Calculus of Variations.

Preliminary Integration by Parts.

$$\frac{d}{dx} u(x)v(x) = u'(x)v(x) + u(x)v'(x)$$

$$\therefore \quad u'(x)v(x) = \frac{d}{dx} u(x)v(x) - u(x)v'(x)$$

$$\int. \quad \therefore \int u'(x)v(x)\,dx = u(x)v(x) - \int u(x)v'(x)\,dx.$$

Example Find $\int x \sin x \, dx$.

Let $v(x) = x \quad u'(x) = \sin x$.

Note u' gets integrated, v gets differentiated. So we choose for v something that simplifies when differentiated. $v(x) = x$ gives $v'(x) = 1$ which is simple.

$u'(x) = \sin x \quad u(x) = -\cos x$ will do.

$$\int x \sin x \, dx = (-\cos x)x - \int (-\cos x)\, 1 \, dx$$

$$= -x\cos x + \int \cos x \, dx$$

$$= \underline{-x\cos x + \sin x + c}$$

Check $\frac{d}{dx}\{-x\cos x + \sin x\} = (-1)\cos x - x(-\sin x) + \cos x$
$= x \sin x$ as required.

Find $y = f(x)$ to make join $(0,0)$ to $(1,2)$ by the shortest path.

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$
$$\therefore \quad s = \int_{x=0}^{x=1} \sqrt{1+f'(x)^2}\, dx$$

$f(x) + \varepsilon g(x)$ would give

$$s + \delta s = \int_{x=0}^{1} \sqrt{1+\{f'(x)+\varepsilon g'(x)\}^2}$$

We are only interested in getting zero coefficient for ε, so we neglect ε^2.

$$s + \delta s = \int_{x=0}^{1} \sqrt{1+ f'(x)^2 + 2\varepsilon f'(x) g'(x)}\, dx$$

$$\sqrt{a+b\varepsilon} = \sqrt{a}\sqrt{1+\frac{b\varepsilon}{a}} = \sqrt{a}\left\{1+\frac{b\varepsilon}{2a} + \dots\right. \ ?$$
$$\doteqdot \sqrt{a} + \frac{b\varepsilon}{2\sqrt{a}}$$

so $\sqrt{1+f'^2+2\varepsilon f'g'} \doteqdot \sqrt{1+f'^2} + \dfrac{\varepsilon f'g'}{\sqrt{1+f'^2}}$

Hence $\quad \delta s \doteqdot \displaystyle\int_{x=0}^{1} \frac{\varepsilon f'(x)}{\sqrt{1+f'(x)^2}}\, g'(x)\, dx.$

Now $g(x)$ has to be zero for $x=0$ and $x=1$. Subject to this, $g(x)$ is arbitrary but this puts restrictions on $g'(x)$, so we integrate by parts.

We want $\quad 0 = \displaystyle\int_{x=0}^{1} \frac{f'(x)}{\sqrt{1+f'(x)^2}}\, g'(x)\, dx$

Put $u'(x) = g'(x)$ so $u(x) = g(x)$.

$$v(x) = \frac{f'(x)}{\sqrt{1+f'(x)^2}}$$

We require $\quad 0 = \left[\dfrac{f'(x)}{\sqrt{1+f'(x)^2}}\, g(x)\right]_{x=0}^{1} - \displaystyle\int_0^1 g(x) \frac{d}{dx}\left[\frac{f'(x)}{\sqrt{1+f'(x)^2}}\right] dx$

As $g(0)=0$ and $g(1)=0$

we want $\quad 0 = \displaystyle\int_0^1 g(x) \frac{d}{dx}\left[\frac{f'(x)}{\sqrt{1+f'(x)^2}}\right] dx \quad *$

Now, in $0 < x < 1$, we can take $g(x)$ different from 0 in some very small interval of x, but zero elsewhere. The only way to make the integral in * zero, is for

$\frac{d}{dx}\left[\frac{f'(x)}{\sqrt{1+f'(x)^2}}\right]$ to be zero throughout this interval.

This means $$\frac{f'(x)}{\sqrt{1+f'(x)^2}} = c$$

$$\therefore \quad f'(x)^2 = c^2(1+f'(x)^2)$$

$$\therefore \quad f'(x)^2 = \frac{c^2}{1-c^2} = \text{constant.}$$

Accordingly $f'(x) = $ constant, if $f'(x)$ is continuous. If it were not constant, it would have to switch suddenly from $+\frac{c}{\sqrt{1-c^2}}$ to $-\frac{c}{\sqrt{1-c^2}}$.

$$\therefore \quad f(x) = a + bx.$$

At $x=0, y=0$; at $x=1, y=2$.

$$\therefore \quad 0 = a$$
$$2 = a+b$$

$a=0, \quad b=2. \qquad y = 2x.$

Calculus of Variations.

It is required to find $f(x)$ such that $y = f(x)$ makes $\int_a^b F(x, y, \frac{dy}{dx})\, dx$ a minimum. It is understood that the end points are fixed: $f(a) = \alpha$, $f(b) = \beta$ where α and β are not to change.

Let $z = \frac{dy}{dx}$. Suppose $f(x)$ changes to $f(x) + \varepsilon g(x)$ where ε is "small", i.e. $\varepsilon \to 0$. The change in $\int_a^b F(x, y, z)\, dx$ is, to first order $\int_a^b \frac{\partial F}{\partial y} \cdot \varepsilon g(x) + \frac{\partial F}{\partial z} \varepsilon g'(x) dx$

since $y = f(x) + \varepsilon g(x)$ implies $\frac{dy}{dx} = f'(x) + \varepsilon g'(x)$

This is to be so for <u>any</u> variation $g(x)$.

Integrate by parts $\int_a^b \frac{\partial F}{\partial z} \varepsilon g'(x) dx = \left[\frac{\partial F}{\partial z} g(x)\right]_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial z}\right) g(x) dx$

so $0 = \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial z}\right)\right] dx$ as variation at a and b is zero.

Condition is $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

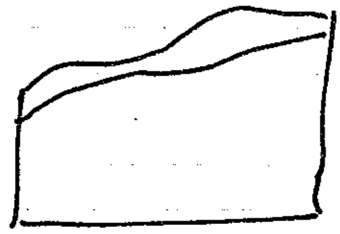
Here y' is treated as a variable quite independent of y.

CV 1. V_5

Calculus of variations. The path $y = f(x)$ from (x_0, y_0) to (x_1, y_1) is to be chosen so that

$$I = \int_{x_0}^{x_1} \phi(x, y, y')\,dx \text{ is stationary.}$$

Suppose $y = f(x) + \varepsilon g(x)$ is a neighbouring path.



Change in I is

$$\int_{x_0}^{x_1} \phi(x, f(x) + \varepsilon g(x), f'(x) + \varepsilon g'(x)) - \phi(x, f(x), f'(x)) \,.\, dx$$

This $=$

$$\varepsilon \int_{x_0}^{x_1} \frac{\partial \phi}{\partial y} g(x) + \frac{\partial \phi}{\partial y'} g'(x)\, dx.$$

$$\int_{x_0}^{x_1} \frac{\partial \phi}{\partial y'} g'(x)\,dx = \left. \frac{\partial \phi}{\partial y'} g(x) \right]_{x_0}^{x_1} - \int_{x_0}^{x_1} \frac{d}{dx}\left[\frac{\partial \phi}{\partial y'}\right) g(x)\,dx$$

Ends are fixed, so $g(x_0) = 0$. $g(x_1) = 0$.

