space rate of change of energy.

$$F\,dx = d\left(\tfrac{1}{2}mv^2\right)$$

work done = increase of kinetic energy

In 2 dimensions, $v^2 = \dot{x}^2 + \dot{y}^2$

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2$$

Force (X, Y) $X = m\ddot{x}$ $Y = m\ddot{y}$

$$X = m\frac{d}{dt}\dot{x} = m\frac{dx}{dt}\frac{d}{dx}\dot{x} = m\dot{x}\frac{d}{dx}\dot{x} = \frac{d}{dx}\left(\tfrac{1}{2}m\dot{x}^2\right)$$

$$X\,dx = d\left(\tfrac{1}{2}m\dot{x}^2\right)$$

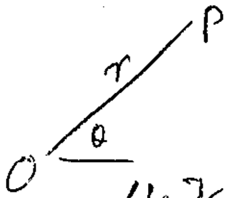
Similarly $Y\,dy = d\left(\tfrac{1}{2}m\dot{y}^2\right)$

$$X\,dx + Y\,dy = d\left(\tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2)\right)$$

Note $X\,dx + Y\,dy = (X, Y)\cdot(dx, dy) = F\cos\theta\,ds$

Mass in vertical circle

POLAR CO-ORDINATES



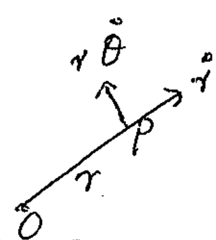
Polar co-ordinates

Required, velocity and acceleration along OP and $\perp$ to it.

Unit vector along OP $(\cos\theta, \sin\theta)$

$\perp$ OP $(-\sin\theta, \cos\theta)$

$$x = r\cos\theta \qquad \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$
$$y = r\sin\theta \qquad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

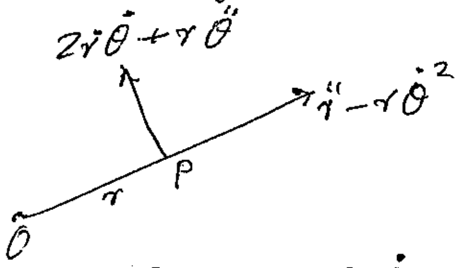


$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \dot{r}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + r\dot{\theta}\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$

$$\ddot{x} = \ddot{r}\cos\theta - \dot{r}\dot{\theta}\sin\theta - \dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta$$
$$= \cos\theta(\ddot{r} - r\dot{\theta}^2) - \sin\theta(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

$$\ddot{y} = \ddot{r}\sin\theta + \dot{r}\dot{\theta}\cos\theta + \dot{r}\dot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta + r\ddot{\theta}\cos\theta$$
$$= \sin\theta(\ddot{r} - r\dot{\theta}^2) + \cos\theta(2\dot{r}\dot{\theta} + r\ddot{\theta})$$

So

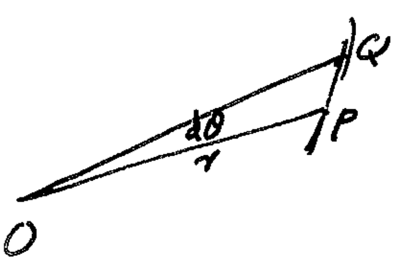


Planetary motion. $2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}(2r\dot{r}\dot{\theta} + r^2\ddot{\theta})$

$$= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}).$$

Area OPQ given by

$\frac{1}{2}r^2 d\theta$



$\therefore$ $\frac{1}{2}r^2\dot{\theta}$ = rate at which area is swept out by line joining sun (at O) to planet at P.

Kepler's Laws of Planetary Motion include assertion that this rate is constant.

$\therefore$ $\frac{d}{dt}(\frac{1}{2}r^2\dot{\theta}) = 0$ $\therefore \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$

$\therefore$ there is no acceleration $\perp$ OP

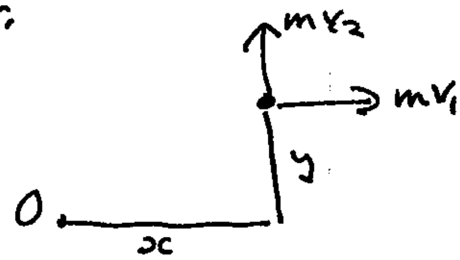
$\therefore$ there is no force $\perp$ OP

$\therefore$ The force is in the direction PO.

Angular momentum vector.

We have a system of moving particles, m at (x, y, z). Consider the angular momentum about the axis OZ. The coordinate z and the velocity $\dot{z}$ are irrelevant.

Let velocity be $(v_1\ v_2\ v_3)$



Angular momentum about OZ is $\sum x . mv_2 - y . mv_1$

$= \cancel{\Sigma m}$

$=$ z component of $\Sigma(\underline{x} \wedge m\underline{v})$

where $\underline{x} = (x, y, z)$

$\underline{V} = (v_1\ v_2 . v_3)$.

Now $\underline{x} \wedge m\underline{v}$ has a geometrical meaning independent of the choice of axes. For any line we could choose OZ along it.

Hence the angular momentum about any axis through O is the component of $\Sigma \underline{x} \wedge m\underline{V}$ along it.

$\sum \underline{x} \wedge m\underline{v}$ is known as the <u>angular momentum vector</u> for any system.

The moment of a force XYZ at xyz about OZ is $xY - yX$ = component of $\underline{x} \wedge \underline{F}$ where $F = (X\,Y\,Z)$

The vector $\sum x \wedge F$ thus represents the moment of forces acting: its component along any axis through O is the moment about that axis.

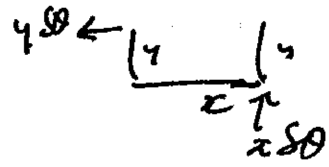
Moments of internal forces

$(X_I\ Y_I\ Z_I)$ internal force on m at $(x\ y\ z)$

In any admissible displacement, internal forces do no work

A rotation $\delta\theta$ about OZ causes displacement $(-y\delta\theta, x\delta\theta, 0)$ at (x, y, z)

$\therefore 0 = \Sigma(-yX_I + xY_I)$



<u>Contribution of internal forces to moment vector is zero.</u>

D 18

Rate of change of angular momentum vector.

$$\frac{d}{dt} \sum \underline{r} \wedge m\underline{v} = \sum \dot{\underline{r}} \wedge m\underline{v} + \sum \underline{r} \wedge m\dot{\underline{v}}$$

As $\dot{r} = v$, $\sum \dot{r} \wedge m\underline{v} = 0$.

$m\dot{v}$ is acceleration of mass, hence $= \underline{F}$

$$\frac{d}{dt} \sum r \wedge m\underline{v} = \sum r \wedge F$$

Rate of change of angular momentum vector
= moment of forces vector.

In a system with internal forces, $F = F_E + F_I$
We saw above that $\sum r \wedge F_I = 0$.

Hence rate of change of angular momentum vector
= moment of external forces vector.

D 19

A rigid body moves under the influence of any system of forces whatever.

(I) The C.G moves as if the total mass were concentrated there and all forces acted on this concentrated mass.

(II) The rotational motion can be found by considering the C.G to be at rest and taking moment of forces about it.

(I) If x, y, z are co-ords. of any mass m, let

$x = \bar{x} + \xi$, $y = \bar{y} + \eta$, $z = \bar{z} + \zeta$. $(\bar{x}\bar{y}\bar{z})$ being C.G.

C.G is defined by $\bar{x}\sum m = \sum mx$ etc.

$\therefore \bar{x}\sum m = \sum m(\bar{x} + \xi) = \sum m\bar{x} + \sum m\xi = \bar{x}\sum m + \sum m\xi$

$\therefore \sum m\xi = 0$. Similarly $\sum m\eta = 0$ $\sum m\zeta = 0$.

These equations hold at all times, so they can be differentiated as many times as we wish.

We have $X = m\ddot{x} = m(\ddot{\bar{x}} + \ddot{\xi})$

$\therefore \sum X = \sum m\ddot{\bar{x}} + \sum m\ddot{\xi}$

$\sum m\ddot{\xi} = 0$ $\therefore \sum X = \sum m\ddot{\bar{x}} = \ddot{\bar{x}}\sum m = M\ddot{\bar{x}}$ cas.

$\therefore \sum X = M\ddot{\bar{x}}$

Similarly for $\sum Y = M\ddot{\bar{y}}$ $\sum Z = M\ddot{\bar{z}}$. Q.E.D.

(II) Form vectors for moment of forces about origin.

~~and~~ z-component is $\sum xY - yX = \sum(xm\ddot{y} - ym\ddot{x})$

Now $x = \bar{x} + \xi$, $y = \bar{y} + \eta$

$\therefore \sum(\bar{x}Y + \xi Y - \bar{y}X - \eta X)$

$= \sum m(\bar{x} + \xi)(\ddot{\bar{y}} + \ddot{\eta}) - m(\bar{y} + \eta)(\ddot{\bar{x}} + \ddot{\xi})$

RHS contains $\sum m\bar{x}\ddot{\eta} = \bar{x}\sum m\ddot{\eta} = \bar{x}\frac{d^2}{dt^2}\sum m\eta = 0$.

Similarly $\sum m\xi\ddot{\bar{y}} = 0$, $\sum m\bar{y}\ddot{\xi} = 0$ $\sum m\eta\ddot{\bar{x}} = 0$

$\therefore \sum\bar{x}Y + \sum\xi Y - \sum\bar{y}X - \sum\eta X$

$= \bar{x}M\ddot{\bar{y}} + \sum m\xi\ddot{\eta} - \bar{y}M\ddot{\bar{x}} - \sum m\eta\ddot{\xi}$

Now $\sum\bar{x}Y = \bar{x}\sum Y = \bar{x}.M\ddot{\bar{y}}$ by (I) above.

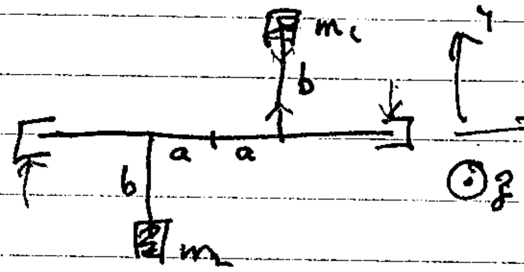
$-\sum\bar{y}X = -\bar{y}\sum X = -\bar{y}M\ddot{\bar{x}}$ Cancelling.

$\therefore \sum(\xi Y - \eta X) = \sum(m\xi\ddot{\eta} - m\eta\ddot{\xi})$

$= \frac{d}{dt}\sum m(\xi\dot{\eta} - \eta\dot{\xi})$ as required

Unbalanced machine.

$(0, 0, \omega y)$



Angular momentum vector

is $\sum \underline{r} \wedge m\underline{v}$ $\qquad \begin{array}{ccc} x & y & z \\ v_1 & v_2 & v_3 \end{array}$

$\sum m(yv_3 - zv_2),\ \sum m(zv_1 - xv_3),\ \sum m(xv_2 - yv_1)$

	$v_1\ v_2\ v_3$	$x\ y\ z$
m_1	$(0, 0, b\omega)$	$(a, b, 0)$
m_2	$(0, 0, -b\omega)$	$(-a, -b, 0)$

$m(b^2\omega),\ -mab\omega,\ 0$

$m(b^2\omega),\ -mab\omega,\ 0$

Angular momentum vector is $(2mb^2\omega, -2mab\omega, 0)$

This vector lies in a plane that is rotating with angular velocity ω about OX.

Rate of change is $(0, 0, -2mab\omega^2)$.

Requires moment force $\otimes$ $2mab\omega^2$ $\uparrow$ $\downarrow$

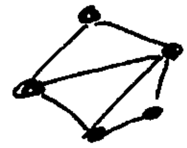
So bearings have to exert downward force on right and upward on left, when system is in the plane of the paper.

This checks. m_1, going round a circle, produces tension in the bar holding it, and tends to drag right-hand end upwards.

Note. In such a system, angular momentum vector is not in the direction of the vector representing rotation.

Motion of a rigid body or system.

§1. We may picture a rigid body as consisting of masses linked together by inextensible rods.



Each mass satisfies Newton's Laws. The force acting on it consists partly of known external force (X_E, Y_E, Z_E) and an internal force, due to the rods, (X_I, Y_I, Z_I). This is not known initially. Thus, for masses 1 to N, with coordinates $(x_i\ y_i\ z_i)$ $i = 1$ to N we have

$$m_i\ddot{x}_i = X_{Ei} + X_{Ii},\quad m_i\ddot{y}_i = Y_{Ei} + Y_{Ii},\quad m_i\ddot{z}_i = Z_{Ei} + Z_{Ii}.$$

We need to eliminate the internal forces X_I, Y_I, Z_I. This appears a formidable task. However in any admissible displacement (one that does not destroy the rigidity of the system) <u>the internal forces do ~~not~~ no work</u>. This can be deduced from the model above with rods, or, without making any assumption about the nature of the body, it may be taken as a basic assumption.

Thus, if $\delta x_i, \delta y_i, \delta z_i$ is an admissible displacement

(I) $$\sum(m_i\ddot{x}_i\delta x_i + m_i\ddot{y}_i\delta y_i + m_i\ddot{z}_i\delta z_i) = \sum_i X_{Ei}\delta x_i + Y_{Ei}\delta y_i + Z_{Ei}\delta z_i.$$

Now we suppose the state of the system may be specified by $q_1 \dots q_n$ which can vary freely. We can get an equation by supposing a change δq_r in q_r, the others remaining fixed. In this way we get n equations, which is what we need to determine the behaviour of the n quantities $q_1 \dots q_r$.

On the right hand side of (I) we then get

(II) $$\sum_i\left(X_{Ei}\frac{\partial x_i}{\partial q_r} + Y_{Ei}\frac{\partial y_i}{\partial q_r} + Z_{Ei}\frac{\partial z_i}{\partial q_r}\right)\delta q_r,$$ the work done in the displacement δq_r.

Note. We have not assumed potential energy.

$$m\ddot{x} = X_E + X_I$$

$$\therefore \sum m\ddot{x}\,\delta x + m\ddot{y}\,\delta y + m\ddot{z}\,\delta z$$

$$= \sum X_E\,\delta x + Y_E\,\delta y + Z_E\,\delta z \quad \text{as } \sum X_I\,\delta x + \text{etc} = 0.$$

$$= \sum Q_i\,\delta q_i$$

$$\delta x = \sum_i \frac{\partial x}{\partial q_i}\,\delta q_i$$

$$\therefore \text{LHS} = \sum m\ddot{x}\frac{\partial x}{\partial q_i}\,\delta q_i + \text{etc.}$$

$$x = f(q_1 - - \; q_n)$$

Differentiate w.r.t. time

$$\dot{x} = \sum \frac{\partial x}{\partial q_i}\dot{q}_i$$

$$T = \sum \tfrac{1}{2}m\dot{x}^2 + \dots$$

$$\frac{\partial T}{\partial \dot{q}} = m\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}} + \dots$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) = m\ddot{x}\frac{\partial \dot{x}}{\partial \dot{q}} + m\dot{x}\frac{d}{dt}\frac{\partial \dot{x}}{\partial \dot{q}}$$

$$= m\ddot{x}\frac{\partial x}{\partial q} + m\dot{x}\,\cancel{\partial x}\,\frac{d}{dt}\frac{\partial x}{\partial q}$$

$$= m\ddot{x}\frac{\partial x}{\partial q} + m\dot{x}\frac{\partial \dot{x}}{\partial q}$$

$$= m\ddot{x}\frac{\partial x}{\partial q} + \frac{\partial}{\partial q}T$$

The forces could, for instance, be due to friction, air resistance etc. Whatever they are, we take them in (4), which is usually denoted by $Q_r \delta q_r$.
In any displacement, the work done is $\sum_{r=1}^{n} Q_r \delta q_r$.

Lagrange's equations are obtained by showing that

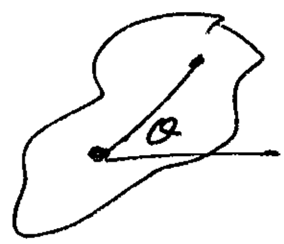
$$\sum\left(m_i \ddot{x}_i \frac{\partial x_i}{\partial q_r} + m_i \ddot{y}_i \frac{\partial y_i}{\partial q_r} + m_i \ddot{z}_i \frac{\partial z_i}{\partial q_r}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r}.$$

Thus we have

$$Q_r \delta q_r = \left(\sum m_i \ddot{x}_i \frac{\partial x_i}{\partial q_r} + m_i \ddot{y}_i \frac{\partial y_i}{\partial q_r} + m_i \ddot{z}_i \frac{\partial z_i}{\partial q_r}\right)\delta q_r$$
$$= \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r}\right)\delta q_r$$

so $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} = Q_r.$$

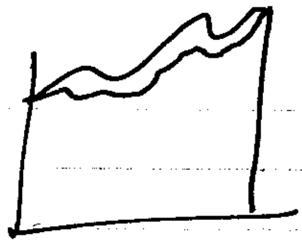
§2. For a rigid body rotating about an axle we have only one q_r, namely $q_1 = \theta$
The velocity of a point at distance r is



Relation of Lagrange to Calculus of Variations

D24

$$\int f(x,y,y')\,dx.$$



Let y change to $y+\varepsilon u$

y' becomes $y'+\varepsilon u'$

$f(x,y,y')$ becomes $f(x, y+\varepsilon u, y'+\varepsilon u')$

$$= f(x,y,y') + \varepsilon u \frac{\partial f}{\partial y} + \varepsilon u' \frac{\partial f}{\partial y'}$$

$$\delta \int f(x,y,y')\,dx = \int \left[\varepsilon u \frac{\partial f}{\partial y} + \varepsilon u' \frac{\partial f}{\partial y'}\right] dx$$

$\int$ by Parts $u' \frac{\partial f}{\partial y'}$: it gives $\left[u \frac{\partial f}{\partial y'}\right] - \int u \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) . dx$

We require $u=0$ at the ends.

$$\delta \int f(x,y,y')\,dx = \varepsilon \int u \frac{\partial f}{\partial y} - u \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) . dx$$

$$= \varepsilon \int u \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] dx$$

u is arbitrary. Considering a variation

$[\ ]$ must be zero throughout.

$$\therefore \quad \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0.$$

In some books εu called δy and we have $\delta y' = \delta \frac{dy}{dx}$

and we show $\delta d = d \delta$.

F1

D25

An illustration of transformation theory.

$\mu = \frac{1}{x}$ Light follows the path for which

time $= \int \mu\, ds$ is a minimum: $\int \frac{1}{x}\sqrt{1+y'^2}\, dx$

$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y}$ gives

$$\frac{d}{dx}\left(\frac{1}{x}\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\therefore \quad \frac{1}{x}\frac{y'}{\sqrt{1+y'^2}} = a \quad \text{(constant)}$$

$$y'^2 = a^2x^2 + a^2x^2y'^2$$

$$y'^2 = \frac{a^2x^2}{1-a^2x^2}$$

$$\frac{dy}{dx} = \pm\frac{ax}{\sqrt{1-a^2x^2}}$$

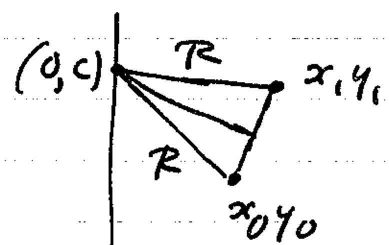
$$\therefore \quad y - c = \mp\frac{1}{a}\sqrt{1-a^2x^2}$$

$$(y-c)^2 = \frac{1}{a^2} - x^2$$

$$x^2 + (y-c)^2 = \frac{1}{a^2}$$

Thus light travels along a circle, with centre on OY.

If it goes from $(x_0 y_0)$ to $(x_1 y_1)$,



$$x_0^2 + (y_0-c)^2 = x_1^2 + (y_1-c)^2$$

so $$c = \frac{x_1^2 + y_1^2 - x_0^2 - y_0^2}{2(y_1-y_0)}$$

$$R^2 = x_1^2 + (y_1-c)^2$$

$$= \frac{x_0^4 + x_1^4 + (y_1-y_0)^4 + 2x_0^2(y_1-y_0)^2 + 2x_1^2(y_1-y_0)^2 - 2x_0^2x_1^2}{4(y_1-y_0)^2}$$

$$= \frac{[x_0^2 + x_1^2 + (y_1-y_0)^2]^2 - 4x_0^2x_1^2}{4(y_1-y_0)^2}$$

The numerator is difference of squares, and the factors are of interest. They confirm the accuracy of the calculator.

