Continued fraction.
An expression such as $a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}$

To express a number as a continued fraction.
Example, $\frac{23}{16}$. \[ \frac{23}{16} = 1 + \frac{7}{16} \]
\[ \frac{7}{16} = 1 + \frac{1}{2} \]
\[ \frac{1}{2} = \frac{1}{2} \]

\[ \therefore \frac{23}{16} = 1 + \frac{1}{1 + \frac{1}{2}} \]

More interesting, express $\sqrt{2}$ as a continued fraction.
\[ \sqrt{2} = 1 + (\sqrt{2} - 1) = 1 + \frac{1}{\sqrt{2} + 1} \]
\[ \sqrt{2} + 1 = 2 + (\sqrt{2} - 1) = 2 + \frac{1}{\sqrt{2} + 1} \]
\[ \sqrt{2} + 1 = 2 + (\sqrt{2} - 1) \] From now on, we get 2 at each step.
\[ \sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}} \] for ever.

It is of course necessary to check that the R.H.S. does converge to a definite limit.

We can break off, as 1, or $1 + \frac{1}{2}$, or $1 + \frac{1}{2 + \frac{1}{2}}$, etc.
\[ 1 + \frac{1}{2} = \frac{3}{2} \]
\[ 1 + \frac{1}{2 + \frac{1}{2}} = 1 + \frac{2}{3} = \frac{5}{3} \]

More \( \left( \frac{3}{2} \right)^2 = \frac{9}{4} = 2 + \frac{1}{4} \), \( \left( \frac{7}{5} \right)^2 = \frac{49}{25} = 2 + \frac{1}{25} \).

The sequence continues \( \frac{17}{12}, \frac{401}{29}, \frac{99}{70} \), ...
\[ \left( \frac{17}{12} \right)^2 = \frac{289}{144} = 2 + \frac{1}{144} \]
\[ \left( \frac{401}{29} \right)^2 = \frac{1681}{841} = 2 + \frac{1}{841} \]
\[ \left( \frac{99}{70} \right)^2 = \frac{9801}{4900} = 2 + \frac{1}{4900} \]

It seems that we get ever better approximations, alternately above and below.
Convergent.

If we break off \( a_1 + a_2 + a_3 + \ldots + a_n \) points from the nth point we have

\[
\begin{align*}
P_1 &= a_1; & P_2 &= a_1a_2 + a_2; & P_3 &= a_1a_2a_3 + a_1 + a_3 \\
q_1 &= 1; & q_2 &= a_2; & q_3 &= a_2a_3 + 1
\end{align*}
\]

We observe

\[
\begin{align*}
p_3 &= a_3P_2 + P_1 \\
q_3 &= a_3q_2 + q_1
\end{align*}
\]

This suggests that perhaps

\[
\begin{align*}
p_n &= a_n p_{n-1} + p_{n-2} \quad (Ia) \\
q_n &= a_n q_{n-1} + q_{n-2} \quad (Ib)
\end{align*}
\]

Proof by induction. This holds for \( n = 3 \). If it holds as far as \( n = N \), then \( \frac{P_n}{q_n} \) is found by part a for \( n = N+1 \). This means that we replace

\[
a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4 + \frac{1}{a_5}}
\]

by

\[
a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4 + \frac{1}{a_5 + \frac{1}{a_6}}}
\]

So we get \( \frac{P_{n+1}}{q_{n+1}} \) by changing \( a_{n+1} \) to \( a_n + \frac{1}{a_{n+1}} \) in the equation for \( \frac{P_n}{q_n} \).

i.e.

\[
\begin{align*}
p_{n+1} &= \frac{(a_n + \frac{1}{a_{n+1}})p_n + p_{n-1}}{q_n} \\
&= a_n \frac{q_n}{a_{n+1}} p_{n-2} + \frac{1}{a_{n+1}} p_{n-1} + \frac{1}{a_{n+1}q_{n-1}} q_n \\
&= a_n \frac{q_n}{a_{n+1}} p_{n-2} + \frac{1}{a_{n+1}q_{n-1}} q_n + \frac{1}{a_{n+1}} p_{n-1} \\
&= \frac{a_n + p_{n-1}}{a_{n+1} q_{n-1}} q_{n+1} + q_{n-1}
\end{align*}
\]

Thus, if the formula is correct up to \( n = N \), it is correct up to \( n = N+1 \).
The convergents (i.e., \( p_n/q_n \)) to \( \sqrt{2} \) are:

\[
\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \ldots
\]

\[-\frac{1}{2} = \pm \frac{1}{2}, \quad -\frac{7}{5} = \pm \frac{1}{2}, \quad -\frac{17}{12} = \pm \frac{1}{60}, \quad -\frac{41}{29} = \pm \frac{1}{70}, \quad -\frac{99}{70} = \pm \frac{1}{70}
\]

These results suggest:
1. The numerator is always \( \pm 1 \),
2. The denominator is the product of two denominators,

i.e., \( t \) is suggested as:

\[
p_n - p_{n-1} = (-1)^n. \quad \text{(IIa)}
\]

**Proof by Induction:**

Suppose this true for \( n = k \).

This means:

\[
p_k q_{k-1} - q_k p_{k-1} = (-1)^k. \quad \text{(IIb)}
\]

**Proof by Induction:**

If this is true for \( n = k \), then the next expression is:

\[
p_{k+1} q_k - q_{k+1} p_k = (\text{LHS}) - \text{RHS} = (-1)^k - (-1)^{k+1}.
\]

**Corollary.** The convergents, as found by the rule, \( \text{III} \), are \( \pm \) their least terms.

For if \( p_n \) and \( q_n \) had a common factor, it would be a factor of \( (-1)^n \).

The equation:

\[
p_n - p_{n-1} = (-1)^n
\]

shows that:

If \( n \) is even, \( \frac{p_n}{q_n} > \frac{p_{n-1}}{q_{n-1}} \); if \( n \) is odd

\[
\frac{p_n}{q_n} < \frac{p_{n-1}}{q_{n-1}}.
\]

**Corollary.** If \( p, q \) are two integers with no common factor, we can find integers \( p', q' \) such that \( p'q' - pq = 1 \).

**Proof.** Express \( pq \) as a continued fraction. Let

\( p/q \) be the convergent before the final \( p/q \).
It follows from Ia and Ib that $p_n + p_{n-1}$ and $q_n > q_{n-1}$

(V)

$$p_n q_{n-2} - p_{n-2} q_n = (-1)^{n-1} a_n.$$  

**Proof**  

From I

$$p_n q_{n-2} - p_{n-2} q_n = (a_n p_{n-1} + p_{n-2}) q_{n-2} - p_{n-2} (a_{n-1} q_{n-2} + q_{n-3})$$

$$= a_n (p_{n-1} q_{n-2} - p_{n-2} q_{n-1}) = a_n (-1)^{n-1}.$$  

(VII) It follows from (V) that

$$\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = \frac{(-1)^{n-1} a_n}{q_{n-2} q_{n-3}}.$$  

(VIII) Hence the even convergents continually decrease, while the odd convergents continually increase.

(IX) Hence even $p_n/q_n$ decrease to a limit, the odd $p_n/q_n$ increase to a limit.

(X) These limits are the same, for

$$\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = \frac{(-1)^n}{q_n q_{n-2}}$$  and  RHS $\to 0$ by (V).