Hence change $= \varepsilon \int_{x_0}^{x_1} \frac{\partial \phi}{\partial y} - \frac{d}{dx}\left(\frac{\partial \phi}{\partial y'}\right) \cdot g(x)\,dx$

This is to be zero for any variation $g(x)$

$$\therefore \quad \frac{\partial \phi}{\partial y} - \frac{d}{dx}\left(\frac{\partial \phi}{\partial y'}\right) = 0.$$

Example 1. Formal derivation of shortest path from A to B

Length of path $= \int ds = \int \sqrt{1+y'^2}\,.\,dx$.

So $\phi(x, y, y') = \sqrt{1+y'^2}$

$\frac{d}{dx}\frac{\partial \phi}{\partial y'} = \frac{\partial \phi}{\partial y}$ gives

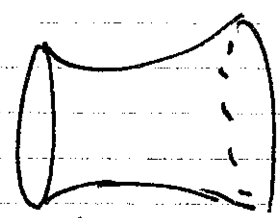
$$\frac{d}{dx}\left\{\frac{y'}{\sqrt{1+y'^2}}\right\} = 0$$

i.e. $\frac{y'}{\sqrt{1+y'^2}} = $ constant.

so y' is constant.

Example 2 Form of bubble.

The curve $y = f(x)$ is to be rotated around $y=0$ to give the surface with minimum area.



$$\text{Area} = \int_a^b 2\pi y\, ds = \int_a^b 2\pi y \sqrt{1+y'^2}\, dx.$$

Take $\phi(x, y, y') = y\sqrt{1+y'^2}$

$$\frac{\partial \phi}{\partial y} = \sqrt{1+y'^2} \qquad \frac{\partial \phi}{\partial y'} = \frac{yy'}{\sqrt{1+y'^2}}$$

$$\therefore \quad \frac{d}{dx}\left(\frac{yy'}{\sqrt{1+y'^2}}\right) = \sqrt{1+y'^2}$$

$$\frac{1}{\sqrt{1+y'^2}}(yy'' + y'^2) - \frac{yy'^2y''}{(1+y'^2)^{3/2}} = \sqrt{1+y'^2}$$

Multiply by $(1+y'^2)^{3/2}$

$$(1+y'^2)(yy''+y'^2) - yy'^2 = (1+y'^2)^2$$
$$= (1+2y'^2+y'^4)$$

$$-yy'^2 + yy''(1+y'^2) = 1+y'^2$$

Multiply by $(1+y'^2)^{3/2}$

$$(1+y'^2)(yy''+y'^2) - yy'^2y'' = (1+y'^2)^2$$

$$y''[y+yy'^2-yy'^2] = 1+y'^2$$

So $$yy'' = 1+y'^2$$

$$\frac{y''}{1+y'^2} = \frac{1}{y}$$

$$\frac{y'y''}{1+y'^2} = \frac{y'}{y}$$

$\int$. $\therefore$ $$\frac{1}{2}\ln(1+y'^2) = \ln y + \text{const.}$$

$$1+y'^2 = ky^2$$

$$y'^2 = ky^2-1$$

$$\frac{dx}{dy} = \frac{1}{\sqrt{ky^2-1}}$$

It is convenient to put $k = \frac{1}{c^2}$

$$\frac{dx}{dy} = \frac{c}{\sqrt{y^2-c^2}}$$

Let $y = c\cosh\theta$ $\quad y^2-c^2 = c^2\sinh^2\theta$

$$x = \int \frac{c\,dy}{\sqrt{y^2-c^2}} = \int \frac{c^2\sinh\theta\, d\theta}{c\sinh\theta}$$

$$= c\theta + \text{const.}$$

We can choose origin of x so that const $=0$.

Then $$x = c\cosh^{-1}\frac{y}{c}$$

$\therefore$ $$y = c\cosh\frac{x}{c}.$$ Catenary.

Catenary

$T\cos\psi = T_0$

$T\sin\psi = ws$

$\tan\psi = \dfrac{w}{T_0}s = \dfrac{s}{c}$

$s = \int \sqrt{1+y'^2}\,dx$

$\dfrac{ds}{dx} = \sqrt{1+y'^2}$

$\tan\psi = dy/dx$

$\dfrac{1}{c}\dfrac{ds}{dx} = y''$

$\sqrt{1+y'^2} = cy''$

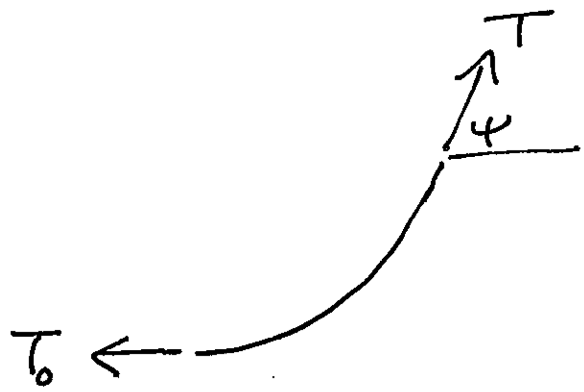
$\dfrac{1}{c} = \dfrac{y''}{\sqrt{1+y'^2}}$

$\therefore\ \dfrac{x}{c} = \int \dfrac{dy'}{\sqrt{1+y'^2}} = \sinh^{-1} y'$

$\therefore\ y' = \sinh\dfrac{x}{c}$

$\therefore\ y = c\cosh\dfrac{x}{c} + \text{constant}$

By suitable choice of axes $y = c\cosh\dfrac{x}{c}$



$\int \dfrac{dx}{\sqrt{1+x^2}}$

let $x = \sinh t$

$1+x^2 = \cosh^2 t$

$dx = \cosh t\,dt$

$\int \dfrac{\cosh t\,dt}{\cosh t} = t = \sinh^{-1} x$

If x changes to $x+\varepsilon$, $f(x)$ changes to $f(x)+\varepsilon f'(x)$ approximately.

$(1+x^2)^{\frac{1}{2}}$
$\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\cdot 2x$

If $f(x) = \sqrt{1+x^2}$

$$f(x+\varepsilon) - f(x) \doteqdot \varepsilon f'(x) = \frac{\varepsilon x}{\sqrt{1+x^2}}.$$

The length of arc is $\int_a^b \sqrt{1+y'^2}\,dx$.

Suppose $y=\varphi(x)$ and it change to

$y = \varphi(x) + \varepsilon\psi(x)$.

The $y' = \varphi'(x) + \varepsilon\psi'(x)$.

The change in y' is $\varepsilon\psi'(x)$.

The change in $\sqrt{1+y'^2}$ is $\frac{y'}{\sqrt{1+y'^2}}\cdot\varepsilon\psi'(x)$.

So $\int_a^b\sqrt{1+y'^2}\,dx$ changes by

$$\int_a^b \frac{y'}{\sqrt{1+y'^2}}\cdot\varepsilon\psi'(x)\,dx$$

$\psi(x)$ is to be 0 at $x=a$, $x=b$.