For motion of a mass, the principle of Least Action states that $\int T\,dt$ is minimum subject to conservation of energy. As $E = T + V$, $\frac{1}{2}mv^2 = E - V$

$$\int \tfrac{1}{2}mv^2\,dt = \int \tfrac{1}{2}mv\,ds = \int \sqrt{\tfrac{m}{2}(E-V)}\,ds.$$

We can identify the path of the particle with that of a ray of light by taking $\sqrt{E-V}$ proportional to $\frac{1}{x}$. Thus $E - V = \frac{k}{x^2}$. Taking $E=0$, we have $V = -\frac{k}{x^2}$.

Hence $L = T - V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{k}{x^2}$

Equations of motion. $\frac{d}{dt}(m\dot{x}) = -\frac{2k}{x^3}$ $\quad \frac{d}{dt}(m\dot{y}) = 0.$

$\therefore \dot{y} = h$, constant. The equation of energy gives an integral of the first equation of motion

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}mh^2 - \frac{k}{x^2} = 0$$

$$\dot{x}^2 = \frac{2k/m}{x^2} - h^2$$

$$\therefore \int dt = \pm\int \frac{dx}{\sqrt{\frac{2k/m}{x^2} - h^2}}$$

$$= \pm\int \frac{x\,dx}{\sqrt{\frac{2k}{m} - h^2x^2}}$$

$$t + g = \mp \frac{1}{h^2}\sqrt{\frac{2k}{m} - h^2x^2}$$

$$\therefore h^4(t+g)^2 = \frac{2k}{m} - h^2x^2$$

$$\therefore (ht + hg)^2h^2 = \frac{2k}{m} - h^2x^2$$

Now $y = ht +$ constant.

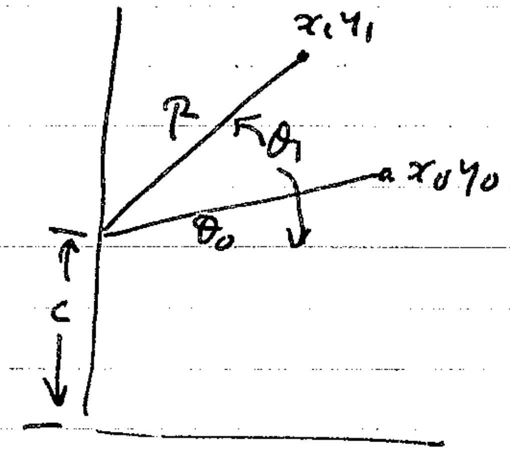
$$\therefore (ht + hg)^2 = \frac{2k}{mh^2} - x^2$$

writing $hg = -c$

$$(y-c)^2 + x^2 = \frac{2k}{mh^2}$$

which will be identical with the path of the ray of light if we suppose $R^2 = \frac{2k}{mh^2}$.

Returning to the ray of light at a point of the circle



$x = R\cos\theta,\ y - c = R\sin\theta$

$ds = R\,d\theta$

Hence the time from $(x_0 y_0)$ to (x_1, y_1) is given by

$$\int \mu\, ds = \int_{\theta_0}^{\theta_1} \frac{1}{R\cos\theta} \cdot R\,d\theta$$

$$= \int_{\theta_0}^{\theta_1} \frac{d\theta}{\cos\theta}.$$

This can be evaluated as

$$\frac{1}{2}\left[\ln(1+\cos\theta_1) - \ln(1+\cos\theta_0) - \ln(1-\cos\theta_1) + \ln(1+\cos\theta_0)\right]$$

and this can be expressed as $F(x_0\, y_0\, x_1\, y_1)$ by finding $\cos\theta_0$ and $\cos\theta_1$ in terms of the coordinates $x_0\, y_0\, x_1\, y_1$. We shall not need this formula.

For the moving particle, consider $S = \int L\,dt$.

We have $y - c = R\sin\theta$

so $h = \dot{y} = R\cos\theta\,\dot{\theta}$

The velocity of the particle is $R\dot{\theta}$.

$L = T - V = E - 2V = \frac{2k}{x^2}$ as $E = 0$, $V = -\frac{k}{x^2}$

$$\therefore S = \int \frac{2k}{x^2}\,dt = \int \frac{2k}{x^2}\frac{dt}{d\theta}\,d\theta = \int \frac{2k}{x^2}\frac{d\theta}{\dot{\theta}}$$

$$= \int \frac{2k}{R^2\cos^2\theta}\,\frac{R\cos\theta}{h}\,d\theta = \frac{2k}{Rh}\int_{\theta_0}^{\theta_1}\frac{d\theta}{\cos\theta}.$$

We can make the two results identical by making $k = \frac{1}{2}Rh$. As $R = \sqrt{\frac{2k}{mh^2}}$ this means $k = \frac{1}{2}h\sqrt{\frac{2k}{mh^2}} = \sqrt{\frac{k}{2m}}$

$k^2 = k/2m$, $k = \frac{1}{2m}$.

$\mu = \frac{1}{r}$

D_{28}

Polar co-ordinates $\mu = \frac{1}{r}$ $\int \mu\, ds$ stationary

$ds^2 = dr^2 + r^2 d\theta^2$

$f = \frac{1}{r}\sqrt{r'^2 + r^2}$ $\int f\, d\theta$ stationary

$\frac{d}{d\theta}\frac{\partial f}{\partial r'} = \frac{\partial f}{\partial r}$ $\frac{\partial f}{\partial r'} = \frac{r'}{r}\frac{1}{\sqrt{r'^2+r^2}}$

$\frac{rr'' - r'^2}{r^2}(r'^2+r^2)^{-\frac{1}{2}} + \frac{r'}{r}\left[-\frac{1}{2}(r'^2+r^2)^{-\frac{3}{2}}2(r'r''+rr')\right]$

$= -\frac{1}{r^2}(r'^2+r^2)^{-\frac{1}{2}} + \frac{1}{r}(r'^2+r^2)^{-3/2}\cdot r$

~~Multiply by~~ $\cancel{r^2(r'^2+r^2)^{3/2}}$

$\cancel{(rr''-r'^2)(r'^2+r^2) - rr'^2(r''+r)}$

$\cancel{= r'^2+r^2}$

$= -\frac{1}{r^2}(r'^2+r^2)^{\frac{1}{2}} + \frac{1}{r}(r'^2+r^2)^{-\frac{1}{2}} r$

Multiply by $(r'^2+r^2)^{3/2} r^2$

$(rr''-r'^2)(r'^2+r^2) - rr'(r'r''+rr')$

$= -(r'^2+r^2)^2 + (r'^2+r^2)r^2$

$rr'^2r'' + r^3r'' - r'^4 - r^2r'^2 - rr'^2r'' - r^2r'^2 = -r'^4 - 2r^2r'^2 - r^4 + r^2r'^2 + r^4$

$\therefore$ $\cancel{rr'^2r'' + r^3r'' = 0}$

$r^3r'' - r^2r'^2 = 0$

$rr'' - r'^2 = 0$

Divide by rr' $\frac{r''}{r'} - \frac{r'}{r} = 0$

Integrate $\ln r' - \ln r = \text{const.}$

$\frac{r'}{r} = \text{const} = b$ say

$\therefore r = ae^{b\theta}$ equiangular spiral.

($\mu = \frac{1}{r}$) Lagrangian for wave 2

Φ_{29}

$$\text{Time} = \int \mu \, ds = \int \frac{1}{r}\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$$

$$r = a e^{b\theta} \qquad \frac{dr}{d\theta} = abe^{b\theta} = br.$$

$$\therefore \quad t = \int \sqrt{b^2+1}\, d\theta = \cancel{\sqrt{b^2+1}\,\theta}$$

~~if $\theta = 0$ when $t = 0$~~.

$$= \sqrt{b^2+1}\,(\theta_2 - \theta_1)$$

$$r = ae^{b\theta} \qquad r_1 = ae^{b\theta_1} \quad r_2 = ae^{b\theta_2}$$

$$\therefore \ln r_1 = \ln a + b\theta_1$$

$$\ln r_2 = \ln a + b\theta_2$$

$$\therefore \quad b(\theta_2 - \theta_1) = \ln r_2 - \ln r_1$$

$$b^2(\theta_2-\theta_1)^2 = (\ln r_2 - \ln r_1)^2$$

$$F = \cancel{\xi}\sqrt{b^2+1}\,(\theta_2-\theta_1) = \sqrt{b^2(\theta_2-\theta_1)^2 + (\theta_2-\theta_1)^2}$$

$$= \sqrt{(\ln r_2 - \ln r_1)^2 + (\theta_2-\theta_1)^2}$$

Suppose now $r_1\,\theta_1$ to $r_2\,\theta_2$ corresponds to Δt:

$\theta_2 - \theta_1 \sim \dot\theta \Delta t$

$\ln r_2 - \ln r_1 \sim \frac{1}{r}\dot r \Delta t$.

Thus $$L = \sqrt{\frac{\dot r^2}{r^2} + \dot\theta^2} = \frac{1}{r}\sqrt{\dot r^2 + r^2\dot\theta^2}$$

$\mu = \frac{1}{r}$ 3

D 30

$$L = \frac{1}{r}\sqrt{\dot r^2 + r^2\dot\theta^2} \qquad \frac{\partial L}{\partial \dot r} = \frac{1}{r}\frac{\dot r}{\sqrt{\dot r^2 + r^2\dot\theta^2}}$$

$$\frac{d}{dt}\left[\frac{1}{r}\frac{\dot r}{\sqrt{\dot r^2 + r^2\dot\theta^2}}\right] = -\frac{1}{r^2}\sqrt{} + \frac{1}{r}\frac{1}{\sqrt{\dot r^2 + r^2\dot\theta^2}}\cdot r\dot\theta^2$$

$$\frac{d}{dt}\left[\frac{1}{r}\frac{r^2\dot\theta}{\sqrt{\dot r^2 + r^2\dot\theta^2}}\right] = 0 \qquad \frac{r\dot\theta}{\sqrt{\dot r^2 + r^2\dot\theta^2}} = \text{constant.}$$

$$\frac{r^2\dot\theta^2}{(\dot r^2 + r^2\dot\theta^2)} = \text{const.} \qquad \frac{\dot r^2 + r^2\dot\theta^2}{r^2\dot\theta^2} = \text{constant.}$$

$$\therefore \quad \frac{\dot r^2}{r^2\dot\theta^2} = \text{const.} \qquad \frac{\dot r}{r\dot\theta} = \text{constant}$$

This means $\frac{1}{r}\frac{dr}{d\theta}$ is constant.

Let $r\dot\theta = k\dot r$ First eqn becomes

$$\frac{d}{dt}\left[\frac{\dot r}{r\sqrt{\dot r^2 + k^2\dot r^2}}\right] = -\frac{\sqrt{\dot r^2 + k^2\dot r^2}}{r^2} + \frac{k^2\dot r^2}{r^2\sqrt{\dot r^2 + k^2\dot r^2}}$$

$$\frac{d}{dt}\left[\frac{1}{r\sqrt{k^2+1}}\right] = -\frac{\dot r\sqrt{1+k^2}}{r^2} + \frac{k^2\dot r}{r^2\sqrt{1+k^2}}$$

$$\therefore \quad \frac{-\dot r}{r^2\sqrt{k^2+1}} = -\frac{\dot r\sqrt{1+k^2}}{r^2} + \frac{k^2\dot r}{r^2\sqrt{1+k^2}}$$

$$-\frac{1}{r^2} = -\frac{1+k^2}{r^2} + \frac{k^2}{r^2 \cancel{\sqrt{1+k^2}}}$$

Which is true.

Dynamics, equiangular spiral

D 31

$$\sqrt{E-V} = \frac{1}{r} \quad \text{Take } E=0, \; V = -\frac{K}{r^2}$$

$$L = \tfrac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2) + \frac{K}{r^2}$$

$$\frac{d}{dt} m\dot{r} = mr\dot{\theta}^2 - \frac{2K}{r^3} \qquad \frac{d}{dt} mr^2\dot{\theta} = 0$$

$$\therefore \dot{\theta} = h/r^2$$

Energy equation $\tfrac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2) - \frac{K}{r^2} = 0$

$$\therefore \quad \tfrac{1}{2}m\dot{r}^2 + \frac{m}{2}\frac{h^2}{r^2} - \frac{K}{r^2} = 0$$

$$\dot{r}^2 = \frac{2K/m - h^2}{r^2} = \frac{C^2}{r^2} \text{ say}$$

$$\therefore \dot{r} = \frac{C}{r} \qquad r\dot{r} = C$$

$$\therefore \tfrac{1}{2}r^2 = Ct + G$$

$$\frac{d\theta}{dt} = \frac{h}{r^2} = \frac{h}{2(Ct+G)}$$

$$\therefore \theta = \frac{h}{2C}\ln(Ct+G) + \text{const}$$

$$\therefore \theta = \frac{h}{2C}\ln\frac{r^2}{2} + \text{const}$$

$$= \frac{h}{C}\ln r + \text{const}$$

giving r of the form $ae^{b\theta}$.

I had trouble with this at one time, as I overlooked the fact that we have assumed $E=0$.

$\mu = \frac{1}{r}$

D32

$$L = \frac{1}{r}\sqrt{\dot r^2 + r^2\dot\theta^2}$$

$$\frac{d}{dt}\left[\frac{\dot r}{r\sqrt{\dot r^2+r^2\dot\theta^2}}\right] = \frac{-\dot r^2}{r^2\sqrt{\dot r^2+r^2\dot\theta^2}}$$

$$LHS = \frac{\ddot r}{r}(\dot r^2+r^2\dot\theta^2)^{-\frac12} - \frac{\dot r^2}{r^2}(\ )^{-\frac12} - \frac{\dot r}{r}(\dot r^2+r^2\dot\theta^2)^{-3/2}(\dot r\ddot r + r\dot r\dot\theta^2 + r^2\dot\theta\ddot\theta)$$

$$\times\ (\dot r^2+r^2\dot\theta^2)^{3/2} r^2$$

$$r\ddot r(\cancel{\dot r^2} + r^2\dot\theta^2) - \dot r^2(\cancel{\dot r^2}+\cancel{r^2\dot\theta^2}) - r\dot r(\dot r\cancel{\ddot r} + r\dot r\dot\theta^2 + r^2\dot\theta\ddot\theta) =$$

$$= -\dot r^2(\cancel{\dot r^2}+\cancel{r^2\dot\theta^2})$$

$$0 = \ddot r \cancel{r}\dot\theta^2 - \cancel{r}\dot r^2\dot\theta^2 - \cancel{r}r\dot r\dot\theta\ddot\theta$$

$$\cancel{0 = r^2\ddot r\dot\theta -}$$

$$0 = r\ddot r\dot\theta - \dot r^2\dot\theta - r\dot r\ddot\theta \qquad \frac{\ddot\theta}{\dot\theta} = \frac{r\ddot r - \dot r^2}{r\dot r} = \frac{\ddot r}{\dot r} - \frac{\dot r}{r}$$

$$\ln\dot\theta = \ln\dot r - \ln r + \text{const}$$

$$\therefore\ r\dot\theta/\dot r = \text{constant } a$$

$$\dot\theta = \frac{a\dot r}{r}$$

$$\theta = a\ln r + b$$

$$\frac{d}{dt}\left[\frac{r^2\dot\theta}{r\sqrt{\dot r^2+r^2\dot\theta^2}}\right] = 0 \qquad \frac{r\dot\theta}{\sqrt{\dot r^2+r^2\dot\theta^2}} = \text{constant}$$

$$r^2\dot\theta^2 = p^2(\dot r^2+r^2\dot\theta^2)$$

$$r^2\dot\theta^2(1-p^2) = \dot r^2$$

$$r\dot\theta/\dot r = \text{constant.}$$

D 33

Wave propagation.
Constant velocity, 1.

$(p_1\ p_2)$ gives direction of normal.
Normal does not change direction
so $P_1 = p_1 \quad P_2 = p_2$

$$X = x + \frac{p_1 t}{\sqrt{p_1^2+p_2^2}} \qquad Y = y + \frac{p_2 t}{\sqrt{p_1^2+p_2^2}}$$ after time t.

At first I thought something was wrong, as it looked as if $P_1 dX_1 + P_2 dX_2$ was going to contain dp_1, dp_2 as well as dx_1, dx_2

However $d(p_1(p_1^2+p_2^2)^{-\frac{1}{2}}) = (p_1^2+p_2^2)^{-\frac{1}{2}} dp_1 - p_1(\ )^{-\frac{3}{2}}(p_1dp_1+p_2dp_2)$

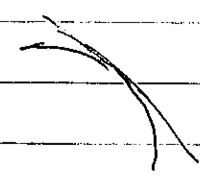
$$= (p_1^2+p_2^2)^{-3/2}(p_2^2 dp_1 - p_1p_2\, dp_2)$$
$$= p_2(p_1^2+p_2^2)^{-3/2}(p_2dp_1 - p_1dp_2)$$

$$d(p_2(p_1^2+p_2^2)^{-\frac{1}{2}}) = (\ )^{-\frac{1}{2}} dp_2 - p_2(\ )^{-3/2}(p_1dp_1+p_2dp_2)$$
$$= (\ )^{-3/2}[(p_1^2+p_2^2)dp_2 - p_2(p_1dp_1+p_2dp_2)]$$
$$= (p_1^2+p_2^2)^{-\frac{3}{2}}[p_1^2dp_2 - p_1p_2dp_1]$$
$$= -p_1(p_1^2+p_2^2)^{-3/2}(p_2dp_1 - p_1dp_2)$$

$$\therefore\ P_1dX + P_2dY = p_1dx + p_2dy$$

and, as expected, we have a contact transformation.

This is good. It means that, if we hold x, y fast and let $p_1\ p_2$ vary, we do get changes dX, dY but these satisfy $p_1dX + p_2dY = 0$ i.e. we go to a



nearby point on the circle spreading out from (x,y), but as this circle touches the new wave front at (X,Y), the displacement is (to the first order) in the direction of the tangent.

E 1

A. Einstein. 1

Zur Elektrodynamik Bewegter Körper. Ann. d. Phys. 17 (1905)

Two assumptions. (1) Velocity of light always same. (2) Principle of Relativity. No difference in laws for systems with constant relative velocity

This leads to Lorentz transformation.

$$t' = \beta\left(t - \frac{v}{c^2}x\right) \quad x' = \beta(x - vt) \quad \eta = y \quad \zeta = z \quad \beta = 1/\sqrt{1 - v^2/c^2}.$$

X, Y, Z electrical force L M N magnetic force

$$(1)\quad \frac{1}{c}\frac{\partial X}{\partial t} = \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} \qquad \frac{1}{c}\frac{\partial L}{\partial t} = \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\quad (4)$$

$$(2)\quad \frac{1}{c}\frac{\partial Y}{\partial t} = \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x} \qquad \frac{1}{c}\frac{\partial M}{\partial t} = \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\quad (5)$$

$$(3)\quad \frac{1}{c}\frac{\partial Z}{\partial t} = \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \qquad \frac{1}{c}\frac{\partial N}{\partial t} = \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\quad (6)$$

Now Einstein considers how these rates of change appear to the other system $(x'\ y'\ z'\ t')$.

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} = \beta\frac{\partial}{\partial x'} - \beta\frac{v}{c^2}\frac{\partial}{\partial t'}$$

$$\frac{\partial}{\partial t} = \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = -\beta v\frac{\partial}{\partial x'} + \beta\frac{\partial}{\partial t'}$$

Equations (2) and (3) are simpler to deal with than (1).