Integrating by parts give

$$\int_a^b \frac{y'}{\sqrt{1+y'^2}}\psi'(x)\,dx = \left[\frac{y'\psi(x)}{\sqrt{1+y'^2}}\right]_a^b - \int_a^b \psi(x)\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}}\cdot dx$$

This is to be zero whatever $\psi(x)$, if the length is stationary for small variations.

$$\therefore \frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} = 0.$$

$$\therefore \frac{y'}{\sqrt{1+y'^2}} = \text{constant}$$

$$y'^2 = K(1+y'^2)$$

$$y'^2(1-K) = K \quad \therefore y'^2 = \frac{K}{1-K}$$

y' constant.

$$L = T - V = \frac{1}{2}m\dot{x}^2 - mgx.$$ Hamilton's Principle. $\int L\,dt$ stationary.

$$\int \frac{1}{2}m\dot{x}^2 - mgx.\,dt$$ a minimum.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{\partial L}{\partial x} = -mg.$$

$$\therefore \quad \frac{d}{dt}(m\dot{x}) = -mg$$

$$\underline{x'' = -g.}$$

Least distance

$$s = \int_a^b \sqrt{1+y'^2}\,dx$$

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}} \qquad \frac{\partial f}{\partial y} = 0$$

$$\therefore \quad \frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} = 0$$

$$\therefore \quad \frac{y'}{\sqrt{1+y'^2}} = c$$

$$y'^2 = c^2(1+y'^2)$$

$$y'^2(1-c^2) = c^2$$

$$y'^2 = \frac{c^2}{1-c^2}$$ y' constant.

Soap bubble. Surface $= \int_a^b 2\pi y\sqrt{1+y'^2}\,dx$ Ignore 2π.

$$\frac{\partial f}{\partial y} = \sqrt{1+y'^2} \qquad \frac{\partial f}{\partial y'} = \frac{yy'}{\sqrt{1+y'^2}}$$

$$\frac{d}{dx}\left[\frac{yy'}{\sqrt{1+y'^2}}\right] = \sqrt{1+y'^2}$$

$$\frac{1}{\sqrt{1+y'^2}}(yy''+y'^2) - \frac{1}{2}(1+y'^2)^{-\frac{3}{2}}\,2y''\cdot yy' = \sqrt{1+y'^2}$$

$$\therefore (1+y'^2)(yy''+y'^2) - \cancel{3\,0y'^2} = (1+y'^2)^2 - yy'^2y''$$

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$$y y'' + \underline{y y'^2 y''} + \dot{y}'^2 + \underline{\underline{y'^4}} - \underline{y y'^2 y''} = 1 + 2\dot{y}'^2 + \underline{\underline{y'^4}}$$

$$y y'' = 1 + y'^2$$

Let $\dot{x} = \frac{dx}{dy} = \frac{1}{y'}$

$$\cancel{\ddot{x} = \frac{d}{dy}\dot{x} = \frac{dx}{dy}\frac{d}{dx}\dot{x} = \frac{dx}{dy}}$$

$$y' = \frac{1}{\dot{x}} \quad y'' = \frac{d}{dx}\left(\frac{1}{\dot{x}}\right) = \frac{dy}{dx}\frac{d}{dy}\left(\frac{1}{\dot{x}}\right)$$

$$= \frac{1}{\dot{x}}\left(-\frac{\ddot{x}}{\dot{x}^2}\right) = -\frac{\ddot{x}}{\dot{x}^3}$$

$$\therefore \quad -\frac{y\ddot{x}}{\dot{x}^3} = 1 + \frac{1}{\dot{x}^2} \quad \therefore -y\ddot{x} = \dot{x}^3 + \dot{x}$$

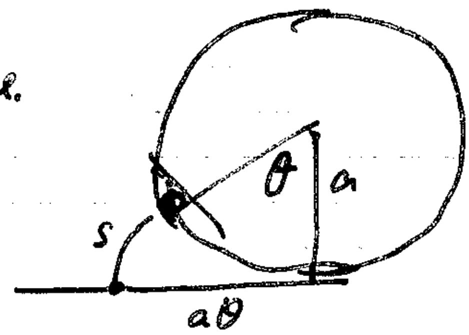
Let $\dot{x} = v \qquad -y\frac{dv}{dy} = v^3 + v$

$$\therefore \quad -\int\frac{dy}{y} = \int\frac{dv}{v^3+v} = \int dv\left(\frac{1}{v} - \frac{v}{1+v^2}\right)$$

$$-\ln y = \ln v - \frac{1}{2}\ln(1+v^2) + \text{const.}$$

$$\therefore \quad y =$$

A wheel of radius $\underline{a}$ rolls on a flat surface. After rotation through θ a speck of mud on the rim has coordinates

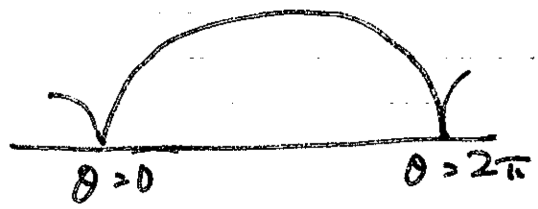


$$x = a\theta - a\sin\theta$$

$$y = a - a\cos\theta.$$

Find the length of the arc described as a function of θ.

This curve, the cycloid, is known as the examinee's curve. Practically all the questions that can be asked about it have simple answers



Calculus of Variations.

Required the condition for $\int_a^b \varphi(x,y,y')dx$ to be stationary for a given path from $x=a, y=y_0$ to $x=b, y=y_1$.

Example. The shortest path from (a,y_0) to (b,y_1) is a straight line. So $\int \sqrt{1+y'^2}.\,dx$ is a minimum for this path

Let $y=f(x)$ be the path in question.
Consider a change to $y=f(x)+\varepsilon g(x)$ where ε is to be indefinitely small (squares neglected)

$$y' = f'(x)+\varepsilon g'(x)$$

Thus we have

$$\varphi(x,\ f(x)+\varepsilon g(x),\ f'(x)+\varepsilon g'(x))$$
$$= \varphi(x, f(x), f'(x)) + \varepsilon g(x)\frac{\partial\varphi}{\partial y} + \varepsilon g'(x)\frac{\partial\varphi}{\partial y'}$$
$$= \varphi(x,y,y') + \varepsilon g(x)\frac{\partial\varphi}{\partial y} + \varepsilon g'(x)\frac{\partial\varphi}{\partial y'}$$

φ, $\frac{\partial\varphi}{\partial y}$, $\frac{\partial\varphi}{\partial y'}$ having x,y,y' values for the undisturbed path.