(2) leads to $\frac{\beta}{c}\left(\frac{\partial Y}{\partial t'} - v\frac{\partial Y}{\partial x'}\right) = \frac{\partial L}{\partial z'} - \beta\left(\frac{\partial N}{\partial x'} - \frac{v}{c^2}\frac{\partial N}{\partial t'}\right)$

$$\frac{1}{c}\frac{\partial}{\partial t'}\beta\left(Y - \frac{v}{c}N\right) = \frac{\partial L}{\partial z'} - \frac{\partial}{\partial x'}\beta\left(N - \frac{v}{c}Y\right).$$

Eqn (2) in $x't'$ system should be $\frac{1}{c}\frac{\partial Y'}{\partial t'} = \frac{\partial L'}{\partial z'} - \frac{\partial N'}{\partial x'}$

and we shall have this if

$$\boxed{Y' = \beta\left(Y - \frac{v}{c}N\right) \quad L' = L \quad N' = \beta\left(N - \frac{v}{c}Y\right)}$$

Eqn ③ gives
$$\frac{\beta}{c}\left(\frac{\partial Z}{\partial t'} - v\frac{\partial Z}{\partial x'}\right) = \beta\left(\frac{\partial M}{\partial x'} - \frac{v}{c^2}\frac{\partial M}{\partial t'}\right) - \frac{\partial L}{\partial y}$$
$$\frac{1}{c}\frac{\partial}{\partial t'}\beta\left(Z + \frac{v}{c}M\right) = \frac{\partial}{\partial x'}\beta\left(M + \frac{v}{c}Z\right) - \frac{\partial L}{\partial y'}$$
This should be $\frac{1}{c}\frac{\partial Z'}{\partial t'} = \frac{\partial M'}{\partial x'} - \frac{\partial L'}{\partial y'}$

so we need
$$\boxed{Z' = Z + \frac{v}{c}M \quad M' = M + \frac{v}{c}Z \quad L' = L.}$$
To deal with ① we need to use $\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0$

so $\beta\left(\frac{\partial X}{\partial x'} - \frac{v}{c^2}\frac{\partial X}{\partial t'}\right) + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0$

so $\frac{\partial X}{\partial x'} = \frac{v}{c^2}\frac{\partial X}{\partial t'} - \frac{1}{\beta}\left(\frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right)$

① gives $\frac{\beta}{c}\left(\frac{\partial X}{\partial t'} - v\frac{\partial X}{\partial x'}\right) = \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}$
$$\frac{\partial X}{\partial t'} - v\frac{\partial X}{\partial x'} = \frac{\partial X}{\partial t'}\left(1 - \frac{v^2}{c^2}\right) + \frac{v}{\beta c}\left(\frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right)$$
$$\therefore \frac{\beta}{c}\left(1 - \frac{v^2}{c^2}\right)\frac{\partial X}{\partial t'} + \frac{v}{c}\left(\frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) = \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}$$
$\beta = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$ so $\left(1 - \frac{v^2}{c^2}\right)\beta = \frac{1}{\beta}$ Hence
$$\frac{1}{\beta c}\frac{\partial X}{\partial t'} = \frac{\partial}{\partial y}\left(N - \frac{v}{c}Y\right) - \frac{\partial}{\partial z}\left(M + \frac{v}{c}Z\right)$$
$$\therefore \frac{1}{c}\frac{\partial X}{\partial t'} = \frac{\partial}{\partial y}\beta\left(N - \frac{v}{c}Y\right) - \frac{\partial}{\partial z}\beta\left(M + \frac{v}{c}Z\right)$$
This is $\frac{1}{c}\frac{\partial X'}{\partial t'} = \frac{\partial N'}{\partial y'} - \frac{\partial M'}{\partial z'}$ if

$$X' = X \quad N' = \beta\left(N - \frac{v}{c}Y\right) \quad M' = \beta\left(M + \frac{v}{c}Z\right)$$

To explain Einstein's work, which is highly condensed.

J680/1 E3

$f\ g\ h$ displacement

Jeans § 680.

(704) $\frac{4\pi}{c}\left(\rho U + \frac{df}{dt}\right) = \frac{\partial\gamma}{\partial y} - \frac{\partial\beta}{\partial z}$ etc.

(705) $-\frac{1}{c}\frac{da}{dt} = \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}$ $H = (a, b, c)$

(706) $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = \rho$ $E = (X\ Y\ Z)$

(707) $\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z} = 0$

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} \qquad x' = \beta(x - ut)$$

$$\frac{\partial}{\partial t} = \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'}, \qquad t' = \beta\left(t - \frac{xu}{c^2}\right)$$

so $\frac{\partial}{\partial x} = \beta\left(\frac{\partial}{\partial x'} - \frac{u}{c^2}\frac{\partial}{\partial t'}\right)$

$\frac{\partial}{\partial t} = \beta\left(\frac{\partial}{\partial t'} - u\frac{\partial}{\partial x'}\right)$

(704) → (709) $\frac{4\pi}{c}\left[\rho U + \beta\left(\frac{\partial f}{\partial t'} - u\frac{\partial f}{\partial x'}\right)\right] = \frac{\partial\gamma}{\partial y'} - \frac{\partial\beta}{\partial z'}$

(704b) is $\frac{4\pi}{c}\left(\rho V + \frac{\partial g}{\partial t}\right) = \frac{\partial\alpha}{\partial z} - \frac{\partial\gamma}{\partial x}$

$$\frac{4\pi}{c}\left[\rho V + \beta\left(\frac{\partial g}{\partial t'} - u\frac{\partial g}{\partial x'}\right)\right] = \frac{\partial\alpha}{\partial z'} - \beta\left(\frac{\partial\gamma}{\partial x'} - \frac{u}{c^2}\frac{\partial\gamma}{\partial t'}\right)$$

Similarly (704c) gives

$$\frac{4\pi}{c}\left[\rho W + \beta\left(\frac{\partial h}{\partial t'} - u\frac{\partial h}{\partial x'}\right)\right] = \beta\left(\frac{\partial\beta}{\partial x'} - \frac{u}{c^2}\frac{\partial\beta}{\partial t'}\right) - \frac{\partial\alpha}{\partial y'}$$

~~For $\rho=0$, 704b leads to~~ Let $g' = \beta\left[g - \frac{u}{4\pi c}\gamma\right]$

~~$\frac{\beta}{c}\left(\partial\right.$~~ $\frac{4\pi}{c}\frac{\partial g'}{\partial t'} = \frac{\partial\alpha'}{\partial z'} - \frac{\partial\gamma'}{\partial x'}$ comes from

$$\frac{4\pi\kappa}{c}\frac{\partial}{\partial t'}\left[g - \frac{u}{4\pi c}\gamma\right] = \frac{\partial\alpha}{\partial z'} - \frac{\partial}{\partial x'}\kappa\left(\gamma - \frac{4\pi u}{c}g\right)$$

$\rho = 0.$ $$\frac{4\pi K}{c}\left(\frac{\partial g}{\partial t'} - u\frac{\partial g}{\partial x'}\right) = \frac{\partial \alpha}{\partial z'} - K\left(\frac{\partial \gamma}{\partial x'} - \frac{u}{c^2}\frac{\partial \gamma}{\partial t'}\right)$$

$$\frac{4\pi}{c}\frac{\partial g}{\partial t'} = u\frac{\partial g}{\partial x'} + \frac{1}{K}\frac{\partial \alpha}{\partial z'} - \frac{\partial \gamma}{\partial x'} + \frac{u}{c^2}\frac{\partial \gamma}{\partial t'}$$

$$\frac{4\pi}{c}\frac{\partial}{\partial t'}\left(g - \frac{u}{4\pi c}\gamma\right)$$

$\rho = 0$ $$\frac{4\pi K}{c}\left[\frac{\partial g}{\partial t'} - u\frac{\partial g}{\partial x'}\right] = \frac{\partial \alpha}{\partial z'} - K\left(\frac{\partial \gamma}{\partial x'} - \frac{u}{c^2}\frac{\partial \gamma}{\partial t'}\right)$$

$$\frac{4\pi}{c}\frac{\partial}{\partial t'}K\left[g - \frac{u}{4\pi c}\gamma\right] = \frac{\partial \alpha}{\partial z'} - \frac{\partial}{\partial x'}K\left(\gamma - \frac{4\pi u}{c}g\right)$$

$$\frac{4\pi}{c}\frac{\partial g'}{\partial t'} = \frac{\partial \alpha}{\partial z'} - \frac{\partial \gamma'}{\partial x'}$$

y $$g' = K\left(g - \frac{u}{4\pi c}\gamma\right)$$

$$\gamma' = K\left(\gamma - \frac{4\pi u}{c}g\right).$$

$\rho = 0$. 709 becomes $\frac{4\pi}{c}\cancel{k}\left(\frac{\partial f}{\partial t'} - u\frac{\partial f}{\partial x'}\right) = \frac{\partial \gamma}{\partial y'} - \frac{\partial \beta}{\partial z'}$

$$\frac{4\pi}{c}\left[k^2\frac{\partial f}{\partial t'} - uk^2\frac{\partial f}{\partial x'}\right] = \frac{\partial}{\partial y'}k\gamma - \frac{\partial}{\partial z'}k\beta.$$

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = 0$$

$$k\left(\frac{\partial f}{\partial x'} - \frac{u}{c^2}\frac{\partial f}{\partial t'}\right) + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = 0$$

$$\therefore \quad k\frac{\partial f}{\partial x'} = \frac{ku}{c^2}\frac{\partial f}{\partial t'} - \frac{\partial g}{\partial y} - \frac{\partial h}{\partial z}$$

So $$\frac{4\pi}{c}\left[k\frac{\partial f}{\partial t'} - \frac{ku^2}{c^2}\frac{\partial f}{\partial t'} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right] = \frac{\partial \gamma}{\partial y'} - \frac{\partial \beta}{\partial z'}$$

$$k\frac{\partial f}{\partial t'} - \frac{ku^2}{c^2}\frac{\partial f}{\partial t'} = k\frac{\partial f}{\partial t'}\left(1 - \frac{u^2}{c^2}\right)$$
$$= \frac{1}{k}\frac{\partial f}{\partial t'}$$

So $$\frac{4\pi}{c}\left[\frac{1}{k}\frac{\partial f}{\partial t'} + \frac{\partial g}{\partial y'} + \frac{\partial h}{\partial z'}\right] = \frac{\partial \gamma}{\partial y'} - \frac{\partial \beta}{\partial z'}$$

$$\therefore \quad \frac{4\pi}{ck}\frac{\partial f}{\partial t'} = \frac{\partial}{\partial y'}\left(\gamma - \frac{4\pi}{c}g\right) - \frac{\partial}{\partial z'}\left(\beta + \frac{4\pi}{c}h\right)$$

$$\therefore \quad \frac{4\pi}{c}\frac{\partial f}{\partial t'} = \frac{\partial \gamma'}{\partial y'} - \frac{\partial \beta'}{\partial z'}$$

with $$\gamma' = k\left(\gamma - \frac{4\pi}{c}g\right)$$
$$\beta' = k\left(\beta + \frac{4\pi}{c}h\right).$$

$iE = \mathcal{E} \qquad iV = \varphi.$

$$E = -\frac{1}{c}\frac{\partial \mathcal{A}}{\partial t} - \text{grad } V$$

$$\mathcal{E} = iE = -\frac{i}{c}\frac{\partial \mathcal{A}}{\partial t} - \text{grad } iV$$

$$= \frac{\partial \mathcal{A}}{ic\partial t} = \frac{\partial \mathcal{A}}{\partial T} - \text{grad } \varphi.$$

$$\text{curl } \mathcal{E} = \text{curl } \frac{\partial \mathcal{A}}{\partial T} = \frac{\partial H}{\partial T}$$

This agrees with N53

$\psi_1 = \mathcal{A}_1 \quad \psi_2 = \mathcal{A}_2 \quad \psi_3 = \mathcal{A}_3 \quad \mu_4 = \varphi.$

$F_{ij} = \dfrac{\partial \psi_i}{\partial x_j} - \dfrac{\partial \psi_j}{\partial x_i}$ is 0 if $i = j$.

$$\frac{\partial \psi_3}{\partial x_2} - \frac{\partial \psi_2}{\partial x_3} = \frac{\partial \mathcal{A}_3}{\partial x_2} - \frac{\partial \mathcal{A}_2}{\partial x_3} = H_1$$

$$\frac{\partial \psi_1}{\partial x_3} - \frac{\partial \psi_3}{\partial x_1} = \frac{\partial \mathcal{A}_1}{\partial x_3} - \frac{\partial \mathcal{A}_3}{\partial x_1} = H_2$$

$$\frac{\partial \psi_2}{\partial x_1} - \frac{\partial \psi_1}{\partial x_2} = \frac{\partial \mathcal{A}_2}{\partial x_1} - \frac{\partial \mathcal{A}_1}{\partial x_2} = H_3$$

$$\begin{matrix} . & -H_3 & H_2 & \mathcal{E}_1 \\ H_3 & . & -H_1 & \mathcal{E}_2 \\ -H_2 & H_1 & . & \mathcal{E}_3 \\ -\mathcal{E}_1 & -\mathcal{E}_2 & -\mathcal{E}_3 & . \end{matrix}$$

$$F_{14} = \frac{\partial \psi_1}{\partial x_4} - \frac{\partial \psi_4}{\partial x_1} = \frac{\partial \mathcal{A}_1}{\partial T} - \frac{\partial \varphi}{\partial x_1} = \mathcal{E}_1$$

$$F_{24} = \frac{\partial \psi_2}{\partial x_4} - \frac{\partial \psi_4}{\partial x_2} = \frac{\partial \mathcal{A}_2}{\partial T} - \frac{\partial \varphi}{\partial x_2} = \mathcal{E}_2$$

Ni 58
E7

So $\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}, \frac{dt}{ds}\right) \cdot c$

$$= \left(\frac{v_1}{\sqrt{c^2-v^2}}, \frac{v_2}{\sqrt{c^2-v^2}}, \frac{v_3}{\sqrt{c^2-v^2}}, \frac{1}{\sqrt{c^2-v^2}}\right) \cdot c$$

$$= \left(\frac{v_1}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{v_2}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{v_3}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right)$$

Multiply by ρ_0, invariant. As $\rho = \frac{\rho_0}{\sqrt{1-\frac{v^2}{c^2}}}$

$(\rho v_1, \rho v_2, \rho v_3, \rho)$ is a 4-vector.

We suppose charges at rest in $\bar{x}\bar{t}$ system, with density ρ_0. Charge $= \rho_0 l_0$.

Charge is invariant. $\rho(x_2 - x_1) = \rho_0 l_0$

As $x_2 - x_1 = l_0\sqrt{1 - v^2/c^2}$

$$\rho = \frac{\rho_0}{\sqrt{1 - v^2/c^2}}.$$

Consider vector $0, 0, 0, \rho_0$ in $\bar{x}\bar{t}$ system.

$$x = \frac{\bar{x} + v\bar{t}}{\sqrt{1 - v^2/c^2}} \qquad t = \frac{\frac{v}{c^2}\bar{x} + \bar{t}}{\sqrt{1 - v^2/c^2}}$$

Putting $\bar{x} = 0$, $\bar{t} = \rho_0$ we find

$$x = \frac{v\rho_0}{\sqrt{1 - v^2/c^2}} \qquad t = \frac{\rho_0}{\sqrt{1 - v^2/c^2}}$$

i.e $\quad x = \rho v \quad t = \rho. \quad y = 0 \quad z = 0.$

So in $xyzt$ system vector appears as

$$(\rho v, 0, 0, \rho)$$

The first three coordinates represent j_1, j_2, j_3 the current density.

Thus it seems that (j_1, j_2, j_3, ρ) transform like a vector, for $0\,0\,0\,\rho$ represents this in $\bar{x}\bar{y}\bar{z}\bar{t}$ system

A formal proof is as follows. For a moving point (dx, dy, dz, dt) is a contravariant 4-vector.

$ds^2 = -dx^2 - dy^2 - dz^2 + c^2 dt^2$ is invariant

so $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}, \frac{dt}{ds}$ is a vector.

Let $v_1 = \frac{dx}{dt} \quad v_2 = \frac{dy}{dt} \quad v_3 = \frac{dz}{dt} \quad v^2 = v_1^2 + v_2^2 + v_3^2$

$$\left(\frac{ds}{dt}\right)^2 = -(v_1^2 + v_2^2 + v_3^2) + c^2 = -v^2 + c^2$$

$$\cancel{\frac{dx}{dt}} \quad \frac{dx}{ds} = \frac{dx}{dt} \Big/ \frac{ds}{dt} = \frac{v_1}{\sqrt{c^2 - v^2}} \quad \text{etc}$$

N56

E9

A rod at rest in the $\bar{x}\bar{t}$ system has length l_0. Thus $\bar{x}_2 - \bar{x}_1 = l_0$. Since it is at rest in the $\bar{x}\bar{t}$ system, this is true at whatever times the coordinates are ~~mentioned~~ measured.

Hence
$$l_0 = \frac{x_2 - vt_2}{\surd} - \frac{x_1 - vt_1}{\surd}$$

To measure it in the x, t system, we take $t_1 = t_2$, since measurement supposes ends are observed at the same time t.

$$\text{Hence} \quad l_0 = \frac{x_2 - x_1}{\sqrt{1 - v^2/c^2}}$$

$$\text{so} \quad x_2 - x_1 = l_0 \sqrt{1 - \frac{v^2}{c^2}}.$$

ρ chosen as 4π times density of charge.

$$\rho' = \frac{\partial X'}{\partial x'} + \frac{\partial Y'}{\partial y'} + \frac{\partial Z'}{\partial z'}$$

$$X' = X \quad Y' = \beta\left(Y - \frac{v}{c}N\right) \quad Z' = \beta\left(Z + \frac{v}{c}M\right)$$

$$t' = \beta\left(t - \frac{v}{c^2}x\right) \quad x' = \beta(x - vt)$$

$$vt' + x' = \beta\left(x - \frac{v^2}{c^2}x\right) = x\sqrt{1-\frac{v^2}{c^2}} \quad \therefore x = \beta(x' + vt')$$

$$t' + \frac{v}{c^2}x' = \beta t\left(1 - \frac{v^2}{c^2}\right) = t\sqrt{1-\frac{v^2}{c^2}} \qquad t = \beta\left(t' + \frac{v}{c^2}x'\right)$$

$$\rho' = \left(\frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t}\right)X' + \left(\cancel{\frac{\partial x}{\partial y'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial y'}\frac{\partial}{\partial t}}\right)Y + \left(\cancel{\frac{\partial x}{\partial z'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial z'}\frac{\partial}{\partial t}}\right)Z$$

$$= \beta\frac{\partial X'}{\partial x} + \frac{\beta v}{c^2}\frac{\partial X'}{\partial t} + \frac{\partial Y'}{\partial y} + \frac{\partial Z'}{\partial z}$$

$$= \beta\frac{\partial X}{\partial x} + \frac{\beta v}{c^2}\frac{\partial X}{\partial t} + \beta\left(\frac{\partial Y}{\partial y} - \frac{v}{c}\frac{\partial N}{\partial y}\right) + \beta\left(\frac{\partial Z}{\partial z} + \frac{v}{c}\frac{\partial M}{\partial z}\right)$$

$$= \beta\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) + \frac{\beta v}{c^2}\frac{\partial X}{\partial t} - \frac{\beta v}{c}\frac{\partial N}{\partial y} + \frac{\beta v}{c}\frac{\partial M}{\partial z}$$

Now $-\dfrac{u_x\rho}{c} = \dfrac{1}{c}\dfrac{\partial X}{\partial t} - \dfrac{\partial N}{\partial y} + \dfrac{\partial M}{\partial z}$

Last terms above are $\dfrac{\beta v}{c}\cdot\left(-\dfrac{u_x\rho}{c}\right)$

$$\therefore \quad \rho' = \beta\rho - \frac{\beta v}{c^2}u_x\rho$$

$$= \beta\left(\rho - \frac{v}{c^2}\cdot u_x\rho\right)$$

$$2F = \ln\left[\frac{x_0\sqrt{\alpha^2+\beta^2}}{\beta}\left(1+\frac{\alpha\Delta t}{2x_0}\right) \cancel{\frac{\beta}{\beta}} + \frac{\alpha x_0}{\beta} + \frac{1}{2}\beta\Delta t\right]$$ E_{11}

$$-\ln\left[\frac{x_0\sqrt{\alpha^2+\beta^2}}{\beta}\left(1+\frac{\alpha\Delta t}{2x_0}\right)\right]$$

$x = vt$ represents a point moving with velocity v in our system.

$$t' = \beta\left(t - \frac{v}{c}x\right) \quad x' = \beta(x - vt)$$

In $x'y'z't'$ system the point is at rest.

$\rho' dx' dy' dz'$ denotes charge in a small region.

$= \rho\, dx\, dy\, dz$. $\qquad dx' = \beta\, dx$. (logic ?)