The new value for the integral is

$$\int_a^b \varphi(x,y,y') + \varepsilon g(x)\frac{\partial\varphi}{\partial y} + \varepsilon g'(x)\frac{\partial\varphi}{\partial y'}.\,dx$$
$$= \int_a^b \varphi(x,y,y') + \varepsilon g(x)\frac{\partial\varphi}{\partial y}.\,dx + \left[g(x)\frac{\partial\varphi}{\partial y'}\right]_a^b - \int_a^b \varepsilon g(x)\frac{d}{dx}\left(\frac{\partial\varphi}{\partial y'}\right)dx$$

We suppose no variation of ends, so $g(a)=0$. $g(b)=0$

Change in integral

$$= \varepsilon\int_a^b g(x)\left[\frac{\partial\varphi}{\partial y} - \frac{d}{dx}\left(\frac{\partial\varphi}{\partial y'}\right)\right]dx$$

This is to be zero for any continuous and continuously differentiable $g(x)$

so $$\frac{\partial\varphi}{\partial y} - \frac{d}{dx}\left(\frac{\partial\varphi}{\partial y'}\right) = 0.$$

Example Shortest path $\varphi(x,y,y') = \sqrt{1+y'^2}$

$$\frac{\partial\varphi}{\partial y} = 0 \qquad \frac{\partial\varphi}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$$

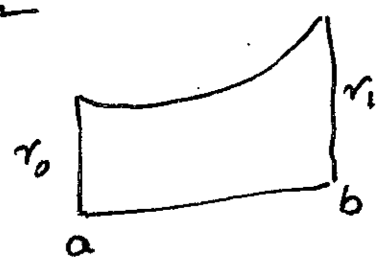
$$\therefore \quad \frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} = 0$$

$$\therefore \quad \frac{y'}{\sqrt{1+y'^2}} = c$$

$$y'^2 = c^2(1+y'^2)$$

$$y'^2 = \frac{c^2}{1-c^2}.$$

Example Soap bubble formed by rotation about Ox.



$$\text{Area} = \int_a^b 2\pi y\,ds$$

$$= \int_a^b 2\pi y\sqrt{1+y'^2}\,dx. \qquad \text{(ignore } 2\pi.$$

$$\frac{\partial\varphi}{\partial y} = \sqrt{1+y'^2} \qquad \frac{\partial\varphi}{\partial y'} = \frac{yy'}{\sqrt{1+y'^2}}$$

$$\sqrt{1+y'^2} = \frac{d}{dx}\frac{yy'}{\sqrt{1+y'^2}}$$

$$= \frac{yy''+y'^2}{\sqrt{1+y'^2}} - \frac{yy'\cdot y'}{\sqrt{(1+y'^2)^3}}$$

$$\therefore \quad (1+y'^2)^2 = (yy''+y'^2)(1+y'^2) - yy'^2$$

$$1+2y'^2+\underline{\underline{y'^4}} = yy''+y'^2+yy'^2y''+\underline{\underline{y'^4}} - yy'^2$$

$$1+y'^2 = yy''(1+y'^2) - yy'^2$$

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$$\sqrt{1+y'^2} = \frac{yy''+y'^2}{\sqrt{1+y'^2}} - yy' \cdot \frac{y'y''}{\sqrt{(1+y'^2)^3}}$$

$$(1+y'^2)^2 = (yy''+y'^2)(1+y'^2) - yy'^2y''$$

$$= yy'' + y'^2(1+y'^2)$$

$$1+y'^2 = yy''$$

$$\frac{y'}{y} = \frac{y'y''}{1+y'^2}$$

$$\ln y = \frac{1}{2}\ln(1+y'^2) + \text{constant}$$

$$y^2 = c^2(1+y'^2)$$

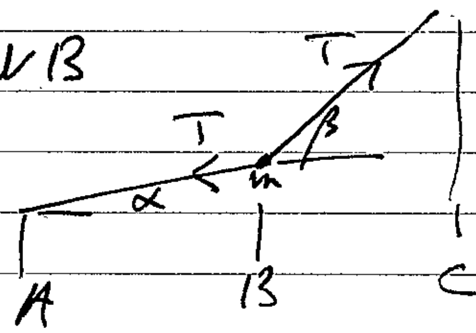
$$\left(\frac{dx}{dy}\right)^2 = \frac{1}{y'^2} = \frac{1}{\frac{y^2}{c^2}-1} = \frac{c^2}{y^2-c^2}$$

$$\frac{dx}{dy} = \frac{c}{\sqrt{y^2-c^2}}$$

(V 16)

Last time we considered vibrating string with masses attached, at intervals h.

Lateral force on mass m at B is $T\sin\beta - T\sin\alpha$



$$\doteqdot T\tan\beta - T\tan\alpha$$

$$= T\left(\frac{y_C - y_B}{h} - \frac{y_B - y_A}{h}\right)$$

$$= \frac{T}{h}\left(y_C - 2y_B + y_A\right) \doteqdot hT\frac{\partial^2 y}{\partial x^2}$$

$$\therefore\ m\frac{\partial^2 y}{\partial t^2} = hT\frac{\partial^2 y}{\partial x^2}$$

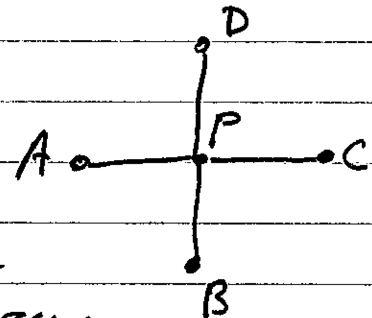
If density ρ, $\rho = m/h$.

$$\frac{\partial^2 y}{\partial x^2} = \underline{\frac{m}{hT}\frac{\partial^2 y}{\partial t^2}} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \text{ say.}$$

By change of variable we found the general solution to be $y = f(x - ct) + \varphi(x + ct)$
waves moving to right or left with velocity c.

Sheet of 2 dimensions

Let V be displacement at P, $\perp$ plane of sheet.



T now has to be <u>the tension per unit length</u>. As the strings occur at intervals of h, there is one string in this interval, so its tension is Th.

The transverse force is made up of $\frac{V_C - 2V_P + V_A}{h}(Th)$ from tension in AC, and $\frac{V_B - 2V_P + V_D}{h}(Th)$ in BD.

The total force is $\frac{V_A + V_B + V_C + V_D - 4V_P}{h}(Th)$

$$\doteqdot Th^2\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)$$

$$\therefore\ \frac{\partial^2 V}{\partial t^2} = \frac{Th^2}{m}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)$$

Now there is one mass m in each square $h \times h$. V/17

So $\rho = m/h^2$

$$\therefore \frac{\partial^2 V}{\partial t^2} = \frac{T}{\rho}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)$$

Putting $T/\rho = c^2$ we have

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$$ in 2 dimensions

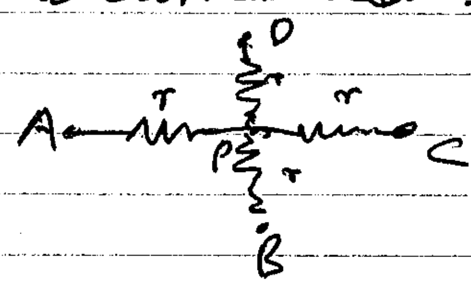
for wave propagation.

Note In the grid network, the system is at rest if, and only if $V_A + V_B + V_C + V_D - 4V_P = 0$ i.e the displacement at each point is the average of that at the 4 neighbours.

In the limit, for a continuous sheet, we no longer have 2 directions singled out. The displacement at any point is the average of that on a circle around the point.

For a sheet in equilibrium $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$.
Equations of this type turn up in several branches of physics and in one important branch of pure maths.

c) Flow of electricity in a network.
Kirchoff's two laws. (1) Ohm's Law holds for any part of the network (2) As much current leaves any junction as enters it.



Current in PC is $\frac{V_C - V_P}{r} \leftarrow$

In AP is $\frac{V_A - V_P}{r} \rightarrow$

In BP is $\frac{V_B - V_P}{r} \uparrow$

In PD is $\frac{V_D - V_P}{r} \downarrow$

All inwards $\therefore$ sum $= 0$.

∴ $V_A + V_B + V_C + V_D - 4V_P = 0$.
Potential at P is average of potentials at ABCD.
In the limit $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$. $\nabla^2 V = 0$ for sheet.

So we can represent the potential distribution in a copper sheet by the shape of an elastic sheet, slightly changed from plane, by imposing some shape on the boundary.

A theorem for either of these – a maximum cannot occur at an internal point. Clear for current flow – it would be away from such a point in every direction. Also for membrane. Also for electric potential. Gauss' Theorem. → Incompressible fluid.

b) Pure mathematics.

$\frac{\partial^2 V}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial y^2}$ has general solution

$\phi(x - cy) + f(x + cy)$.

In particular $V = f(x + cy)$ satisfies this equation

if $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$ $\quad \frac{1}{c^2} = -1$ (Verify by direct calculation.)

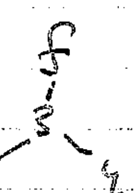
$c = i$. So $f(x + iy)$ satisfies this PDE.

If $f(x+iy) = u(x,y) + iv(x,y)$ where u and v are real, it follows that

$0 = \nabla^2 \phi = \nabla^2 u + i\nabla^2 v$.

Hence $\nabla^2 u = 0$ $\quad \nabla^2 v = 0$.

<u>Question 1.</u> Find $\partial f/\partial x$ and $\partial f/\partial y$ from where $z = x + iy$. What relations must hold between $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$?



~~Question 2 Deduce from these that Angle~~ ~~$\nabla^2 u = 0$ $\nabla^2 v = 0$~~

<u>Question 2</u> Also derive these from $df = f'(z)\,dz$.
Take $z = x + iy$, $f = u + iv$, $f'(z) = a + ib$.
Angle between $u = C$, $v = K$.

Conformal Transformation

4 ✓
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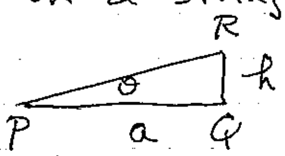
Question 3. From the relations between derivatives of u, v deduce $\nabla^2 u = 0$ $\nabla^2 v = 0$.

Small vibrations of masses on a string.

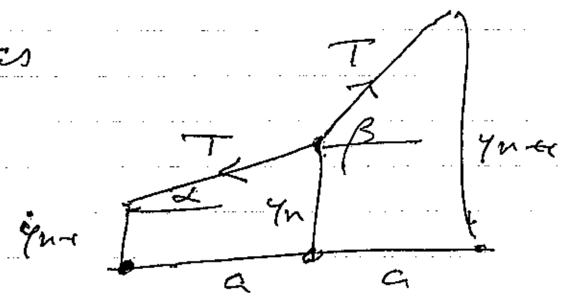


If PQ is originally level, and is displaced to PR so that R is at a height h over Q, the new length of the string is $\sqrt{a^2+h^2} = a\sqrt{1+\frac{h^2}{a^2}} \simeq a\left(1+\frac{h^2}{2a^2}\right)$

The length of the string is thus increased by an amount of order h^2. If h is small, we approximate by considering only quantities at least $O(h)$. Then the change in the length of the string is ignored, and thus tension T is unchanged.

As we are taking PR to have length $= a$, to the order of approximation, $\sin\theta \cong h/a$ is not distinguished from $\tan\theta = h/a$.

If string is displaced as shown, the horizontal forces acting on n^{th} mass are $T\cos\beta - T\cos\alpha$.



To the degree of approximation used, $\cos\alpha = \cos\beta = 1$.

Hence there is no (appreciable) force on n^{th} mass in a horizontal direction.

The vertical force is $T\sin\beta - T\sin\alpha$

$$\cong T\tan\beta - T\tan\alpha = T\left(\frac{y_{n+1}-y_n}{a}\right) - T\left(\frac{y_n - y_{n-1}}{a}\right)$$

$$= \frac{T}{a}\left(y_{n+1} - 2y_n + y_{n-1}\right).$$

$$\therefore \quad m\frac{d^2 y_n}{dt^2} = \frac{T}{a}\left(y_{n+1} - 2y_n + y_{n-1}\right)$$

Choose units so that $T/ma = 1$.

Then $\ddot{y}_n = y_{n+1} - 2y_n + y_{n-1}$

If there are N masses on the string, $y_0 = 0$ and $y_{N+1} = 0$, so

$$\ddot{y}_1 = y_2 - 2y_1$$

$$\ddot{y}_N = -2y_N + y_{N-1}$$

2 masses. It is convenient to write the equations as

$$\ddot{y}_1 = -2y_1 + y_2$$
$$\ddot{y}_2 = y_1 - 2y_2$$

Thus we have the matrix $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$. The problem will be simplified if we introduce new co-ordinates with eigenvectors as axes.

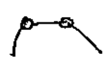
If
$$\lambda y_1 = -2y_1 + y_2$$
$$\lambda y_2 = y_1 - 2y_2$$

$$0 = -(\lambda+2)y_1 + y_2$$
$$0 = y_1 - (\lambda+2)y_2$$

$$0 = \begin{vmatrix} -(\lambda+2) & 1 \\ 1 & -(\lambda+2) \end{vmatrix} = (\lambda+2)^2 - 1$$

So $\lambda + 2 = 1$ or -1
$\lambda = -1$ or -3.

Matrix, in x words must be $\begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$

If $\lambda + 2 = 1$ $\quad \left.\begin{aligned} 0 &= -y_1 + y_2 \\ 0 &= y_1 - y_2 \end{aligned}\right\}$ so $y_1 = y_2$. 

If $\lambda + 2 = -1$ $\quad \left.\begin{aligned} 0 &= y_1 + y_2 \\ 0 &= y_1 + y_2 \end{aligned}\right\}$ $y_1 + y_2 = 0$. 

Thus we may take axes as $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Note $\perp$.

$y_1 = x_1 - x_2 \qquad x_1 = \frac{1}{2}(y_1 + y_2)$
$y_2 = x_1 + x_2 \qquad x_2 = \frac{1}{2}(y_2 - y_1)$

$$\ddot{x}_1 = \tfrac{1}{2}\ddot{y}_1 + \tfrac{1}{2}\ddot{y}_2 = (-y_1 + \tfrac{1}{2}y_2) + (\tfrac{1}{2}y_1 - y_2)$$
$$= -\tfrac{1}{2}y_1 - \tfrac{1}{2}y_2 = -x_1.$$

$$\ddot{x}_2 = \tfrac{1}{2}\ddot{y}_2 - \tfrac{1}{2}\ddot{y}_1$$
$$= \tfrac{1}{2}(y_1 - 2y_2) - \tfrac{1}{2}(-2y_1 + y_2) = \tfrac{3}{2}y_1 - \tfrac{3}{2}y_2 = -3x_2.$$

So
$$\ddot{x}_1 = -x_1$$
$$\ddot{x}_2 = -3x_2.$$

3 masses

$$\ddot{y}_1 = -2y_1 + y_2$$
$$\ddot{y}_2 = y_1 - 2y_2 + y_3$$
$$\ddot{y}_3 = y_2 - 2y_3$$

$$\begin{vmatrix} -(2+\lambda) & 1 & 0 \\ 1 & -(2+\lambda) & 1 \\ 0 & 1 & -(2+\lambda) \end{vmatrix} = 0$$

$$-\mu y_1 + y_2 = 0$$
$$y_1 - \mu y_2 + y_3 = 0$$
$$y_2 - \mu y_3 = 0$$

Let $\mu = 2+\lambda$. $-\mu^3 + 2\mu = 0$

$\mu = -\sqrt{2}$ $\mu = 0$ $\mu = +\sqrt{2}$.