$\therefore \quad \rho'\beta\, \cancel{dx}\, dx\, dy\, dz = \rho\, dx\, dy\, dz$

$$\rho'\beta = \rho \qquad \rho' = \rho_0$$

$$\rho_0 = \frac{\rho}{\beta}. \quad \cancel{\beta}\ \rho = \beta\rho_0.$$

$(x\,y\,z\,t) = \cancel{(x, y, z, t)}\ (vt, 0, 0, t)$ for the moving point

$(dx, dy, dz, dt) = (v\,dt, 0, 0, dt)$

$\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}, \frac{dt}{ds}\right)$ is a vector $\qquad ds^2 = dt^2 - dx^2$

$$= dt^2(1 - v^2)$$

$\left(\frac{dx}{dt}\frac{dt}{ds}, \frac{dy}{dt}\frac{dt}{ds}, \frac{dz}{dt}\frac{dt}{ds}, \frac{dt}{ds}\right)$ $\qquad \frac{dt}{ds} = \beta.$

is a vector

It is $(\beta v, 0, 0, \beta)$

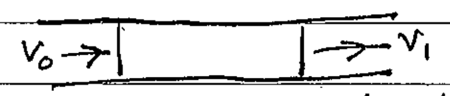
ρ_0 times it is $(\rho_0\beta v, 0, 0, \beta\rho_0)$

$$= (\rho v, 0, 0, \rho).$$

29. i. 94. Pohl? + Derry. Fluid, up to Gauss' Th^m.

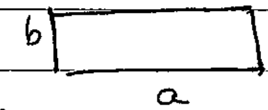
$\textcircled{1}$ E 12

The inverse square law for gravitation seems to have been guessed by 17th century mathematicians from the idea that the sun might be emitting some incompressible fluid, the velocity of which would accordingly fall off with the square of the distance. Incompressible fluids provide a model for much in electromagnetic theory, and we start with this topic

We have one dimensional 

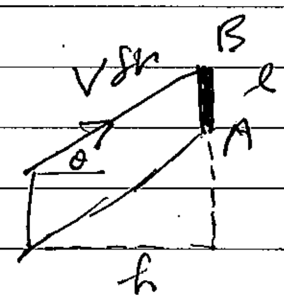
flow when liquid moves along a tube. Clearly $v_0 = v_1$, and so the velocity satisfies the condition $\frac{dv}{dx} = 0$.

Two dimensions

We consider a region in the form of a rectangle, as here, 

and consider the rate at which fluid crosses the various lines.

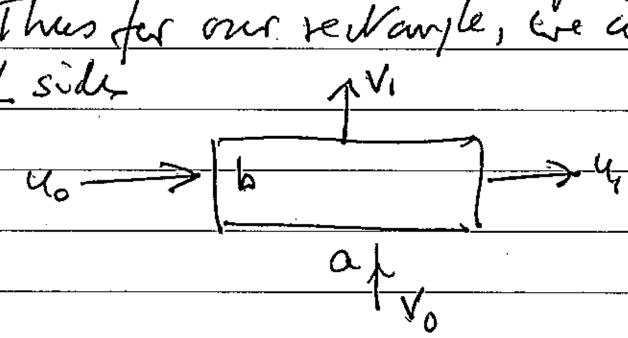
If fluid has velocity V, all the fluid in this region will cross AB in time δt.



The area of the parallelogram is lh.

$h = V\delta t . \cos\theta$

$\therefore$ Area crossing in δt is $l . V\cos\theta . \delta t = lu\,\delta t$ where u is the component of $V \perp AB$. The rate/sec is lu, if we take unit area as unit for fluid quantity. Thus for our rectangle, we consider the velocities $\perp$ sides



If the region is very small, we may take

$$\frac{\partial u}{\partial x} = \frac{u_1 - u_0}{a} \qquad \frac{\partial v}{\partial y} = \frac{v_1 - v_0}{b}$$

Now fluid enters from left at the rate $u_0 b$ and leaves on the right at rate $u_1 b$. The net rate, for fluid going out of region is $b(u_1 - u_0) = ba\frac{\partial u}{\partial x}$.

In the same way, the net rate for the other pair of sides is $ab\frac{\partial v}{\partial y}$.

Together we have $ab\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$. If the fluid is incompressible this must be zero. So the condition for incompressible fluid in 2 dimensions is

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}.$$

This is called the <u>divergence</u>. If $\underline{V} = (u, v)$ we write $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ as $\operatorname{div} \underline{V}$.

<u>SET as QUESTIONS</u>

Symmetrical flow in a plane

Suppose fluid is introduced at the origin, an amount ε in time δt.

The fluid in a circle of radius r receives an extra area ε in δt, so it must increase its area by ε in time δt.

If radius increases by dr, increase of area is $2\pi r\, dr$.

So $\quad \varepsilon = 2\pi r\, \delta r$

$$\frac{\varepsilon}{\delta t} = \frac{2\pi r\, \delta r}{\delta t}$$

$\varepsilon/\delta t$ does not depend on r, so $2\pi r\, dr/dt$ must be the same for all r

$$\therefore \frac{dr}{dt} = \frac{k}{r}.$$

E
14

We can verify that this velocity satisfies the condition div $V = 0$.

V is k/r radially. Its components are $\left(\frac{k}{r}\cos\theta, \frac{k}{r}\sin\theta\right)$.

$$\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}.$$

$$\therefore V = \left(\frac{kx}{r^2}, \frac{ky}{r^2}\right) = \left(\frac{kx}{x^2+y^2}, \frac{ky}{x^2+y^2}\right)$$

$$u = \frac{kx}{x^2+y^2} \quad \frac{\partial u}{\partial x} = \frac{k(x^2+y^2) - 2x.kx}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial x} = \frac{k(y^2-x^2)}{(x^2+y^2)^2}$$

Similarly $\frac{\partial v}{\partial y} = \frac{k(x^2-y^2)}{(x^2+y^2)^2}$ $\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$

In 3-D, the area of a sphere is proportional to r^2, and velocity $= k/r^2$

The unit vector in line from origin to (x,y,z) is $\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$

$$\text{So } V = \left(\frac{kx}{r^3}, \frac{ky}{r^3}, \frac{kz}{r^3}\right)$$

Now $r^2 = x^2+y^2+z^2$. Apply $\frac{\partial}{\partial x}$ $\; 2r\frac{\partial r}{\partial x} = 2x$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial}{\partial x}(kxr^{-3}) = kr^{-3} - 3kxr^{-4}\frac{\partial r}{\partial x}$$

$$= k(r^{-3} - 3x^2r^{-5})$$

$$= \frac{k}{r^5}(r^2 - 3x^2) = \frac{k}{r^5}(-2x^2+y^2+z^2)$$

Similarly $\frac{\partial}{\partial y}(kyr^{-3}) = \frac{k}{r^5}(x^2-2y^2+z^2)$

$$\frac{\partial}{\partial z}(kzr^{-3}) = \frac{k}{r^5}(x^2+y^2-2z^2)$$

Sum $= 0$.

$\frac{E}{15}$

Inverse square law.

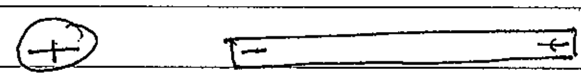
We take as an experimentally shown fact that charges e, e' at distance r repel each other with a force $F = kee'/r^2$, k constant.

Many logical questions involved – how is charge measured? etc. Plausible assumptions – if two bodies have charges e_1, e_2 and are brought into contact the resulting body will have charge $e_1 + e_2$; if electricity flows from A to B, then B will gain just as much charge as A loses.

k in the eqn. for F will depend on units used. It's agreed that the units are so chosen as to make $k = 1$.

Then $F = ee'/r^2$. If e and e' have opposite signs, the force will be one of attraction.

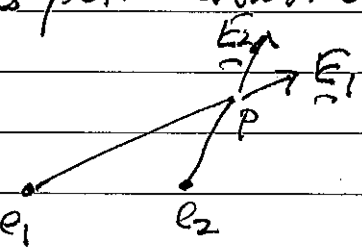
Induction. If a +ve charged body is held near a conductor then negative charges on the conductor will be drawn towards the positive charge, and positive charges repelled



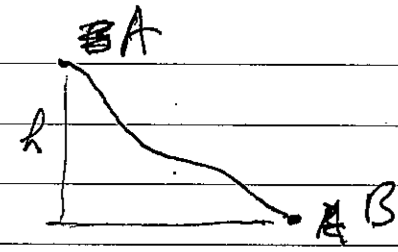
The unlike charges, + and −, are nearer together than the like charges + and +, so the attraction will exceed the repulsion.

Electric intensity at a point, $\mathbf{E}$, is the force a unit charge placed at this point would experience.

If charges e_1 and e_2 produce $\mathbf{E}_1$ and $\mathbf{E}_2$ at P when acting separately, together they produce the VECTOR SUM $\mathbf{E}_1 + \mathbf{E}_2$.



Potential.



For gravity, if A is at a height h above B a mass m sliding smoothly to B will acquire kinetic energy mgh. This is called the difference of potential energy between A and B. An essential point is that any ~~the~~ smooth path from B to A gives the same value.

In electrostatics, we define the potential difference between A and B as the work done by the field on a unit charge going from A to B. Naturally, this definition will be meaningless if the result depends on the particular path chosen.

We first show that this is so for the field of a single ~~unit~~ charge placed at the origin.

The force being $\underset{\frown}{E}$ and the displacement $d\underset{\frown}{s}$ the work done is $\underset{\frown}{E} \cdot d\underset{\frown}{s}$ (the component of the force in the direction of the displacement multiplied by the magnitude of the displacement.

If $\underset{\frown}{E} = (E_1\ E_2\ E_3)$ and $d\underset{\frown}{s} = (dx, dy, dz)$. $\underset{\frown}{E} \cdot d\underset{\frown}{s} = E_1 dx + E_2 dy + E_3 dz$. The work along a path is $\int \underset{\frown}{E} \cdot d\underset{\frown}{s}$.

For a single charge at the origin, the force is $\frac{e}{r^2}$ acting radially outwards. ~~Unit vector~~ $(x, y, z) = P$ is a vector in the direction OP. Its magnitude is $\sqrt{x^2+y^2+z^2} = r$, so $\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$ is a unit vector along OP. The vector of magnitude $\frac{e}{r^2}$ along OP is thus $\frac{ex}{r^3}, \frac{ey}{r^3}, \frac{ez}{r^3}$. In any displacement dx, dy, dz the work done is

$$E_1 dx + E_2 dy + E_3 dz = \frac{e(x\,dx + y\,dy + z\,dz)}{r^3}$$

As $r^2 = x^2+y^2+z^2$ $\quad 2r\,dr = 2x\,dx + 2y\,dy + 2z\,dz$ [?]

$\therefore x\,dx + y\,dy + z\,dz = r\,dr$

$\therefore$ Work done $= \dfrac{e\,r\,dr}{r^3} = \dfrac{e\,dr}{r^2}$

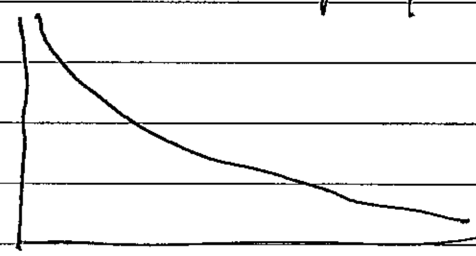
Work done in force from A at distance r_1 to B at distance r_2 is thus $\displaystyle\int_{r_1}^{r_2} \frac{e\,dr}{r^2} = \left(-\frac{e}{r}\right)_{r_1}^{r_2}$

$= \dfrac{e}{r_1} - \dfrac{e}{r_2}$

This is the change in $\frac{e}{r}$.

So the potential at distance r is $\frac{e}{r}$.

The path followed has not had to be specified.



16 Oct 93

If we have two fields, with $\underset{\frown}{E_1}$ and $\underset{\frown}{E_2}$ the potential are found by calculating $V_1 = \int \underset{\frown}{E_1}\, d\underset{\frown}{s}$ and $V_2 = \int \underset{\frown}{E_2}\, d\underset{\frown}{s}$. If they are combined, the electrical intensity becomes $\underset{\frown}{E_1} + \underset{\frown}{E_2}$ and the work done is $V = \int (\underset{\frown}{E_1} + \underset{\frown}{E_2}) . d\underset{\frown}{s}$

$= \int \underset{\frown}{E_1}\, d\underset{\frown}{s} + \int \underset{\frown}{E_2}\, d\underset{\frown}{s} = V_1 + V_2$.

Thus combining fields corresponds to adding the potentials. As a potential is a single number (a scalar) this is much simpler than adding the vectors $\underset{\frown}{E_1}$ and $\underset{\frown}{E_2}$.

It is easy to see that there will be a potential, independent of the path, for ~~an~~ the field due to any number of charges

(JEANS justifies this by appeal to CONSERVATION OF ENERGY)

23. Oct. 93.

As $V = \int E_1 dx + E_2 dy + E_3 dz$

$$\frac{\partial V}{\partial x} = E_1, \quad \frac{\partial V}{\partial y} = E_2, \quad \frac{\partial V}{\partial z} = E_3.$$

The vector $\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)$ is known a grad V.
(grad ← gradient).

So $\quad \underset{\frown}{E} = \text{grad } V.$

$\underset{\frown}{E}$ is $\perp$ equipotential, $V =$ constant.

For $\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt}$

when point (x, y, z) moves as time t passes.

If the motion is on $V =$ constant, $dV/dt = 0$,

so $\quad 0 = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt}$

$\therefore \quad 0 = E_1 \frac{dx}{dt} + E_2 \frac{dy}{dt} + E_3 \frac{dz}{dt}$

i.e motion is $\perp \underset{\frown}{E}$.

(It is well known that $\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}$ gives a vector normal to $V =$ constant.)

<u>The space rate of change of V in any direction equals component of $\underset{\frown}{E}$ in that direction</u>

<u>Proof</u> Let (dx, dy, dz) be of length ds.

Then $\quad dx^2 + dy^2 + dz^2 = ds^2$

So $\quad \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1.$

Thus $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ is a unit vector. Call it $\underset{\frown}{u}$.

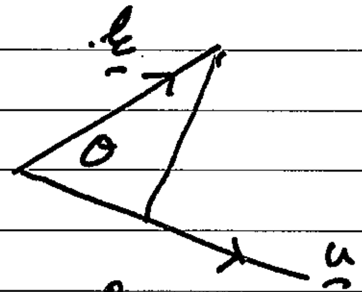
The space rate of change of V is $\frac{dV}{ds}$

$$\frac{dV}{ds} = \frac{\partial V}{\partial x}\frac{dx}{ds} + \frac{\partial V}{\partial y}\frac{dy}{ds} + \frac{\partial V}{\partial z}\frac{dz}{ds}$$

$$= E_1 \frac{dx}{ds} + E_2 \frac{dy}{ds} + E_3 \frac{dz}{ds}$$

$$= (\underset{\frown}{E} \cdot \underset{\frown}{u}) \ \cancel{\text{where}}$$

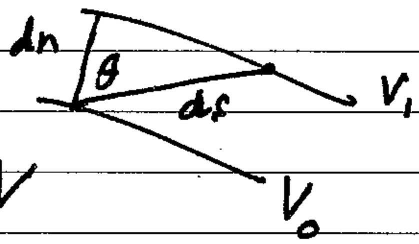
The component of $\underset{\sim}{E}$ in direction of $\underset{\sim}{u}$ is of magnitude



$$\|E\|\cos\theta = \|E\|\cdot 1\cdot\cos\theta$$
$$= \|E\|\cdot\|u\|\cdot\cos\theta = (\underset{\sim}{E}\cdot\underset{\sim}{u})$$ ~~if $\hat{u}$ unit vector.~~

Thus the rate of change of V in the direction of $\underline{u}$ is the component of E in this direction

Geometrically.



Rate of change of V is dV/ds

Now $dn = ds\cos\theta$

$\therefore\ ds = dn/\cos\theta$

$$\therefore\ \frac{dV}{ds} = \frac{dV}{dn}\cos\theta .$$

The differential equation for V in empty space.

We know
$$0 = \operatorname{div} \underline{E} = \frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z}$$

and
$$\underline{E} = \operatorname{grad} V = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)$$

$$\therefore \quad 0 = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}.$$

Often abbreviated to $\nabla^2 V = 0$, and known as Laplacian V. (Nabla squared).

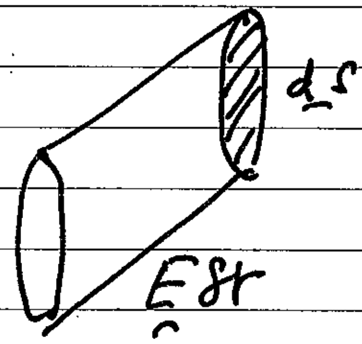
Gauss Theorem.

Textbooks often give analytical proofs. However this result can be seen directly from the idea that E can be seen as the velocity of an incompressible fluid.

By flux $\underline{E}$ for any surface we understand the rate at which fluid would cross this surface, when moving with velocity $\underline{E}$. The quantity of fluid is measured by the volume containing it.

In 3 dimensions we have an argument similar to that on page 1.

In time δt, the liquid inside the figure shown will cross the region dS. The height is the component of $\underline{E}\delta t$ $\perp$ dS.



An element of area is usually represented by a vector $\perp$ the ~~area~~ element, of magnitude equal to the area.

Thus volume crossing $= (\underline{E} \cdot \underline{dS})\delta t$.

Rate of crossing $= \underline{E} \cdot \underline{dS}$ (~~The theorem does not depend on this formula.~~)

Flux $= \iint \underline{E} \cdot \underline{dS}$

The field of a single charge, e, gives a radial electric intensity of e/r^2. The flux over a sphere of radius r is $e/r^2 \times$ area of sphere $= \frac{e}{r^2} \cdot 4\pi r^2 = 4\pi e$

Hence the flux of E through any closed surface containing this charge is $4\pi e$.

If we have charges $e_1\ e_2 \dots e_n$ producing $\underline{E}_1\ \underline{E}_2 \dots \underline{E}_n$, flux $= \int (\underline{E}_1 + \underline{E}_2 + \dots + \underline{E}_n)\, d\underline{S}$

$= \int E_1 dS + \int E_2 dS + \dots \quad \int E_n dS$

$= 4\pi e_1 + 4\pi e_2 \qquad + 4\pi e_n.$

Gauss Theorem. Flux is 4π times total charge enclosed.

Oct. 30

Consequences

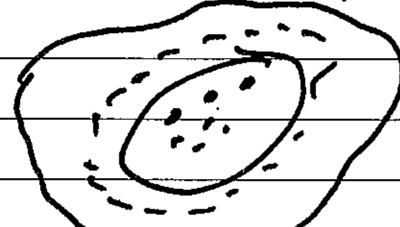
At any point inside a conductor, the electrical intensity is zero. (If it were not, charges would move)

a) If a closed surface is drawn inside conducting material, the total charge in it is zero. For $4\pi \Sigma e = \int \underline{E}.dS$ and $\underline{E} = 0$.

b) The charges on a conductor are all on the surface. For we can take a sphere as small as we like around any internal point, and by (a) the charge inside is 0.

c) If a number of charges are inside a hollow closed conductor, the total charge on the inside of the shell has equal magnitude, but opp. sign, to sum of charges.

Proof Consider flux through a surface inside conductor.

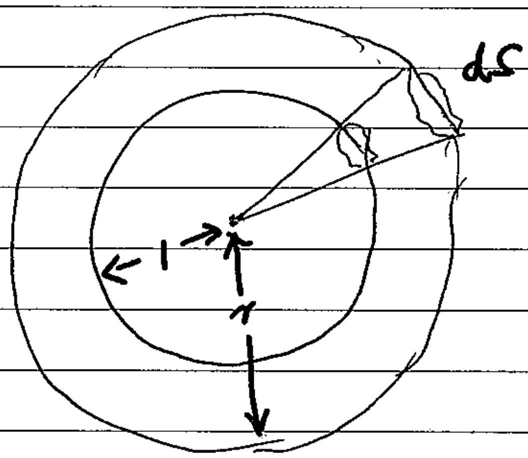


(d) In charge free empty space, the potential at any point is the average of the potential on a sphere with it as centre.

Proof The surface of the sphere has area $4\pi r^2$.