$\mu = -\sqrt{2}$ $y_2 = -\sqrt{2}\,y_1$ $y_2 = -\sqrt{2}\,y_3$

$y_1 = 1$, $y_2 = -\sqrt{2}$, $y_3 = 1$.

Second eqn, $1 + \sqrt{2}(-\sqrt{2}) + 1 = 0$ ✓.

Better $(\frac{1}{\sqrt{2}}, -1, \frac{1}{\sqrt{2}})$

$\mu = 0$ $y_2 = 0$ $y_1 + y_3 = 0$ $y_2 = 0$. $1, 0, -1$

$\mu = \sqrt{2}$. $y_2 = \sqrt{2}\,y_1$ $y_2 = \sqrt{2}\,y_3$.

$y_1 = \frac{1}{\sqrt{2}}$ $y_2 = 1$ $y_3 = \frac{1}{\sqrt{2}}$

$y_1 - \sqrt{2}y_2 + y_3 = \frac{1}{\sqrt{2}} - \sqrt{2} + \frac{1}{\sqrt{2}}$.

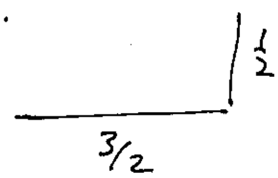
4 masses

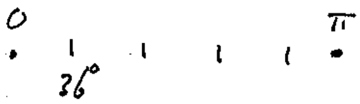
$$\begin{pmatrix} -\mu & 1 & 0 & 0 \\ 1 & -\mu & 1 & 0 \\ 0 & 1 & -\mu & 1 \\ 0 & 0 & 1 & -\mu \end{pmatrix}$$

$$-\mu \begin{vmatrix} -\mu & 1 & 0 \\ 1 & -\mu & 1 \\ 0 & 1 & -\mu \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 \\ 0 & -\mu & 1 \\ 0 & 1 & -\mu \end{vmatrix}$$

$$-\mu(-\mu^3 + 2\mu) - (\mu^2 - 1) = \mu^4 - 3\mu^2 + 1.$$

$$\mu^2 = \frac{3 \pm \sqrt{5}}{2}.$$ $\cancel{\mu^2 = \frac{3 \pm i}{2}}$





$$-\mu \sin 36^\circ + \sin 72^\circ = 0$$
$$-\mu \sin 36^\circ + 2\sin 36^\circ \cos 36^\circ = 0$$
$$\mu = 2\cos 36^\circ.$$

$$\sin 36^\circ - 2\cos 36^\circ \sin 72^\circ + \sin 108^\circ$$
$$\sin 36^\circ + \sin 108^\circ = 2\sin\frac{36+108}{2}^\circ \cos\frac{108-36}{2}^\circ$$
$$= 2\sin 72^\circ \cos 36^\circ.$$

$$\mu^2 = \frac{6 \pm 2\sqrt{5}}{4} = \frac{(\sqrt{5}\pm 1)^2}{4} \quad \mu = \frac{\sqrt{5}+1}{2},\ -\frac{\sqrt{5}+1}{2},\ \frac{\sqrt{5}-1}{2},\ -\frac{\sqrt{5}-1}{2}$$

23

Vibrating wires.
Density $\rho(x)$, tension $T(x)$

Lateral force is $T\sin\psi \doteqdot T\tan\psi = T\frac{\partial y}{\partial x}$
Net lateral force on piece of length dx
is $dx\frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right)$.

Mass × accel^n = $\rho\, dx \frac{\partial^2 y}{\partial t^2}$

$$\therefore \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right) = \rho\frac{\partial^2 y}{\partial t^2}$$

If $y = \sin\omega t$

$$\frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right) + \omega^2 \rho y = 0.$$

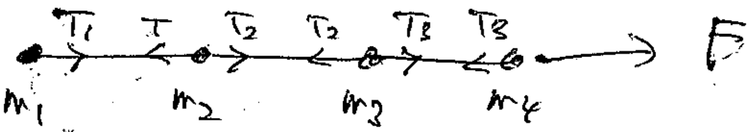
Let $\lambda = \omega^2$

$$\frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right) + \lambda\rho y = 0.$$

Orthogonality, I believe, corresponds to the fact that forces in one mode do not excite motion in another. So we expect this to be a general property.

The pressure of winds. Momentum = mass x velocity

Force = rate of change of momentum.



If we have masses connected like this, tensions will appear in the wires connecting them

If v is the common velocity

$$T_1 = m_1 \frac{dv}{dt}$$

$$T_2 - T_1 = m_2 \frac{dv}{dt}$$

$$T_3 - T_2 = m_3 \frac{dv}{dt}$$

$$F - T_3 = m_4 \frac{dv}{dt}$$

Add these $F = (m_1 + m_2 + m_3 + m_4)\frac{dv}{dt}$

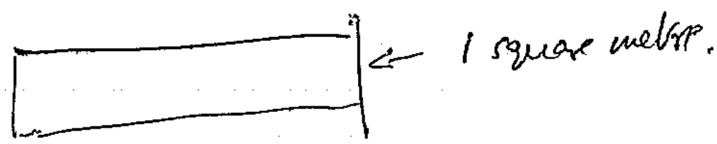
$= M\frac{dv}{dt}$ M, total mass

$= \frac{d}{dt}(Mv)$.

So force = rate of change of momentum can be applied to systems with many masses, and a formal proof is possible for systems more complicated than that shown above.

Let ρ = mass/air, in Kg/m^3.
v = velocity in metres/sec. of wind

In a second a volume v cu. metres will meet surface and (we assume) be brought to rest.



(If it bounces back the pressure will be greater)

This volume has a mass $M = \rho v$

and momentum $Mv = \rho v^2$

Each second, the pressure of 1 square metre of surface a

W_2

the air destroys this amount of momentum.
If N newtons is this pressure

$$N = \rho v^2.$$

ρ density kg/metre^3
v velocity
N newtons pressure on $1\,m^2$

For air $\rho \doteq 1\frac{1}{3}$ kg/m^3.

100 mph $\doteq$ 45 metres/sec

If $v = 45$

$$N = 2700$$

1 newton $\doteq \frac{1}{10}$ kg weight

So 270 kg weight on 1 square metre.

1
W3

Ruark & Urey. Chap XV Wave mechanics.

p 519. It is believed that $\psi(q_1 \dots q_n)\psi^*(q_1 \dots q_n)\,dv$ represents probability that $(q_1 \dots q_n)$ lies in an element of phase space of volume dv
eg. for particle in plane, described by polar coords r, θ $\quad dv = r\,dr\,d\theta$.

ψ must be finite, continuous, single valued.

Rigid rotator with 1 degree of freedom.

$$H = \tfrac{1}{2} I \dot{\theta}^2 \qquad \frac{\partial H}{\partial \dot\theta} = I\dot\theta = p \quad \dot\theta = \frac{p}{I}$$

$$H = \frac{1}{2I} p^2$$

$$p \to \frac{\hbar}{i}\frac{d}{d\theta} \qquad -\frac{\hbar^2}{2I}\frac{d^2\psi}{d\theta^2} = E\psi \quad E \text{ energy}$$

$$\frac{d^2\psi}{d\theta^2} = -\frac{2EI}{\hbar^2}\psi$$

$$\text{Let } a^2 = \frac{2EI}{\hbar^2} \qquad \frac{d^2\psi}{d\theta^2} = -a^2\psi$$

$$\psi = c_1 e^{ia\theta} + c_2 e^{-ia\theta}$$

If E were negative, $a = ik$ $\quad \psi = c_1 e^{-k\theta} + c_2 e^{k\theta}$
As $\theta \to +\infty$ or $-\infty$, $|\psi| \to \infty$, so E cannot be negative.
E being +ve, a is real.

$$\psi = A\cos a\theta + B\sin a\theta.$$

ψ is to be single valued, so $\underline{a}$ must be an integer m. By suitable choice of $\theta = 0$ we can have $\psi = A\sin m\theta \qquad m^2 = \frac{2EI}{\hbar^2}$.

Hence $$E = \frac{\hbar^2 m^2}{2I}.$$

This problem is unreal as rotation in a plane is not physically meaningful.

p 527.

Simple Harmonic Oscillator.

$$T = \tfrac{1}{2} m\dot{x}^2 \qquad V = \tfrac{1}{2} kx^2$$

$$p = \frac{\partial T}{\partial \dot{x}} = m\dot{x} \quad \dot{x} = \frac{p}{m} \qquad T = \frac{p^2}{2m}$$

$$\text{So } E = \frac{p^2}{2m} + \tfrac{1}{2} kx^2$$

$$p \to \frac{\hbar}{i}\frac{d}{dx} \qquad E\psi = -\frac{\hbar^2}{2m}\psi'' + \tfrac{1}{2}kx^2\psi$$

$$\psi'' + \frac{2m}{\hbar^2}\left[E - \tfrac{1}{2}kx^2\right]\psi = 0$$

For large x, $\tfrac{1}{2}kx^2 \Rightarrow \infty$.

Try $\psi = e^{P(x)} u$ $\quad P(x)$ polynomial.

We find $\psi' = e^P u' + P' e^P u$

$$\psi'' = e^P u'' + 2P' e^P u' + (P'' e^P + P'^2 e^P) u$$

so, after removing factor e^P that occurs in each term

$$u'' + 2P'u' + u\left[P'' + P'^2 + \frac{2m}{\hbar^2}E - \frac{mk}{\hbar^2}x^2\right] = 0$$

Highest power in coeff of u comes from $P'^2 - \frac{mk}{\hbar^2}x^2$.

Make this zero by taking $P' = \pm \sqrt{\frac{mk}{\hbar}}\, x$

So $P = \pm \frac{\sqrt{mk}}{\hbar}\frac{x^2}{2}$ $\quad$ + gives ∞ for $x \to \pm\infty$

So choose $P = -\frac{\sqrt{mk}}{2\hbar}x^2$. $\quad P' = -\frac{\sqrt{mk}}{\hbar}x \quad P'' = -\frac{\sqrt{mk}}{\hbar}$

Thus

$$0 = u'' - \frac{2\sqrt{mk}}{\hbar}xu' + u\left[-\frac{\sqrt{mk}}{\hbar} + \frac{2m}{\hbar^2}E\right] = 0$$

$$= u'' - bxu' + uc \qquad \text{say.}$$

It can be proved that an infinite series solution of this will lead to an unacceptable ψ (one that $\to \infty$ for large x). The only escape is for u to be a polynomial.

Let $u = x^\rho(a_0 + a_1 x + \dots)$

$u'' - bxu' + uc$ contains only one term in $x^{\rho-2}$ namely $a_0\rho(\rho-1)x^{\rho-2}$. We suppose $a_0 \neq 0$ so $\rho = 0$ or 1.

<u>If $\rho = 0$</u> $u = \sum_0^\infty a_r x^r$. u is a polynomial if $a_r = 0$ for $r > N$.

Now $0 = \sum a_r r(r-1)x^{r-2} - \sum_0^\infty b r a_r x^r + \sum_0^\infty c a_r x^r$

From the coefficient of x^{r-2}

$$0 = a_r r(r-1) - b(r-2)a_{r-2} + c a_{r-2}$$

so

$$a_r = \frac{c - b(r-2)}{r(r-1)} a_{r-2}$$

Suppose a_N is highest coeft. $\neq 0$.

Put $r = N+2$

$$a_{N+2} = \frac{c - bN}{(N+1)(N+2)} a_N.$$

$$a_{N+2} = 0,\ a_N \neq 0. \therefore c - bN = 0.$$

That is $$\frac{2mE}{\hbar^2} - \frac{\sqrt{mk}}{\hbar} - \frac{2\sqrt{mk}}{\hbar}N = 0$$

$$\therefore 2mE = \hbar\sqrt{mk}(2N+1)$$

$$E = \hbar\sqrt{\frac{k}{m}}\left(N + \tfrac{1}{2}\right).$$

Classically, $m\ddot{x} = -kx$ so $x = \sin\sqrt{\frac{k}{m}}\,t$ is a solution. If frequency is ν_0

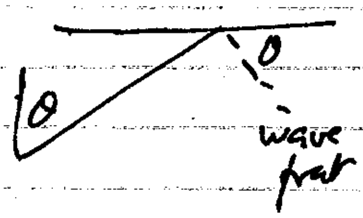
$$\sqrt{\frac{k}{m}} = 2\pi\nu_0,\ \text{so } E = \hbar\cdot 2\pi\nu_0\left(N + \tfrac{1}{2}\right)$$

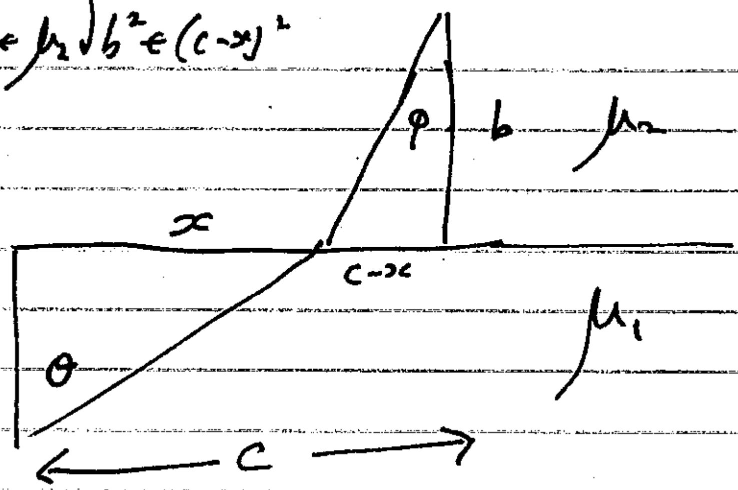
$$= h\nu_0\left(N + \tfrac{1}{2}\right)$$

Note. Zero point energy.