$\therefore$ Average, $a = \frac{1}{4\pi r^2} \iint V\, dS \qquad$ (Scalar dS)

Let $d\Omega$ be area of unit sphere corresponding to dS on sphere radius r



Then $dS = r^2 d\Omega$

$\therefore \frac{1}{4\pi r^2}\iint V\,dS$

$a = \frac{1}{4\pi}\iint V\,d\Omega$

Now $d\Omega$ does not depend on r. If r grows, a grows at rate $\frac{1}{4\pi}\iint \frac{\partial V}{\partial r}\,d\Omega = \frac{1}{4\pi r^2}\iint \frac{\partial V}{\partial r}\,dS$

$= \frac{1}{4\pi r^2}\iint \underset{\frown}{E}\cdot \underset{\frown}{dS}$ for $\frac{\partial V}{\partial r}$ a radial component of $\underset{\frown}{E}$

$= 0$ by Gauss Theorem.

$\therefore$ $\underline{a}$ is the same, whatever the radius.

Letting radius $\to 0$, theorem follows.

(e) V cannot be a maximum or a minimum at any point in charge free empty space.

Proof. Consider average for a small sphere around the point.

6 Nov 93 David. 20 Nov 93 Robert

(f) At a point just outside the surface of a conductor $\underset{\frown}{E} = 4\pi\sigma$ where σ is the surface density of charge.

$\underset{\frown}{E}$ will be normal to the surface, as the surface is an equipotential. Consider an area dS of the surface. Apply Gauss Theorem to the region shown

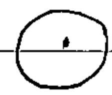
There is no flux for the surface inside, nor for the sides of the tube of force. $\therefore$ Flux $= \|E\|dS$. The charge inside is σdS $\therefore \|E\|\,dS = 4\pi\,\sigma dS$ $\|E\| = 4\pi\sigma$. Q.E.D.

(g) In empty space there cannot be 2 distinct fields of force ~~that are equal on the surface~~ inside a surface that are equal on the surface. If there were, let V_1 and V_2 be the potentials. Then $V_1 - V_2$ is the potential of some field and $V_1 = V_2$ on the boundary, so $V_1 - V_2 = 0$. There cannot be a maximum or a minimum inside the boundary. So $V_1 - V_2$ must be zero throughout. $\therefore V_1 = V_2$. QED.

Capacity of a conducting sphere.

We tend to think of capacitor in connection with capacitors, but a single body has a definite capacity. Capacity is defined by $V = Q/C$, Q = charge. V = potential. C = capacity. If we put a charge on any conductor, it will as a result have a certain potential and Q/V gives its capacity.

Suppose a charge Q is put on a sphere of radius a. Place a sphere of radius r around it. The field will be radial (by symmetry) and flux over the sphere will be $E(r) \cdot 4\pi r^2$



$E(r)$ being the intensity at distance r from the centre.

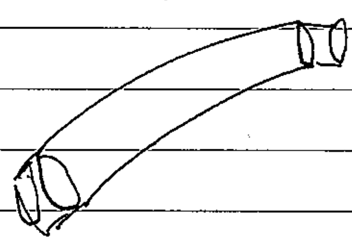
$\therefore E(r) \cdot 4\pi r^2 = 4\pi Q \quad E(r) = Q/r^2$.

This is exactly the intensity that would result from a charge Q at the centre. $V = Q/r$, so the potential on the surface will be Q/a. $V = Q/a$ so the capacity $C = a$.

David + Derry 12 ii 94

The charges on the ends of a tube of force are equal and opposite

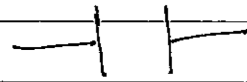
Take ends inside conductors. No force at ends: no flow ~~ac~~ across sides. QED.



17 Nov This part only. Mention EdS. Of fluid.

Spherical capacitor

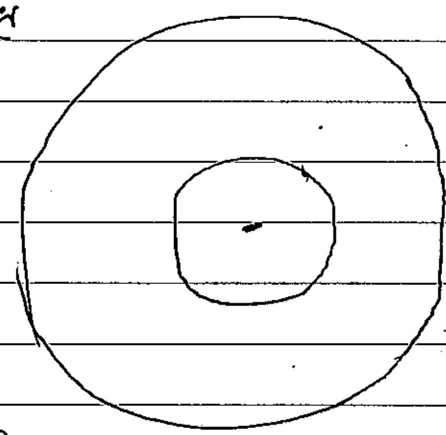
Equal and opposite charges appear. If these are $+e$, $-e$ and potential difference is V, $C = e/V$.



Capacity of spherical capacitor

Radii a, b.

We have seen that potential at distance r, $a < r < b$ is e/r.



$\therefore$ Potentials of the conductors are e/a and e/b

$$V = \frac{e}{a} - \frac{e}{b} = e\left(\frac{1}{a} - \frac{1}{b}\right)$$

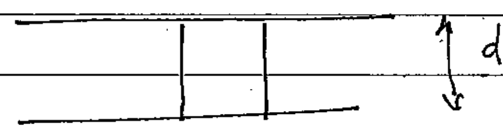
$$C = \frac{e}{V} = \frac{1}{\frac{1}{a} - \frac{1}{b}}.$$

Note If $b \to \infty$, this $\to a$ as we would expect.

Parallel plate capacitor.

We suppose no error of appreciable size introduced by supposing lines of force $\perp$ plate.

Let σ be density of charge on one plate, its area A.



Then $e = \sigma A$.

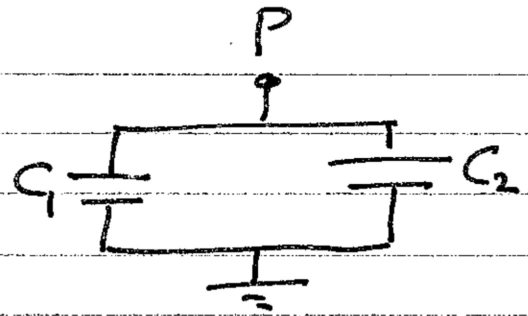
Intensity given by $E = 4\pi\sigma$.

If plates separated by d, $V = Ed$

$$\frac{e}{V} = \frac{\sigma A}{4\pi\sigma d} = \frac{A}{4\pi d}.$$

Not valid if plates separated by substance other than vacuum (or something resembling vacuum in its electrical properties (e.g. air).

Capacitors in parallel.



Let V be potential at P.
Let e_1 and e_2 be charges on capacitors C_1, C_2.

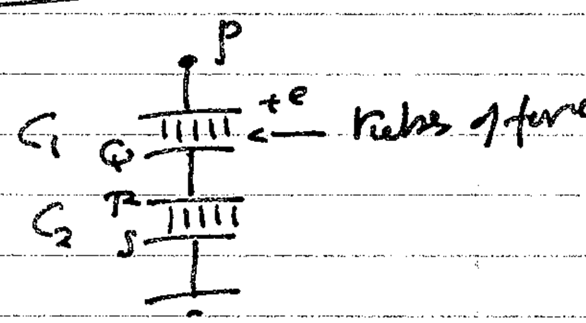
$e_1 = VC_1 \qquad e_2 = VC_2$

$e_1 + e_2$ is the total charge that must be brought to P to produce potential P.

$\frac{e_1 + e_2}{V}$ is capacity of the system

$K = C_1 + C_2$.

Capacitors in series.



We apply charge +e to P.
It attracts −e to Q, as tubes end on opposite charges
But QR has no way of changing its total charge. ∴ +e is on R, and so −e on S.
Thus both the capacitors have charge e.
This will produce potential difference $\frac{e}{C_1}$ on PQ and $\frac{e}{C_2}$ on RS.

∴ Potential of P is $\frac{e}{C_1} + \frac{e}{C_2} = V$

Capacity of system $= \frac{e}{V} = \frac{e}{\frac{e}{C_1} + \frac{e}{C_2}}$

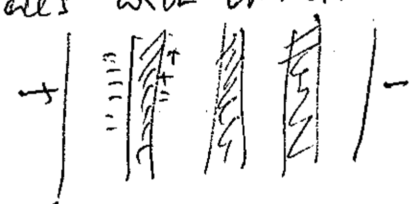
$$C = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

Thus $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$ reciprocals of capacity add in this situation.
cf Resistors in parallel, taps into bath

Field of an infinite charged plane.

Can be found easily by Gauss principle.
In dielectric theory, it is stated that inverse square law becomes ee'/Kr^2. But in fact ee'/r^2 still applies if we take account of extra charges that are induced.

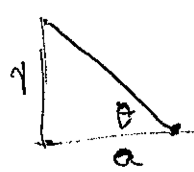
As an approx^n we may consider a dielectric as a set of conducting surfaces with intervals between. The effect of these is to increase the capacity by reducing the potential.



Induced charges appear on surface, and hence form charged plane

$$V = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} \frac{1}{\sqrt{x^2+y^2+a^2}}\,\sigma\,dx\,dy$$

$$= \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} \frac{r}{\sqrt{r^2+a^2}}\,\sigma\,dr\,d\theta$$



$$= \int_{r=0}^{\infty} \frac{2\pi\sigma r\,dr}{\sqrt{r^2+a^2}} = \infty.$$

$$\text{Force} = \frac{e}{r^2+a^2}\cos\theta = \frac{ea}{(r^2+a^2)^{3/2}}$$

$$\int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} \frac{ea\sigma}{(r^2+a^2)^{3/2}}\cdot r\,d\theta\,dr$$

Let $r = at$

$$\iint \frac{ea\sigma}{a^3(1+t^2)^{3/2}}\cdot at\,d\theta\,a\,dt$$

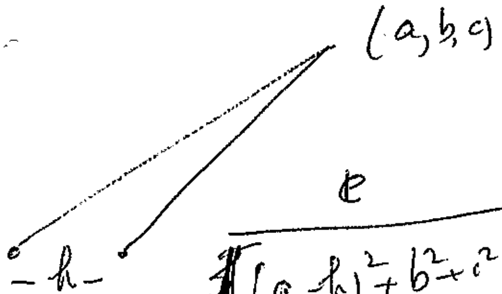
$$= 2\pi\int_{t=0}^{\infty} \frac{\sigma t\,dt}{(1+t^2)^{3/2}} \quad \text{ind of } a$$

$(1+t^2)^{-\frac{1}{2}}$

$-\frac{1}{2}(1+t^2)^{-3/2}\cdot 2t$

$$= 2\pi\left[-\frac{\sigma}{\sqrt{1+t^2}}\right]_0^{\infty}$$

$$= \underline{2\pi\sigma}$$



$$\frac{e}{\sqrt{(a-h)^2+b^2+c^2}} - \frac{e}{\sqrt{a^2+b^2+c^2}}$$

$$[(a-h)^2+b^2+c^2]^{-\frac{1}{2}}$$

$$= [a^2+b^2+c^2-2ah+h^2]^{-\frac{1}{2}}$$

$$= (a^2+b^2+c^2)^{-\frac{1}{2}}\left(1-\frac{2ah+h^2}{a^2+b^2+c^2}\right)^{-\frac{1}{2}}$$

$$= (a^2+b^2+c^2)^{-\frac{1}{2}}\left\{1+\frac{ah}{a^2+b^2+c^2}+O(h^2)\right\}$$

So $V \approx eah\,(a^2+b^2+c^2)^{-3/2}$

$$= \frac{a.he}{r^3}$$

If $(a,b,c) \to (x,y,z)$

$$\frac{\mu x}{r^3}$$

Dipole

Gaussian Elimination. Worked example G1

Escape at line 30

```
>
GAUSS. Data example, Hilbert 5 x 5. 3DIAG.
>         LIST
    1PRINT"HAVE YOU INSERTED DATA,LINE 10000 ON?"
    2 INPUT W$
    3 IF W$="Y" THEN GOTO 10 ELSE END
   10INPUT"M=",M
   20INPUT "N=",N
   30DIM C(M,N): DIM D(N)
   40A=1:B=1
   50FOR R=1 TO M:FOR S=1 TO N
   60READ C(R,S):NEXT S:NEXT R
   70FOR R=1 TO M:FOR S=1 TO N
   80PRINT;C(R,S);" ";:NEXT
   90PRINT" "
  100NEXT
  110P=A
  120PROCNIL
  130IF C(P,B)<>0 THEN GOTO 2000
  140IF P<M THEN P=P+1:GOTO 130
  150IF B<N THEN P=A:B=B+1:GOTO 130
  160IF P=M AND B=N THEN PROCPRINT
  170END
 2000IF P>A THEN PROCSORT
 2010IF B<N THEN PROCDIV:C(A,B)=1
 2020PRINT"---"::PROCPRINT
 2030IF A<M THEN PROCBELOW
 2040PRINT"...":PROCPRINT
 2050IF A>1 THEN PROCABOVE
 2060PRINT"---"
 2070PROCPRINT
 2080IF A<M AND B<N THEN A=A+1:B=B+1:GOTO 110
 2100END
 2999 DEFPROCNIL
 3000R=1:REPEAT:S=1:REPEAT
 3010IF ABS(C(R,S))<.000001 THEN C(R,S)=0
 3020 S=S+1: UNTIL S>N
 3030R=R+1 :UNTIL R>M
 3040ENDPROC
 4000DEFPROCSORT
 4010FOR S=1 TO N
 4020D(S)=C(A,S)
 4030C(A,S)=C(P,S)
 4040C(P,S)=D(S)
 4050NEXT
 4060ENDPROC
 5000DEFPROCDIV
 5010FOR K=B+1 TO N
```

PROCNIL 2999

G2

```
5020C(A,K)=C(A,K)/C(A,B)
5030NEXT
5040ENDPROC
6000DEFPROCBELOW
6010IF B=N THEN ENDPROC
6020FOR J=A+1 TO M
6040FOR K=B+1 TO N
6060C(J,K)=C(J,K)-C(J,B)*C(A,K)
6070NEXT:NEXT
6080FOR J=A+1 TO M
6090C(J,B)=0:NEXT
6100ENDPROC
6900DEFPROCABOVE
6910FOR J=1 TO A-1
6920IF B=N THEN 6960
6930FOR K=B+1 TO N
6940C(J,K)=C(J,K)-C(J,B)*C(A,K)
6950NEXT:NEXT
6960FOR J=1 TO A-1
6970C(J,B)=0:NEXT
6980ENDPROC
8000DEFPROCPRINT
8010PRINT"===="
8020FOR J=1 TO M
8030FOR K=1 TO N
8040PRINT;C(J,K);" ";:NEXT
8050PRINT" "
8060NEXT
8070ENDPROC
10000DATA-1/2,1/2,0,0,0,2/3,1/3,0,0
10010DATA-1/3,0,1/3,0,0,-1/4,1/2,1/4,0
10020DATA-1/4,0,0,1/4,0,-1/5,0,1/3,1/5
10030DATA-1/5,0,0,0,1/5,-1/6,0,0,1/4
10040DATA0,-1/4,1/4,0,0,-1/3,2/15,1/5,0
10050DATA0,-1/5,0,1/5,0,-1/4,-1/6,1/4,1/6
10060DATA0,-1/6,0,0,1/6,-1/5,-1/7,0,1/5
10070DATA0,0,-1/6,1/6,0,0,-1/5,2/35,1/7
10080DATA0,0,-1/7,0,1/7,0,-1/6,-1/8,1/6
10090DATA0,0,0,-1/8,1/8,0,0,-1/7,2/63
```

Example ~~2 6 22~~

2	1	7	3	37
3	2	12	1	30
5	3	19	1	17

$M=5$ $N=3$.

A=1. B=1. 110 P=A. $C(P,B)=C(1,1)=2 \neq 0$
so GOTO 2000
$P \neq A$ so SORT not needed.
$B=1<N$
PROCDIV. Divide row 1 by $C(1,1)$, i.e 2.
New row: 1 $\frac{1}{2}$ $3\frac{1}{2}$ $1\frac{1}{2}$ $18\frac{1}{2}$
(PRINT is simply check)
2030. $A=1<M$ so PROCBELOW.
From rows 2 and 3 we subtract a multiple of the first row that will give 0 in first column.

	1	$\frac{1}{2}$	$3\frac{1}{2}$	$1\frac{1}{2}$	$18\frac{1}{2}$
($-3\times$row 1)	0	$\frac{1}{2}$	$1\frac{1}{2}$	$-3\frac{1}{2}$	$-25\frac{1}{2}$
(-5 row 1)	0	$\frac{1}{2}$	$1\frac{1}{2}$	$-6\frac{1}{2}$	$-75\frac{1}{2}$

2050 A=1 so PROCABOVE not needed.
2080 $A=1<M$, $B=1<N$ so $A\to A+1$, $B\to B+1$. 2,2
GOTO 110. P=A. $C(P,B)=C(2,2)=\frac{1}{2}\neq 0$
GOTO 2000. $P\neq A$. No SORT needed.
$B=2<N$ so PROCDIV Divide row 2 by $C(2,2)$, i.e $\frac{1}{2}$
Row 2 becomes 0 1 3 −7 −51
$A=2<M$ so PROCBELOW Subtract from row 3 a multiple of row 2 that gives 0 at beginning. i.e subtract $\frac{1}{2}$ row 2
New row 3. 0 0 0 −3 −50

played a decisive role. In analysis we are much concerned with questions about limits. For both real and complex numbers, a sequence of numbers z_n is tending to a limit L if the distance of z_n from L is tending to zero. He decided that, if it was possible to find a satisfactory definition of distance between two mathematical objects (of any kind), it would be possible to find theorems about these objects analogous to the theorems about real and complex numbers.

The first question then is - what is a satisfactory definition of distance? He looked at the traditional proofs and found the only properties of distance used were the following very simple ones;-

1. Distance is measured by a real number, which is never negative.
2. A distance is zero if, and only if, it is the distance between a point and itself.
3. The distance from A to B is the same as the distance from B to A.
4. You cannot shorten your journey by breaking it. If you go from A to C, and then from C to B, the total distance cannot be less than the distance from A to B. (It may of course be equal, if C lies on the direct route from A to B.) This is known as the triangle axiom. It corresponds to Euclid's remark, that the sum of the lengths of two sides of a triangle must exceed the length of the third side.

Frechet's investigation was extraordinarily fruitful. It was found possible to find a satisfactory definition for the distance between two matrices, two transformations, two functions, two operations that may involve differentiation and integration. At one blow, this opens the door to a whole series of results concerning the most varied situations.

It is often possible to find more than one definition of distance for given objects. For instance, on a chessboard we can define distance as the minimum number of moves a king needs to get from one square to another. We get a different definition if we consider a rook instead of a king. Options can equally well arise in more serious mathematical contexts.

The distance from A to B is the length of AB. We now consider defining length. Of the many possible definitions we shall here consider only those that lead to our usual geometry, or to a geometry very similar to it. In all the spaces now to be listed, Pythagoras Theorem is true in some sense. All these spaces are vector spaces.

The symbol $||\mathbf{u}||$ will be used for the length of the vector **u**. As **v**-**u** is the vector that goes from the point **u** to the point **v**, $||\mathbf{v}-\mathbf{u}||$ gives the distance of the point **u** from the point **v**.

LENGTH.

1. Euclidean space of 2 dimensions.

If $\mathbf{u} = (u_1, u_2)$, we define length by

$$||\mathbf{u}||^2 = u_1^{\,2} + u_2^{\,2}.$$

With the help of Pythagoras Theorem. we can define "perpendicular" . We shall say **v** is perpendicular to **u**, if the points **0**, **u**, **v** form a right-angled triangle with a right-angle at the origin, **0**. Whether it is right-angled or not we test by using Pythagoras. The length of the hypotenuse is the distance from **u** to **v**, that is, $||\mathbf{v}-\mathbf{u}||$. As $\mathbf{v}-\mathbf{u}$ is (v_1-u_1, v_2-u_2), c, the length of the hypotenuse is given by

$$c^2 = (v_1-u_1)^2 + (v_2-u_2)^2 \quad \ldots(1)$$

The sum of the squares on the other two sides is

$$||\mathbf{u}||^2+||\mathbf{v}||^2 = (u_1^2+u_2^2) + (v_1^2+v_2^2) \quad \ldots(2)$$

The condition for a right-angle is that the expressions in (1) and (2) are equal. On writing the equation, we find there is cancelling of all squared terms. Dividing what remains by -2 we reach the equation

$$u_1v_1+u_2v_2 = 0.$$

We recognize the expression here as the dot product, **u.v**. Accordingly, in each geometry below we shall define **u.v** as the expression that turns up in this way. Thus, in each geometry, **u.v** = 0 will be the condition for **u** being perpendicular to **v**.

2. Euclidean space of 3 dimensions.

We follow essentially the same lines as for 2 dimensions. We define length by

$$||\mathbf{u}||^2 = u_1^2 + u_2^2 + u_3^2.$$

The square on the hypotenuse will be

$$c^2 = (v_1-u_1)^2 + (v_2-u_2)^2 + (v_3-u_3)^2$$

The sum of the squares on the other two sides is

$$(u_1^2 + u_2^2 + u_3^2) + (v_1^2 + v_2^2 + v_3^2) .$$

Equating these, cancelling the squares and dividing by -2 we obtain the definition of **u.v** as

$$\mathbf{u.v} = u_1v_1 + u_2v_2 + u_3v_3 .$$

3. Euclidean space of n dimensions.

The argument follows exactly the same lines as for 3 dimensions. the only difference is that, instead of letting the numbers run through 1,2,3, we let them run through 1,2,3,... up to n. At the end we arrive at the definition

$$\mathbf{u.v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n.$$

4. Function space.

Suppose we have a continuous function, defined on the interval [p,q]. We could get a good idea of its nature by dividing the interval into a large number, n, of parts, taking x_1, x_2, ... x_n as the midpoints of each part and looking at the values $f(x_1)$, $f(x_2)$, ... $f(x_n)$. These would specify a vector in n dimensions; the square of its length would be

G6

The usefulness of perpendicularity.

It is often found in attacking a problem that the axes of co-ordinates in which we have started are not the best for further work, and it becomes necessary to go over to some other system. If **u**, **v**, **w** are to be unit vectors in the directions of the new axes, the point with co-ordinates X,Y,Z for the new system will be **Xu+Yv+Zw**. By equating this to (x,y,z), the specification of the point in the old system, we obtain 3 equations by solving which we can find the new co-ordinates X,Y,Z. In 3 dimensions this may not be too bad; the corresponding problem in 10 dimensions, for instance, could be rather trying.

However, if the vectors along the axes are perpendicular in both the old and the new system of axes, a much simpler method is available.

Suppose, for example, that in 3 dimensions the new axes are to have the vectors **u**,**v**,**w** where **u** = (1,1,1), **v** =(0,1,-1), and **w** = (-2,1,1). It is easily verified from the dot products that these vectors are perpendicular to each other. Let the vector **s** be (x,y,z) in the old system. In the new system we are to have

$$\mathbf{s} = X\mathbf{u} + Y\mathbf{v} + Z\mathbf{w} .$$

Take the dot product of this equation with **u**.

On the right-hand side **u.v** and **u.w** are zero, so the result is simply

$$\mathbf{s.u} = X\,(\mathbf{u.u}) \qquad .....(3)$$

The dot products are easy to work out; **s.u** = x+y+z and **u.u**=3. We thus find x+y+z =3X, which gives X immediately. In the same way, by taking dot products with **v** and **w** we can single out Y and Z.

Fourier Series.

For many problems it is important to expand a function in a series consisting of sines. We may be given a function in the interval $(0,\pi)$ and wish to find the constants c_1, c_2, c_3, ... in the series

$$f(x) = c_1 \sin x + c_2 \sin 2x + c_3 \sin 3x + ... (4)$$

A device which has been known since about 1750 for finding these constants is the following. (There are certain logical difficulties in this procedure which will not be discussed here. It is not always allowable to integrate an infinite series term by term.) It was noticed that, if m and n are different whole numbers, then

$$\int_0^{\pi} \sin mx \sin nx \, dx = 0 . \qquad ...(5)$$

To find, say, c_3, we multiply equation (3) by sin 3x and integrate from 0 to π. This will wipe out all the terms except that containing c_3, and we shall have

$$\int_0^{\pi} f(x) \sin 3x \, dx = c_3 \int_0^{\pi} \sin^2 3x \, dx. . (6)$$

As soon as we have worked out the integrals we shall have the value of c_3.

Here we have singled out c_3 in much the same way that we singled out X in the 3-dimensional problem. The analogy in

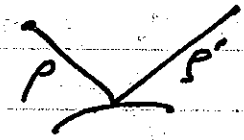
H_1

Notes on Hamilton.

Collected Works. Optics - p9.

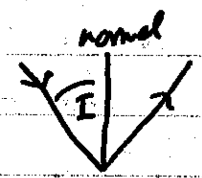
§1, §2, §3 For reflections Hamilton calls "action" the distance from one point to another.

Shows $\delta\rho + \delta\rho' = 0$



Section II deals with the problem of bringing to a focus a given set of rays. He shows this is only possible if rays are perpendicular to a system of surfaces.

§1. If unit forces acted along incident and reflected rays, their resultant would be $2\cos I$ along normal



§2 If l, m, n and $l' m' n'$ are direction cosines of incident and reflected rays and $l_N\ m_N\ n_N$ of normal

$$l + l' = 2\cos I . l_N, \quad m + m' = 2\cos I . m_N, \quad n + n' = 2\cos I . n_N$$

§3 If $\delta x, \delta y, \delta z$ is a variation in mirror

$$(l+l')\delta x + (m+m')\delta y + (n+n')\delta z = 0. \quad (C)$$

This implies $\delta\rho + \delta\rho' = 0$ (D) Principle of least Action.

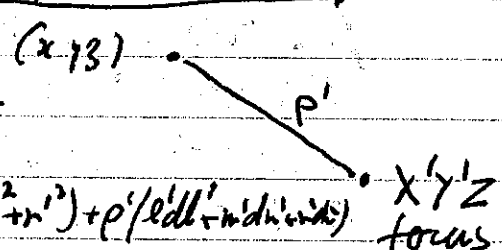
II Theory of focal mirrors

In eqn (C), (l, m, n) are functions of x, y, z determined by the nature of the incident system and $(l' m' n')$ are functions of (x, y, z) fixed when we know the focus to which the reflected rays are to be reflected. Then (C) is a differential eqn to be satisfied by the reflecting surface.

$l' dx + m' dy + n' dz$ is always an exact differential.

For $X' - x = l'\rho'$ etc.

$\therefore$ $-dx = l' d\rho' + \rho' dl'$ etc



Multiply by $l' m' n'$ and add

$$-(l'dx + m'dy + n'dz) = d\rho'(l'^2 + m'^2 + n'^2) + \rho'(l'dl' + m'dm' + n'dn')$$

$$= d\rho' \text{ since } l'^2 + m'^2 + n'^2 = 1$$

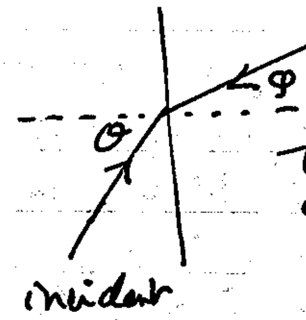
and its variation $2(l'dl' + m'dm' + n'dn') = 0$.

~~Hence, for the~~ This leads to $\operatorname{curl}(l, m, n) = 0$.

#2

Hamilton. Collected Works. Optics.
p88. XIV On ordinary systems of refracted rays.

§77 $\sin\theta = \mu \sin\varphi$



If forces of magnitude 1 and μ act in direction of incident and refracted rays, resultant is $\perp$ surface of refraction. and has magnitude $\cos\theta + \mu\cos\varphi$.

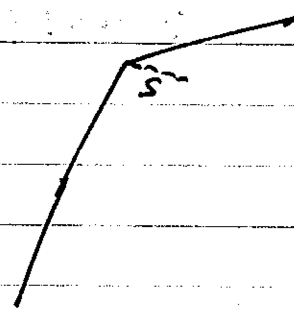
The 'direction' of the incident ray is opposite to the arrow

Writing ρ, ρ', n for directions of incident, and refracted and normal rays and L for direction of any line
$$\cos\rho L + \mu\cos\rho' L = (\cos\rho n + \mu\cos\rho' n)\cos n L. \quad (A')$$

§78. Take L in turn as OX, OY, OZ. We have
$$l + \mu l' = (\cos\rho n + \mu\cos\rho' n)\,\sigma\frac{\partial f}{\partial x}$$
$$m + \mu m' = (\cos\rho n + \mu\cos\rho' n)\,\sigma\frac{\partial f}{\partial y}$$
$$n + \mu n' = (\cos\rho n + \mu\cos\rho' n)\,\sigma(-1)$$
as normal has direction of vector $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right)$
He writes p, q for $\frac{\partial f}{\partial x}$, $\partial f/\partial y$.

By division
$$\frac{l + \mu l'}{n + \mu n'} = -p \qquad \frac{m + \mu m'}{n + \mu n'} = -q$$
(Incident ray is still reversed)

If point of incidence moves a distance s in direction of L, and $\delta\rho, \delta\rho', \delta n$ are the projections of s on directions ρ, ρ', n we have



$$\delta\rho + \mu\delta\rho' = (\cos\rho n + \mu\cos\rho' n)\,\delta n$$
If $\delta n = 0$, i.e. if point of incidence moves on surface of refraction, we have
$$\delta\rho + \mu\delta\rho' = 0$$
<u>the principle of least action</u>

#3

p 110.

§2. To derive (A) $\delta\int v\,ds = \frac{\partial v}{\partial\alpha}\delta x + \frac{\partial v}{\partial\beta}\delta y + \frac{\partial v}{\partial\gamma}\delta z$.

$$\delta\int v\,ds = \int \delta v . ds + v . \delta ds$$

$$\delta v = \frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y + \frac{\partial v}{\partial z}\delta z + \frac{\partial v}{\partial\alpha}\delta\alpha + \frac{\partial v}{\partial\beta}\delta\beta + \frac{\partial v}{\partial\gamma}\delta\gamma .$$

α, β, γ being direction cosines we have

$$\delta\alpha . ds + \alpha\,\delta . ds = \delta\,\alpha ds = \delta dx = d\,\delta x \quad \text{etc.}$$

$$\therefore \delta\int v ds = \int\left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y + \frac{\partial v}{\partial z}\right)ds + \int\left(\frac{\partial v}{\partial\alpha}d\,\delta x + \frac{\partial v}{\partial\beta}d\delta y + \frac{\partial v}{\partial\gamma}d\delta z\right)$$

For we have $ds\left(\frac{\partial v}{\partial\alpha}\delta\alpha + \frac{\partial v}{\partial\beta}\delta\beta + \frac{\partial v}{\partial\gamma}\delta\gamma\right)$ from $\delta v . ds$

and $v\,\delta ds = \delta ds\left(\alpha\frac{\partial v}{\partial\alpha} + \beta\frac{\partial v}{\partial\beta} + \gamma\frac{\partial v}{\partial\gamma}\right)$ since v is homog. degree 1.

Coefficient of $\frac{\partial v}{\partial\alpha}$ is $ds . \delta\alpha + \alpha\delta ds = \delta . \alpha ds = \delta dx$

Integrate $\int\frac{\partial v}{\partial\alpha}d\delta x$ by parts. This gives $\left[\frac{\partial v}{\partial\alpha}\delta x\right] - \left(\delta x . d . \frac{\partial v}{\partial\alpha}\right.$

We suppose lower limit of integration fixed, so no contribution to $\left[\frac{\partial v}{\partial\alpha}\delta x\right)$. We thus find

$$\delta\int v ds = \Sigma\,\frac{\partial v}{\partial\alpha}\delta x + \text{etc} + \int\delta x\left(\frac{\partial v}{\partial x}ds - d\,\frac{\partial v}{\partial x}\right) + \text{etc}$$

Principle of Least Action requires that terms with $\int$ should vanish.

$$\therefore \frac{\partial v}{\partial x}ds = d\frac{\partial v}{\partial\alpha} \quad \text{etc.}$$

Those are DEs for a ray.

Hamilton's note. If we put $\alpha = \frac{dx}{ds} = \dot{x}$ we get Lagrange form $\frac{d}{ds}\frac{\partial v}{\partial\dot{x}} = \frac{\partial v}{\partial x}$

The terms free of $\int$ give (A).

§3. Write $V = \int v\,ds$, $\frac{\partial V}{\partial x} = \frac{\partial v}{\partial\alpha}, \frac{\partial V}{\partial y} = \frac{\partial v}{\partial\beta}, \frac{\partial V}{\partial z} = \frac{\partial v}{\partial\gamma}$ (C)

V is the <u>characteristic function of the system</u>. If we are given V, as fn of x, y, z, we can find $\partial V/\partial x, \partial V/\partial y, \frac{\partial V}{\partial z}$. Eqn (C) will enable us to determine α, β, γ as fn of x, y, z. Thus <u>we can find the directions of the rays passing through any point of the system</u>

26.1.79 (4)

$$\int_{-1}^{1}\int_{-1}^{1} \frac{x}{1-\alpha xy}\left\{\ln(1+x)-\ln(1-x)\right\} dx\,dy$$ (given on 31.1.79 (+))

$$= \int_{-1}^{1}\int_{-1}^{1}\left(1+\alpha xy+\alpha^2x^2y^2+\alpha^3x^3y^3+\alpha^4x^4y^4+\dots\right)\times$$
$$2\left\{x^2+\frac{x^4}{3}+\frac{x^6}{5}+\dots\right\} dx\,dy$$

$$= \cancel{8}\left[\frac{1}{3}+\frac{1}{3.5}+\frac{1}{5.7}+\dots\right]$$
$$+\ \alpha^2.2.\frac{2}{3}2\left[\frac{1}{1.5}+\frac{1}{3.7}+\frac{1}{5.9}+\dots\right]$$
$$+\ \alpha^4.2.\frac{2}{5}.2\left[\frac{1}{1.7}+\frac{1}{3.9}+\frac{1}{5.11}+\dots\right]$$
$$+$$

Sum by means of partial fractions.

$$= \cancel{4}+\frac{8}{9}\alpha^2+\alpha^4\frac{8}{5}\frac{\left(1+\frac{1}{3}+\frac{1}{5}\right)}{6}+\alpha^6.\frac{8}{7}\frac{\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}\right)}{8}$$
$$= 4\left\{1+\frac{2}{3}\alpha^2+\frac{1}{5}.\frac{23}{45}\alpha^4+\alpha^6.\frac{1}{7}.\frac{44}{105}+\dots\right\}$$
$$= 4\mu_2.$$

So $$\iiint\left(\frac{x}{1-\alpha xy}-\frac{z}{1-\alpha yz}\right)\frac{1}{x-z}\,dx\,dy\,dz$$ (see top of 26.1.79 (3)).

$$= 2\iiint\frac{x}{(1-\alpha xy)(x-z)}\,dx\,dy\,dz = 8\mu_2.$$

A) #5

Hamilton's Equations. Collected Works Vol. II

p.107 He considers n masses, $m_1 \ldots m_n$ with co-ordinates $(x_i\ y_i\ z_i)$ $i = 1 - n$. Forces between them create a potential energy $V(x_1 y_1 z_1 \ldots x_n y_n z_n)$.

$S = \int_0^t 2T\,dt$ which he expresses in terms of x_i, y_i, z_i and energy E. He finds

$$\frac{\partial S}{\partial x_i} = m_i \dot{x}_i \quad \frac{\partial S}{\partial y_i} = m_i \dot{y}_i \quad \frac{\partial S}{\partial z_i} = m_i \dot{z}_i$$

$$\frac{\partial S}{\partial E} = t.$$

Writing $W = S - Et$, we have $\frac{\partial W}{\partial t} = -E$

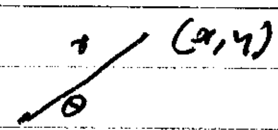
W being considered as $W(x_i\ y_i\ z_i\ t)$. (p. 210)

Example Particle in plane, force free, $V = 0$. Mass 1.

$T = \frac{1}{2}v^2 = E$

At distance r, $S = \int_0^t 2E\,dt = 2Et$. $\quad t = \frac{r}{v} = \frac{r}{\sqrt{2E}}$

$$\therefore S = \sqrt{2E(x^2+y^2)}$$



$$\frac{\partial S}{\partial x} = \sqrt{2E}\,\frac{x}{\sqrt{x^2+y^2}} = v\cos\theta$$
$$= p_x \text{ momentum } \parallel x\text{-axis}.$$

$$\frac{\partial S}{\partial y} = \sqrt{2E}\,\frac{y}{\sqrt{x^2+y^2}} = v\sin\theta = p_y.$$

$$\frac{\partial S}{\partial E} = \sqrt{2(x^2+y^2)}\cdot\frac{1}{2}E^{-1/2} = \frac{\sqrt{x^2+y^2}}{\sqrt{2E}} = \frac{r}{v} = t.$$

$$W = \sqrt{2E(x^2+y^2)} - Et$$

When $x=0, y=0, t=0$, $W=0$. At a later time we shall find this value 0 where

$$0 = r\sqrt{2E} - Et \quad \text{i.e. } r = \sqrt{\frac{E}{2}}\,t.$$

Thus the contours do travel outwards but not at the rate v of the mass.

We now consider a mass m moving in a plane with potential energy $V(x,y)$.

$$p_x = m\dot{x} \qquad p_y = m\dot{y}$$

$$\text{Energy} = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2 + V(x,y)$$

$$= \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + V$$

$$S = \int_0^t 2T\,dt = \int_0^t m(\dot{x}^2+\dot{y}^2)\,dt$$

$$= \int_0^t m\dot{x}\,dx + m\dot{y}\,dy = \int_0 p_x\,dx + p_y\,dy \quad (1)$$

which is to be expressed as $S(x, y, E)$

(2) $\dfrac{\partial S}{\partial x} = p_x \quad \dfrac{\partial S}{\partial y} = p_y$ can be proved.

The energy $H(x,y,p_x,p_y) = \dfrac{p_x^2}{2m} + \dfrac{p_y^2}{2m} + V$ (3)

Energy is conserved

$$\therefore \quad \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + V(x,y) = E \quad (4)$$

during motion.

As $p_x = \dfrac{\partial S}{\partial x}$ and $p_y = \dfrac{\partial S}{\partial y}$

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 = 2m(E-V)$$

Taking $m=1$ as before $\left(\dfrac{\partial S}{\partial x}\right)^2 + \left(\dfrac{\partial S}{\partial y}\right)^2 = 2(E-V)$ (5)

As $\dfrac{\partial S}{\partial x} = p_x = \dot{x} \quad \dfrac{\partial S}{\partial y} = p_y = \dot{y}$

the mass moves $\perp$ S = constant at all times.

$\left(\dfrac{\partial S}{\partial x}\right)^2 + \left(\dfrac{\partial S}{\partial y}\right)^2 = 2(E-V)$ shows that

$\text{grad } S = \sqrt{2(E-V)}$. H. Hertz gave a way to construct the contours for S graphically.

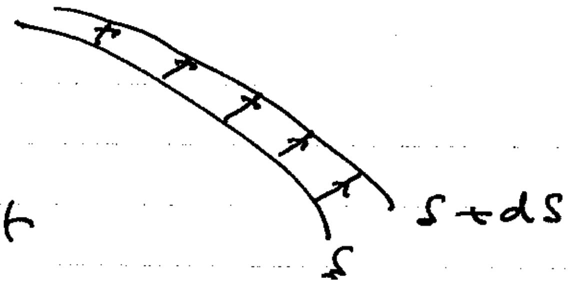
If we have a contour for S, and move a distance ds along normal, S changes by

$$ds\sqrt{2(E-V)} = dS \quad (6)$$

A3

H7

Now the quantity that plays the role of phase for the wave is



$$W = \int_0^t (T - V)\, dt$$

$$= \int_0^t [T - (E - T)]\, dt = \int_0^t (2T - E)\, dt$$

$$= \int_0^t 2T\, dt - Et = S - Et. \qquad (7)$$

Thus the value of W on a fixed contour of S decreases with time. If we want to see a constant W, we must move from one S = constant to another, as time passes.

In a time dt, if the surface with fixed W moves from S = constant to $S + dS$ = constant we must have $dS = E\,dt$. Thus, by (6)

$$ds \sqrt{2(E-V)} = E\,dt$$

$$u = \frac{ds}{dt} = \frac{E}{\sqrt{2(E-V)}} \qquad (8)$$

gives the rate at which the wave surface moves.

Hamilton. Collected Works Vol II. #8

On a General Method in Dynamics. April 1834.

p107-8 Masses $m_1 \ldots m_n$, co-ords $(x_i\ y_i\ z_i)$ $i = 1..n$. Potential V.

$m_i\ddot{x}_i = -\partial V/\partial x_i$ $m_i\ddot{y}_i = -\partial V/\partial y_i$ $m_i\ddot{z}_i = -\partial V/\partial z_i$

$H = T + V$ Change to neighbouring orbit, with different initial conditions and different energy.

$\delta T = \delta H - \delta V$ Apply $\int \ldots dt$

$$T = \sum \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$\delta T = \sum m\dot{x}\,\delta\dot{x} + m\dot{y}\,\delta\dot{y} + m\dot{z}\,\delta\dot{z}$$

$$\delta T.dt = \sum m\,dx\,\delta\dot{x} + m\,dy\,\delta\dot{y} + m\,dz\,\delta\dot{z} \quad \text{so}$$

so $\int \Sigma(m\,dx\,\delta\dot{x} + m\,dy\,\delta\dot{y} + m\,dz\,\delta\dot{z})$ gives $\int \delta T\,dt$

$$-\delta V = \sum m(\ddot{x}\,\delta x + \ddot{y}\,\delta y + \ddot{z}\,\delta z)$$

$$-\delta V dt = \sum m(d\dot{x}\,\delta x + d\dot{y}\,\delta y + d\dot{z}\,\delta z)$$

So

$$\int \Sigma(m\,dx\,\delta\dot{x} + m\,dy\,\delta\dot{y} + m\,dz\,\delta\dot{z}) = \int m(d\dot{x}\,\delta x + d\dot{y}\,\delta y + d\dot{z}\,\delta z) + \int \delta H\,dt$$

If $S = \int_0^t 2T\,dt = \int m\dot{x}^2 dt + \text{etc} = \int m\dot{x}\,dx + \text{etc}.$

$$\delta\int m\dot{x}\,dx = \int m\,\delta\dot{x}\,dx + \int m\dot{x}\,\delta dx = \int m\,\delta\dot{x}\,dx + \int m\dot{x}\,d\delta x$$

$$= \int m\,\delta\dot{x}\,dx + [m\dot{x}\,\delta x] - \int m\,d\dot{x}\,\delta x$$

Thus $\delta S = \int \sum m\,dx\,\delta\dot{x} + \text{etc} - \int \sum m\,d\dot{x}\,\delta x + \text{etc} + \sum m\dot{x}\,\delta x + \text{etc} - \sum m\dot{a}\,\delta a + \ldots$

where $a_i\ b_i\ c_i$ are initial values of $x_i\ y_i\ z_i$

$$\therefore \delta S = \sum m(\dot{x}\,\delta x + \dot{y}\,\delta y + \dot{z}\,\delta z) - \sum m(\dot{a}\,\delta a + \dot{b}\,\delta b + \dot{c}\,\delta c) + t\,\delta H.$$

$$\therefore \frac{\partial S}{\partial x_i} = m\dot{x}_i \quad \frac{\partial S}{\partial a_i} = -m\dot{a}_i \quad \frac{\partial S}{\partial H} = t.$$

#9

Contact groups.

For wave propagation, $F(x,y,z,x',y',z') = t'-t$ (1) gives time for disturbance at xyz at time t to affect $x'y'z'$ at time t'.

Then $dF = \xi' dx' + \eta' dy' + \zeta' dz' - \xi dx - \eta dy - \zeta dz$. (6)

For infinitesimal transformation

$$\frac{dx}{dt} = \frac{\partial H}{\partial \xi} \quad \frac{d\xi}{dt} = -\frac{\partial H}{\partial x} \quad \text{etc.} \qquad (9)$$

In S^n $(q_1 \dots q_n\ p_1 \dots p_n)$, and $(Q_1 \dots Q_n\ P_1 \dots P_n)$ defined by $2n$ equations involving first set of variables

Contact transformation is when $\sum P_i dQ_i - \sum p_i dq_i$, when expressed in terms of p, q and differentials, is a perfect differential.

Obviously 2 successive contact transformations give such a transformation: so also inverse. ∴ form group.

p. 294 If $\sum (P_r dQ_r - p_r dq_r) = dW$

From the eqns for P, Q it may be possible to eliminate P, p completely so as to obtain one or more relations between Q, q. Suppose there are k such relations,

$\Omega_r(q, Q) = 0 \quad r = 1 \dots k$ ..... (A)

Since dq, dQ are conditioned only by the relations (A) we must have

$$P_r = \frac{\partial W}{\partial Q_r} + \sum_1^k \lambda_i \frac{\partial \Omega_i}{\partial Q_r} \quad p_r = -\frac{\partial W}{\partial q_r} - \sum_1^k \lambda_i \frac{\partial \Omega_i}{\partial q_r}. \quad (B)$$

(A) and (B) give $2n+k$ eqns for $Q, P, \lambda_1 \dots \lambda_k$.

Conversely, if $W, \Omega_1 \dots \Omega_k$ are $k+1$ fns that satisfy ~~B~~ (A) and (B), then $pq \to PQ$ is contact transformation for these eqns show that $\sum (P_r dQ_r - p_r dq_r) = dW$.

p 303. §133 The most general infinitesimal transformation is given by $Q_r = q_r + \frac{\partial K}{\partial p_r} \Delta t$, $P_r = p_r - \frac{\partial K}{\partial q_r} \Delta t$.

p 315

The bilinear covariant in detail.

We have $P\,dx + Q\,dy$.

$$a_{11} = \frac{\partial P}{\partial x} - \frac{\partial P}{\partial x} = 0 \qquad a_{22} = 0.$$

$$a_{12} = \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \qquad a_{21} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$a_{ij}\,dx^i \delta x^j = a_{12}(dx\,\delta y) + a_{21}\,dy\,\delta x$$
$$= \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)(dx\,\delta y - dy\,\delta x)$$

Suppose now we change to polars $x = r c\theta$, $y = r s\theta$

$dx = dr\cos\theta - r\sin\theta\,d\theta$, $dy = dr\,s\theta + r c\theta\,d\theta$

$\delta x = \delta r\cos\theta - r\sin\theta\,\delta\theta$ $\quad \delta y = \delta r\,s\theta + r c\theta\,\delta\theta$

$$dx\,\delta y - dy\,\delta x = \begin{cases} dr\,\delta r\sin\theta c\theta - r\sin^2\theta\,d\theta\,\delta r + r\cos^2\theta\,dr\,\delta\theta - r^2\sin\theta\cos\theta\,d\theta\,\delta\theta \\ -[dr\,\delta r\sin\theta c\theta + r\cos^2\theta\,d\theta\,\delta r - r\sin^2\theta\,dr\,\delta\theta - r^2\cos\theta\sin\theta\,d\theta\,\delta\theta] \end{cases}$$

$$= -r\,d\theta\,\delta r + r\,dr\,\delta\theta = r(dr\,\delta\theta - d\theta\,\delta r).$$

$$P\,dx + Q\,dy = P(dr\cos\theta - r\sin\theta\,d\theta) + Q(dr\sin\theta + r c\theta\,d\theta)$$
$$= (P\cos\theta + Q\sin\theta)\,dr + (-Pr\sin\theta + Qr c\theta)\,d\theta$$

$\therefore\ \bar{P} = P\cos\theta + Q\sin\theta \qquad \bar{Q} = -Pr\sin\theta + Qr\cos\theta$

$$b_{12} = \frac{\partial \bar{P}}{\partial\theta} - \frac{\partial \bar{Q}}{\partial r} = \left[-P\sin\theta + \frac{\partial P}{\partial\theta}\cos\theta + Q c\theta + \frac{\partial Q}{\partial\theta}\sin\theta\right]$$
$$- \left[-P\sin\theta - \frac{\partial P}{\partial r} r\sin\theta + Q c\theta + \frac{\partial Q}{\partial r} r c\theta\right]$$
$$= \frac{\partial P}{\partial\theta}\cos\theta + \frac{\partial Q}{\partial\theta}\sin\theta + \frac{\partial P}{\partial r} r\sin\theta - \frac{\partial Q}{\partial r} r\cos\theta$$

$$a_{12} = \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial P}{\partial\theta}\frac{\partial\theta}{\partial y} - \frac{\partial Q}{\partial r}\frac{\partial r}{\partial x} - \frac{\partial Q}{\partial\theta}\frac{\partial\theta}{\partial x}$$
$$= \frac{\partial P}{\partial r}\sin\theta + \frac{\partial P}{\partial\theta}\frac{\cos\theta}{r} - \frac{\partial Q}{\partial r}\cdot\cos\theta + \frac{\partial Q}{\partial\theta}\frac{\sin\theta}{r}$$

So $b_{12} = r a_{12}$, and $a_{ij}\,dx^i \delta x^j$ transforms as expected.

§ 128 The bilinear covariant depends linearly on the diff. form.
The bilinear covariant of an exact differential is 0.
The bilinear covariant of a form, operated upon by any transformation, goes to the bilinear covariant of the transformed form.

(Q, P) is obtained from (q, p) by a contact transformation if $\sum P_r dQ_r - \sum q_r dp_r$ is exact diff!

~~Hence bilinear covariant of $\sum P_r d$~~

$$\sum P_r dQ_r = \sum q_r dp_r + dF$$

Bilinear covariant of $\sum P_r dQ_r$ = bilinear covariant of $\sum q_r dp_r + dF$ = bilinear covariant of $\sum q_r dp_r$.

$\therefore \sum_{r=1}^{n} (\delta p_r dq_r - dp_r \delta q_r)$ is invariant under the transformation.

p294, on analytic expression for a contact transformation, discusses the possibility of 1, 2 or 3 relations of the form $\Omega(q_1 \ldots q_r, Q_1 \ldots Q_r)$. It does not make very clear that there may be none, but I think this is so. To get a particular example, I consider a mass moving in 2-D, under no force.

$$H = \frac{1}{2m}(p_1^2 + p_2^2) = \frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right]$$

so
$$0 = \frac{\partial W}{\partial t} + H\left(x\, y\, \frac{\partial W}{\partial x}\, \frac{\partial W}{\partial y}\right) = \frac{\partial W}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right]$$

Let $\lambda = \partial W/\partial x$, $\mu = \partial W/\partial y$ constants.

Then $W = -\frac{1}{2m}(\lambda^2 + \mu^2)t + \lambda x + \mu y$

say
$$W = -\frac{1}{2m}(\alpha_1^2 + \alpha_2^2)t + \alpha_1 x + \alpha_2 y.$$

p 315.
$$\beta_1 = -\frac{\partial W}{\partial \alpha_1} = -x + \frac{\alpha_1 t}{m} \qquad \beta_2 = -\frac{\partial W}{\partial \alpha_2} = -y + \frac{\alpha_2 t}{m}$$

$$p_1 = \frac{\partial W}{\partial x} = \alpha_1 \qquad p_2 = \frac{\partial W}{\partial y} = \alpha_2.$$

$Q_1 = x$ $\quad Q_2 = y$ $\quad q_1 = x_0 = -\beta_1$ $\quad q_2 = y_0 = -\beta_2$.
$P_1 = p_1$ $\quad P_2 = p_2$

There is no way to eliminate $P_1\, P_2\, p_1\, p_2$ and get an equation connecting $Q_1\, Q_2\, q_1\, q_2$.

In fact $Q_1 - q_1 = p_1 t/m$, $Q_2 - q_2 = p_2 t/m$.

H13

§138. Let contact transformation be defined by

$$P_r = \frac{\partial W}{\partial Q_r} \qquad p_r = -\frac{\partial W}{\partial q_r}$$

where W is any function of $q_1 \ldots q_n \, Q_1 \ldots Q_n$.

Then $$\sum p_r dq_r = \sum P_r dQ_r - \sum_1^n \left(\frac{\partial W}{\partial q_r} dq_r + \frac{\partial W}{\partial Q_r} dQ_r \right)$$
$$= \sum P_r dQ_r + \frac{\partial W}{\partial t} dt - dW$$

where dW represents change in W when $q_1 \ldots q_n Q_1 \ldots Q_n t$ become $q_1 + dq_1 \ldots q_n + dq_n \, Q_1 + dQ_1 \ldots Q_n + dQ_n \, t + dt$.

$$\therefore \sum p_r dq_r - H dt = \sum P_r dQ_r - \left[H - \frac{\partial W}{\partial t} \right] dt - dW$$

We are going to form first Pfaff's system, so dW can be neglected, so contact transformation changes H to $K = H - \frac{\partial W}{\partial t}$. The eqns of motion (via Pfaff's 1st system) are invariantively connected so

$$\dot{q}_r = \frac{\partial H}{\partial p_r}, \; \dot{p}_r = -\frac{\partial H}{\partial q_r} \text{ transforms to}$$

$$\dot{Q}_r = \frac{\partial K}{\partial P_r} \quad \dot{P}_r = -\frac{\partial K}{\partial Q_r}.$$

where $K = H - \frac{\partial W}{\partial t}$ $\quad K(Q_1 \ldots Q_n P_1 \ldots P_n t)$

p315 We change to

$P_r = -\partial W / \partial Q_r \quad p_r = \partial W / \partial q_r$ so $K = H + \frac{\partial W}{\partial t}$.

K will be 0 if $\frac{\partial W}{\partial t} + H = 0$. If we can find $W(q_1 \ldots q_n \, \alpha_1 \ldots \alpha_n, t)$ with n genuinely independent parameters $\alpha_1 \ldots \alpha_n$, ~~this will make~~ and make contact transformation from $(q_1 - q_n \, p_1 - p_n)$ to $(\alpha_1 - \alpha_n \, \beta_1 - \beta_n)$ then $\dot{\alpha}_r = 0 \;\; \dot{\beta}_r = 0$.

and $\beta_r = -\frac{\partial W}{\partial \alpha_r}$, $p_r = \frac{\partial W}{\partial q_r}$ gives soln of the dynamical problem.

§157. Let diff. form be $X\,dx + Y\,dy + T\,dt$

Bilinear covariant

$$a_{12} = \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x} \qquad a_{13} = \frac{\partial X}{\partial t} - \frac{\partial T}{\partial x}$$

$$a_{23} = \frac{\partial Y}{\partial t} - \frac{\partial T}{\partial y}. \qquad a_{ij} = -a_{ji}$$

Pfaff's first system

~~$a_{11}dx_2 + a_{12}d$~~ $a_{11}dx + a_{21}dy + a_{31}dt = 0$

$$a_{12}dx + a_{22}dy + a_{32}dt = 0$$

$$a_{13}dx + a_{23}dy + a_{33}dt = 0$$

$$-\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right)dy + \left(\frac{\partial T}{\partial x} - \frac{\partial X}{\partial t}\right)dt = 0$$

$$\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right)dx + \left(\frac{\partial T}{\partial y} - \frac{\partial Y}{\partial t}\right)dt = 0$$

$$\left(\frac{\partial X}{\partial t} - \frac{\partial T}{\partial x}\right)dx + \left(\frac{\partial Y}{\partial t} - \frac{\partial T}{\partial y}\right)dy = 0$$

Consider $y\,dx - \frac{1}{2}y^2\,dt$ as diff form.

$X = y \quad Y = 0 \quad T = -\frac{1}{2}y^2 - \frac{1}{2}x^2$

$-dy \; \cancel{+} -x\,dt = 0 \qquad \frac{dy}{dt} = -x$

$dx - y\,dt = 0 \qquad \frac{dx}{dt} = y.$

$x\,dx + y\,dy = 0.$

Report on Hilbert matrix problem

A problem I have been working on has led to some procedures that may be of interest to you.

The integral equation ~~λf~~ $\lambda(\sigma)f(x,\sigma) = \int_0^1 \frac{f(y,\sigma)\,dy}{1-\sigma xy}$

resembles a matrix equation $Mv = \lambda v$. The solution $f(x,\sigma)$, for eigenvalue $\lambda(\sigma)$ corresponds to the ~~eigenvector~~ v for eigenvalue λ.
σ is a parameter $0 < \sigma < 1$.

<u>Moment function</u>

$w(t)$ is defined for $0 \le t \le 1$.
$\mu_n = \int_0^1 t^n w(t)\,dt$ is called the n^{th} moment.

For example, if $w(t) \equiv 1$, the n^{th} moment is
$\mu_n = \int_0^1 t^n\,dt = 1/(n+1) \quad n = 0, 1, 2 \ldots$.

There is a simple test for moments corresponding to the interval $[0,1]$. In the example above, $\mu_n = 1/(n+1)$, the difference table is as follows

$$
\begin{array}{ccccccccccc}
1 & & 1/2 & & 1/3 & & 1/4 & & 1/5 & & 1/6 \\
& -1/2 & & -1/6 & & -1/12 & & -1/20 & & -1/30 & \\
& & 1/3 & & 1/12 & & 1/30 & & 1/60 & & \\
& & & -1/4 & & -1/20 & & -1/60 & & & \\
& & & & 1/5 & & 1/30 & & & & \\
& & & & & -1/6 & & & & &
\end{array}
$$

These rows are alternately positive and negative. This happens always for moments corresponding to the interval $[0,1]$.

I have computer evidence which suggests that the largest eigenvalue $\lambda(\sigma) = c_0 + c_1\sigma + c_2\sigma^2 + c_3\sigma^3 + \ldots$ has coefficients $c_0, c_1, c_2, c_3 \ldots$ that are the moments $\mu_0, \mu_1, \mu_2, \mu_3 \ldots$ for some weight function.

If this is so

$$\lambda(\sigma) = \int_0^1 w(t)\,dt + \sigma\int_0^1 t\,w(t)\,dt + \sigma^2\int_0^1 t^2 w(t)\,dt + \dots$$

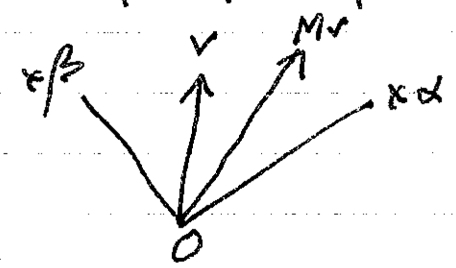
$$= \int_0^1 (1 + \sigma t + \sigma^2 t^2 + \dots)\, w(t)\,dt$$

$$= \int_0^1 \frac{w(t)\,dt}{1-\sigma t} \quad \text{for some } w(t).$$

Solution by iteration.

Consider a matrix in 2 dimensions with eigenvalues α, β, $\alpha > \beta > 0$. We may take the case of a symmetric matrix, in which case the eigenvectors will be perpendicular. Take them as axes. If $v = \begin{pmatrix} x \\ y \end{pmatrix}$ $Mv = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ \beta y \end{pmatrix}$.

As $\alpha > \beta$, Mv will be pulled in the direction of the eigenvector with $\lambda = \alpha$; provided v is not in the direction of eigenvector for $\lambda = \beta$.



If we repeatedly apply the operation M, the vector will be repeatedly pulled in the direction of α-eigenvector, so, for large n, $M^n v$ should be very nearly in the direction of the eigenvector for the larger λ.

This argument holds equally well in any number of dimensions. We can start with any vector that is not perpendicular to the eigenvector with maximum λ.

A similar procedure is possible with integral equations. If the operation M is defined by $g = Mf$ where $g(x,\sigma) = \int_0^1 \frac{1}{1-\sigma xy} f(y,\sigma)\,dy$ iteration should indicate the nature of the solution for the largest $\lambda(\sigma)$.

Let $\varphi_0(x,\sigma) \equiv 1$ and $\varphi_{n+1}(x,\sigma) = M\varphi_n(x,\sigma)$.
We find, for example.

$$\varphi_0(x,\sigma) = 1$$
$$\varphi_1(x,\sigma) = 1 + \tfrac{1}{2}\sigma x + \tfrac{1}{3}\sigma^2 x^2 + \tfrac{1}{4}\sigma^3 x^3 + \dots.$$

$$\varphi_2(x,\sigma) = 1 + \frac{\sigma}{2}\left(x + \frac{1}{2}\right) + \frac{\sigma^2}{3}\left(x^2 + \frac{x}{2} + \frac{1}{3}\right) + \dots.$$

It would help towards expressing $\lambda(\sigma)$ in the form $\int_0^1 \frac{1}{1-\sigma t} w(t)\,dt$ if we could express $\varphi_n(x,\sigma)$ as $\int_0^1 \frac{1}{1-\sigma t} W(x,t)\,dt$ for some function $W(x,t)$.

<u>Note that σ must not appear in W.</u>

If we take $W(x,t) = \begin{cases} \frac{1}{x} & \text{for } 0 \le t \le x \\ 0 & x < t \le 1 \end{cases}$

we find,
$$\int_0^1 \frac{1}{1-\sigma t} W(x,t)\,dt = \int_0^x \frac{1}{1-\sigma t}\cdot\frac{1}{x}\,dt$$
$$= \frac{1}{-\sigma x}\Big[\ln(1-\sigma t)\Big]_0^x = -\frac{1}{\sigma x}\ln(1-\sigma x)$$
$$= -\frac{1}{\sigma x}\left[-\sigma x - \frac{\sigma^2 x^2}{2} - \frac{\sigma^3 x^3}{3} - \dots \right]$$
$$= 1 + \frac{\sigma x}{2} + \frac{\sigma^2 x^2}{3} + \dots = \varphi_1(x,\sigma).$$

We now try to ~~do~~ get a similar result for

$$\varphi_2(x,\sigma) = \int_0^1 \frac{1}{1-\sigma x y}\varphi_1(y,\sigma)\,dy$$
$$= \int_{y=0}^1 \frac{dy}{1-\sigma x y}\int_{t=0}^y \frac{dt}{(1-\sigma t)\,y}$$
$$= \int_{y=0}^1 \int_{t=0}^y \frac{dy\,dt}{(1-\sigma x y)(1-\sigma t)\,y} \qquad *$$

$$\frac{1}{(1-\sigma xy)(1-\sigma t)} = \frac{\left(\frac{xy}{xy-t}\right)}{1-\sigma xy} + \frac{\left(\frac{t}{t-xy}\right)}{1-\sigma t} \qquad **$$

First of these leads to

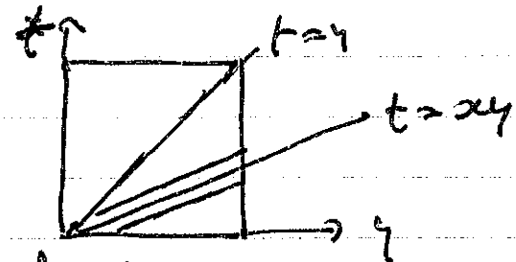
$$I_1 = \int_{y=0}^{1}\int_{t=0}^{y} \frac{x\cancel{y}\,dy\,dt}{(1-\sigma xy)(xy-t)\cancel{y}}$$

Let $v = xy \qquad dv = x\,dy$

$$I_1 = \int_{v=0}^{x}\int_{t=0}^{v/x} \frac{dv\,dt}{(1-\sigma v)(v-t)}$$

$$= \int_{v=0}^{x} \frac{dv}{1-\sigma v}\int_{t=0}^{v/x} \frac{dt}{v-t}$$

Now $t=v$ comes between $t=0$ and $t=\frac{v}{x}$, so we have to deal with an infinity. Look at the double integral * on p. 3.
The integrand is continuous. The value of the integral will be the limit of the value obtained when we exclude a small region around $t=xy$, that is, if we replace $\int_{t=0}^{y}$ by $\int_{t=0}^{xy-\varepsilon} + \int_{t=xy+\varepsilon}^{y}$



We suppose this done in both parts of **.

~~This gives I_1~~ In I_1 we now have

$$\int_{t=0}^{v-\varepsilon} \frac{dt}{v-t} - \int_{v+\varepsilon}^{v/x} \frac{dt}{t-v}$$

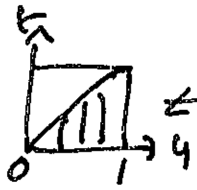
$$= \Big[-\ln(v-t)\Big]_0^{v-\varepsilon} - \Big[\ln(t-v)\Big]_{v+\varepsilon}^{v/x}$$

$$= -\ln\varepsilon + \ln v - \ln\left(\frac{v}{x}-v\right) + \ln\varepsilon$$

$$= -\ln\left(\frac{1}{x}-1\right) = \ln x - \ln(1-x).$$

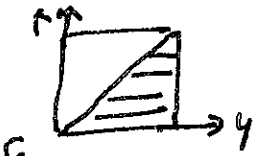
The second part of ** on p 4 leads to

$$I_2 = \int_{y=0}^{1}\int_{t=0}^{y} \frac{t\,dy\,dt}{(1-\sigma t)(t-xy)y}$$



$y = 0$ to 1, $t = 0$ to y corresponds to shaded region.

This can also be shown as



and specified as $t = 0$ to 1, $y = t$ to 1. So

$$I_2 = \int_{t=0}^{1}\frac{dt}{1-\sigma t}\int_{y=t}^{1}\frac{t\,dy}{(t-xy)\,y}$$

$$\int_t^1 \frac{t\,dy}{(t-xy)y} = \int_t^1\frac{x\,dy}{t-xy} + \int_t^1 \frac{1}{y}\,dy.$$

$\int_t^1 \frac{1}{y}\,dy = \Big[\ln y\Big]_t^1 = -\ln t$. The other integral requires separate treatment, depending on whether $t<x$ or $t>x$.

(i) $t<x$ $\quad \int_t^1 = \int_t^{\frac{t}{x}-\varepsilon} + \int_{\frac{t}{x}+\varepsilon}^1 = -\ln(t-xy)\Big]_t^{\frac{t}{x}-\varepsilon} + \Big[-\ln(xy-t)\Big]_{\frac{t}{x}+\varepsilon}^1$

$$= -\ln\varepsilon + \ln(t-xt) - \ln(x-t) + \ln\varepsilon$$
$$= \ln t + \ln(1-x) - \ln(x-t)$$

Hence $\displaystyle\int_t^1 \frac{t\,dy}{(t-xy)\,y} = \ln(1-x)-\ln(x-t)$

(ii) $t>x$ $\quad t-xy>0$ throughout. $\displaystyle\int_t^1 \frac{x\,dy}{t-xy} = -\ln(t-xy)\Big]_t^1$

$$= -\ln(t-x)+\ln t+\ln(1-x)$$

In this case $\displaystyle\int_t^1\frac{t\,dy}{(t-xy)\,y} = \ln(1-x)-\ln(t-x)$.

Thus

$$I_2 = \int_{t=0}^{x} \frac{dt}{1-\sigma t}\{\ln(1-x)-\ln(x-t)\} + \int_{t=x}^{1} \frac{\{\ln(1-x)-\ln(t-x)\}\,dt}{1-\sigma t}$$

$$I_1 = \int_{t=0}^{x} \frac{dt}{1-\sigma t}\{\ln x - \ln(1-x)\}$$

$$I_1+I_2 = \int_{t=0}^{x} \frac{dt}{1-\sigma t}\{\ln x - \ln(x-t)\} + \int_{t=x}^{1} \frac{dt}{1-\sigma t}\{\ln(1-x)-\ln(t-x)\}$$

$$= \varphi_2(x,\sigma)$$

I have verified this result. The work was rather heavy and involved the algebraic identity

$$\sum_{s=0}^{k} \binom{n+1}{s}\binom{n-s}{k-s}(-1)^{k-s} = 1.$$

A neater method is to proceed from the series for φ_2 and obtain the integrals as follows:—

$$\varphi_2(x,\sigma) = \sum_{n=0}^{\infty} \frac{\sigma^n}{n+1}\sum_{i=1}^{n+1}\frac{x^{n+1-i}}{i}$$

$$\sum_{i=1}^{n+1}\frac{x^{n-i+1}}{i} = \sum_{i=1}^{n+1} x^{n+1-i}\int_0^1 u^{i-1}\,du$$

$$= \int_0^1 \sum_{i=1}^{n+1} x^{n+1-i}u^{i-1}\,du$$

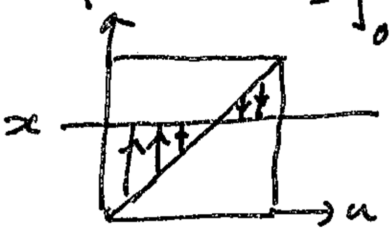
$$= \int_0^1 \frac{x^{n+1}-u^{n+1}}{x-u}\,du$$

$$\therefore\ \varphi_2(x,\sigma) = \int_0^1 \sum_0^{\infty} \frac{\sigma^n}{x-u}\left\{\frac{x^{n+1}}{n+1}-\frac{u^{n+1}}{n+1}\right\}du$$

$$= \int_0^1\frac{du}{x-u}\sum_0^{\infty} \sigma^n\int_u^x t^n\,dt$$

$$= \int_0^1 \frac{du}{x-u}\int_u^x \sum\sigma^n t^n\,dt$$

$$= \int_0^1\frac{du}{x-u}\int_u^x\frac{dt}{1-\sigma t}$$



The region of integration is shown here. Note that in upper triangle arrows point downwards. We have $-\int_x^u$.

We can specify integral as $\int_{t=0}^{x}\int_{u=0}^{t} - \int_{t=x}^{1}\int_{u=t}^{1}$

$$= \int_{t=0}^{x} \frac{dt}{1-\sigma t}\int_{u=0}^{t}\frac{du}{x-u} - \int_{t=x}^{1}\frac{dt}{1-\sigma t}\int_{u=t}^{1}\frac{du}{x-u}$$

$$= \int_{t=0}^{x}\frac{dt}{1-\sigma t}\Big[-\ln(x-u)\Big]_0^t + \int_{t=x}^{1}\frac{dt}{1-\sigma t}\Big[\ln(u-x)\Big]_t^1$$

$$= \int_{t=0}^{x} \frac{dt}{1-\sigma t}\{-\ln(x-t)+\ln x\} + \int_{t=x}^{1}\frac{dt}{1-\sigma t}\{\ln(1-x)-\ln(t-x)\}.$$

$$\frac{dy}{a_{31}} = \frac{dt}{a_{12}} = \frac{dx}{a_{23}}$$

$\frac{\partial}{\partial x}$ $\frac{\partial}{\partial y}$ $\frac{\partial}{\partial z}$ $\quad$ $\operatorname{curl} v)_1 = a_{32} = -a_{23}$

X Y Z $\quad$ $\operatorname{curl} v)_2 = a_{13} = -a_{31}$

$\operatorname{curl} v)_3 = a_{21} = -a_{12}$

Thus 1st Pfaff system here requires

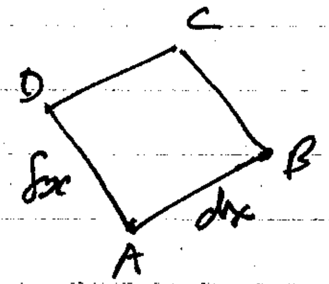
$(dx, dy, dt) = \operatorname{curl} v \cdot d\rho$.

So x, y, t moves along the lines generated by curl v.

$$d f(x_1 .. x_n) = f(x_1 + dx_1, ... x_n + dx_n) - f(x_1 .. x_n)$$

$\underline{v} . d\underline{x}$ = work done in displacement dx

$\delta(v . dx)$ = change in this due to displacement δx.



$\delta(v . dx)$

= work for DC − work for AB.

Thus $\quad a_{ij} dx^i \delta x^j$ is work done in circuit ABCDA.

If this = 0 for any δx^i this means that dx is such that any displacement δx gives a circuit with no work.

Try particular X Y Z.

H22

Try $X = y$ $Y = -x$ $Z = 0$.

$a_{12} = \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x} = 1 + 1 = 2$

$a_{23} = \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y} = 0$

$a_{31} = \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z} = 0$.

$$\begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$(dx\ \ dy\ \ dz)\, A = 0$

$-2dy = 0 \qquad 2dx = 0$.

are the condition.

i.e $du = (0, 0, dz)$.

§6 Force X, Y, Z per unit mass, from outside.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} \quad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} \quad (2)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}. \quad (3)$$

§17 Velocity potential

$$u = -\frac{\partial \varphi}{\partial x} \quad v = -\frac{\partial \varphi}{\partial y} \quad w = -\frac{\partial \varphi}{\partial z}.$$

§20 Consequently $\frac{\partial v}{\partial z} = \frac{\partial w}{\partial y}, \frac{\partial w}{\partial x} = \frac{\partial u}{\partial z}, \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

So we have

$$-\frac{\partial^2 \varphi}{\partial x \partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x} = -\frac{\partial \Omega}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

So $$\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left[\frac{\partial \varphi}{\partial t} - \Omega - \tfrac{1}{2}(u^2+v^2+w^2)\right]$$

Similarly $$\frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{\partial}{\partial y}\left[\frac{\partial \varphi}{\partial t} - \Omega - \tfrac{1}{2}(u^2+v^2+w^2)\right]$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{\partial}{\partial z}\left[\frac{\partial \varphi}{\partial t} - \Omega - \tfrac{1}{2}(u^2+v^2+w^2)\right]$$

Now ρ is constant or a function of p alone.

Suppose $\frac{1}{\rho} = F'(p)$.

$$F'(p)\frac{\partial p}{\partial x} = \frac{\partial}{\partial x} F(p)$$

So $$0 = \frac{\partial}{\partial x}\left\{ F(p) - [\quad]\right\}$$

$$0 = \frac{\partial}{\partial y}\left\{ F(p) - [\quad]\right\}$$

$$0 = \frac{\partial}{\partial z}\left\{ F(p) - [\quad]\right\}$$

$\therefore$ $F(p) - \left[\frac{\partial \varphi}{\partial t} - \Omega - \frac{1}{2}(u^2+v^2+w^2)\right]$ is a function of t alone.

§20. When there is a velocity potential, φ, and the external force per unit mass is gradient of a potential Ω the equations of motion can be integrated explicitly.

We have $u = -\frac{\partial \varphi}{\partial x}$, $v = -\frac{\partial \varphi}{\partial y}$, $w = -\frac{\partial \varphi}{\partial z}$

$$\frac{\partial u}{\partial y} = -\frac{\partial^2 \varphi}{\partial x \partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial^2 \varphi}{\partial x \partial y}$$

So
$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} : \frac{\partial v}{\partial z} = \frac{\partial w}{\partial y} : \frac{\partial w}{\partial x} = \frac{\partial u}{\partial z}$$

Eqns of motion

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}.$$

$$-\frac{\partial^2 \varphi}{\partial x \partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x} = -\frac{\partial \Omega}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left[\frac{\partial \varphi}{\partial t} - \frac{1}{2}(u^2+v^2+w^2) - \Omega\right]$$

Fortunately, we have ρ a function of p only.

Let $\frac{1}{\rho} = F'(p)$

$$LHS = F'(p)\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}F(p)$$

So
$$\left.\begin{aligned} &\frac{\partial}{\partial x}\{F(p) - [\quad]\} = 0 \\ &\frac{\partial}{\partial y}\{F(p) - [\quad]\} = 0 \\ &\frac{\partial}{\partial z}\{F(p) - [\quad]\} = 0 \end{aligned}\right\}$$

$\therefore \{\quad\}$ does not depend on x, or y, or z - $\therefore$ Function of t.

$$F(p) = \frac{\partial \varphi}{\partial t} - \frac{1}{2}(u^2+v^2+w^2) + \Psi(t)$$

Special case. Incompressible fluid ρ = constant.

$\frac{1}{\rho} = F'(p)$ $\quad F(p) = \frac{p}{\rho}$ will do.

$$\frac{p}{\rho} = \frac{\partial \varphi}{\partial t} - \frac{1}{2}(u^2+v^2+w^2) + \Psi(t)$$

§9. Physical properties of fluid.

(a) Incompressible. Then ρ is constant. $\frac{D\rho}{Dt}=0$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad \operatorname{div}(u,v,w)=0.$$

(b) Gas, isothermal. $p/p_0=\rho/\rho_0$

(c) Gas, adiabatic $p/p_0=(\rho/\rho_0)^\gamma$.

§17. Velocity potential. $\varphi(x,y,z)$

"In a large and important class of cases"

$$u=-\frac{\partial\varphi}{\partial x},\quad v=-\frac{\partial\varphi}{\partial y},\quad w=-\frac{\partial\varphi}{\partial z}$$

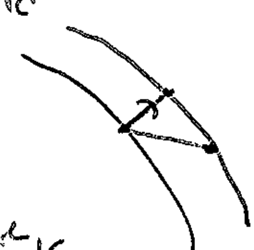
$(u,v,w)=-\operatorname{grad}\varphi$.

§19 Kinematic implication of $E\varphi$.

Stream line is a line such that its direction at any point is that of the velocity there.

For a stream line $\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$

So $\frac{dx}{dt}=ku=-k\frac{\partial\varphi}{\partial x}$ etc



$\left(\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial y},\frac{\partial\varphi}{\partial z}\right)$ is normal to $\varphi=$ constant.

So stream lines are everywhere normal to the surface $\varphi=$ constant.

The component velocity in any direction equal rate of decrease of φ.

Lamb Hydrodynamics. p 151.
§ 118 Moving axes. OX, OY, OZ fixed in body.
O has velocity $u\ v\ w$
Body has velocity of rotation p, q, r.
<u>The components are along the moving axes, OX, OY, OZ</u>

Rotation (p, q, r) produces velocity in the point (x, y, z) of $(p, q, r)\wedge(x\ y\ z) = (qz - ry, rx - pz, py - qx)$
We superpose on this vel^y of O $(u\ v\ w)$
Vel^y of point (x, y, z) ~~relative~~ (specified in the moving axes) is $u + qz - ry,\ v + rx - pz,\ w + py - qx$
The component in direction l, m, n is
$l(u + qz - ry) + m(v + rx - pz) + n(w + py - qx)$

At any point on surface of solid moving in the fluid we have condition $-\partial\phi/\partial n$ must equal normal component of velocity of that point.

So $-\dfrac{\partial\phi}{\partial n} = l(u + qz - ry) + m(v + rx - pz) + n(w + py - qx)$.
We take $\phi \equiv u\varphi_1 + v\varphi_2 + w\varphi_3 + p\chi_1 + q\chi_2 + r\chi_3$

so $-\dfrac{\partial\varphi_1}{\partial n} = l \quad -\dfrac{\partial\varphi_2}{\partial n} = m \quad -\dfrac{\partial\varphi_3}{\partial n} = n$

$-\dfrac{\partial\chi_1}{\partial n} = ny - mz, \quad -\dfrac{\partial\chi_2}{\partial n} = lz - nx, \quad -\dfrac{\partial\chi_3}{\partial n} = mx - ly \quad \Big\}(3)$

[note $\partial/\partial n$ refers to normal, not to l, m, n.]
These functions have to satisfy $\nabla^2(\ ) = 0$ and have derivatives zero at ∞, so they are determined, (§ 41.).

Boundary condition.

§10 At a fixed boundary, the component of velocity of fluid normal to boundary must be zero.

~~For moving boundary, the components normal~~ ~~to the~~ If $F(x, y, z, t) = 0$ is eqn of boundary for boundary surface, $DF/Dt = 0$ at each point of boundary.

§92 Sphere moving with constant velocity U, component of normal velocity at the surface is $Ux/r = U\cos\theta$. ϕ must satisfy

(1) $\nabla^2\varphi = 0$ everywhere.

(2) Derivatives of φ zero at infinity (the fluid is assumed at rest at infinity)

(3) $-\frac{\partial\varphi}{\partial r} = U\cos\theta$

$\nabla^2 x = 0$ so $\nabla^2 r\cos\theta = 0$ and also $\nabla^2 \frac{1}{r^2}\cos\theta = 0$

So $\varphi = -A\frac{\cos\theta}{r^2}$ is suggested.

Condition at ∞ are met.

If sphere of radius a; $\frac{\partial\varphi}{\partial r} = 2A\frac{\cos\theta}{r^3}$

$= 2A\cos\theta/a^3$ at surface.

So $-\frac{2A}{a^3} = U$ $\quad A = -\frac{1}{2}Ua^3$

~~and $\varphi = \frac{2A}{a^3}$~~

$$\phi = \frac{1}{2}U\frac{a^3}{r^2}\cos\theta.$$

Kinetic energy of fluid, T.

$$2T = -\rho\iint \phi\frac{\partial\varphi}{\partial r}dS = \frac{1}{2}\rho a U^2\int_0^\pi \cos^2\theta.\,2\pi a\sin\theta\,d\theta$$

$$= \frac{2}{3}\pi\rho a^3 U^2 = M'U^2$$

Effect of fluid is simply to increase inertia of the sphere. No force required for steady motion