W/6

$I = \int \mu \, ds = \mu_1\sqrt{a^2+x^2} + \mu_2\sqrt{b^2+(c-x)^2}$





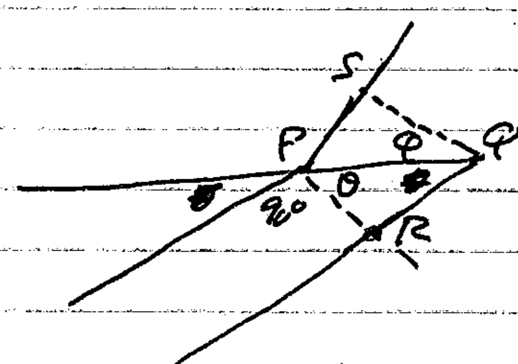
$$0 = \frac{\partial I}{\partial x} = \frac{\mu_1 x}{\sqrt{a^2+x^2}} - \frac{\mu_2(c-x)}{\sqrt{b^2+(c-x)^2}}$$

$$0 = \mu_1 \sin\theta - \mu_2 \sin\varphi.$$

$RQ = PQ \sin\theta$

$PS = PQ \sin\varphi$

$\therefore \frac{RQ}{PS} = \frac{\sin\theta}{\sin\varphi} = \frac{\mu_2}{\mu_1}$



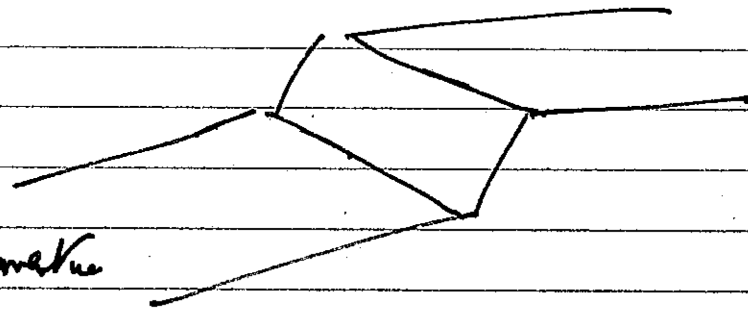
RQ is distance gone in μ_1 in the time that light goes PS in μ_2. Hence $V_1/V_2 = RQ/PS$

So $\frac{V_1}{V_2} = \frac{\mu_2}{\mu_1}$

μ is inversely proportional to speed.

W
7

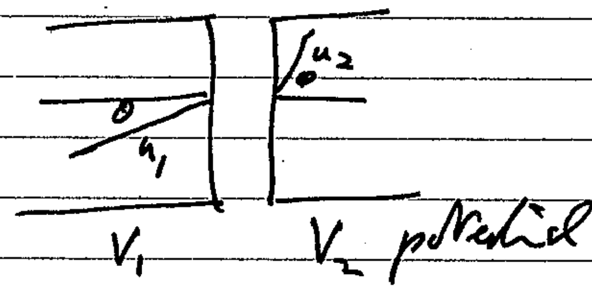
Dynamics.



Conservation of momentum

$u_1 \sin\theta = u_2 \sin\varphi$

Hence $\dfrac{u_1}{u_2} = \dfrac{\mu_1}{\mu_2}$



Origins of Quantum Theory.

On classical theory, any accelerated charged particle radiates energy. So an electron going round a proton should always emit radiation, lose energy and fall into proton. Atoms would be unstable, emit radiation and disappear.

Study of spectra shows that any atom or molecule has certain "spectral terms" $\nu_1, \nu_2, \nu_3 \ldots$ and the emitted frequencies are differences of these terms. $\nu_{mn} = \nu_m - \nu_n$. The "Ritz combination principle".

Black body radiation. Energy between λ and $\lambda + d\lambda$ being $S_\lambda\, d\lambda$, Rayleigh-Jeans predicted $S_\lambda = 2\pi c k T/\lambda^4$. "The Rayleigh Jeans catastrophe" [Rayleigh 1900, Jeans 1905]

Gives infinite total, and infinite energy for very short waves. It agrees with observation for long waves.

Actual curve 

Stefan Boltzmann Law. The total emissive power of any body $\propto T^4$ (T temperature absolute.) Proved by thermodynamics: first conjectured experimentally.

Suggested formulas. Wien $e_\lambda = a\lambda^{-5} e^{-b/\lambda T}$

$$\int_0^\infty e_\lambda\, d\lambda = \int_0^\infty a x^3 e^{-bx/T}\, dx \quad \left(\text{by } x = \tfrac{1}{\lambda}\right) = \int_0^\infty \left(\frac{T}{b}\right)^4 z^3 e^{-z}\, dz = \frac{3!\,T^4}{b^4},$$

in agreement with Stefan Boltzmann

Fits data well for λ large.

It can also be proved by thermodynamics that if e_1 is emissive power for monochromatic light of wavelength λ_1 at temp T_1, then at temperature T_2, for $\lambda_2 = \lambda_1 \frac{T_1}{T_2}$, $e_2 = e_1\left(\frac{T_2}{T_1}\right)^5$.

This enables us to transform curve for distribution at one temperature to that for another temperature.

It follows that if the maximum emissive power at temperature T is e_m for λ_m, then $e_m = BT^5$, $\lambda_m T = A$.

P.T.O.

W_q

Annalen d. Physik - 1926 - Vol. 79.

p 361. E. Schrödinger. Quantisierung als Eigenwertproblem (1st communication)

$$H(q, \partial S/\partial q) = E. \quad \text{Let } S = K \ln \psi.$$

(Later he says K must have dimensions of action, and finds $K = h/2\pi$ gives $-E = 2\pi^2 m e^4/h^2 \ell^2$ as required.)

Then $E = H\left(q, \frac{K}{\psi}\frac{\partial\psi}{\partial q}\right)$

For $V = -e^2/r$ this gives

$$\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2 - \frac{2m}{K^2}\left(E + \frac{e^2}{r}\right)\psi^2 = 0$$

$r = \sqrt{x^2+y^2+z^2}$. He minimizes $\int_0^\infty\int_0^\infty\int_0^\infty$

and gets

$$\nabla^2\psi + \frac{2m}{K^2}\left(E + \frac{e^2}{r}\right)\psi = 0.$$

p 489. Second communication

H.P = Hamilton's P.D.E. Previously he had used "incomprehensible" $S = K \ln \psi$ and the equally incomprehensible $\delta\int\int\int = 0$.

Nothing new in associating mechanics with wave propagation. Hamilton knew it. Hamilton's principle, minimize $\int L\,dt$ corresponds to Fermat's Least Time.

The general problem of classical mechanics.

(1) $\partial W/\partial t + T(q, \frac{\partial W}{\partial q}) + V(q) = 0$

W is action function $\int (T-V)dt$ along an orbit, as function of time and final position. T is K.E. as function of position and momenta, quadratic in momenta, which are replaced by $\partial W/\partial q_k$. To solve it we put $W = -Et + S(q)$ and it becomes

$$^*2T\left(q, \frac{\partial W}{\partial q}\right) = 2(E-V) \qquad (1')$$

The statement (1') can be expressed very simply by an idea of Heinrich Hertz. Define a non-Euclidean metric in configuration space by

*Usually with S, instead of W. No difference.

W
10

Schrödinger $T = \frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)$ Potential V.

$\cancel{\frac{\partial W}{\partial t} + T(q_1^2+}$

$q_1 = \dot{x} \quad q_2 = \dot{y} \quad q_3 = \dot{z}$ momenta.

$$\frac{\partial W}{\partial t} + \frac{1}{2}(q_1^2+q_2^2+q_3^2) + V(x,y,z) = \cancel{E}\ 0$$

$$W = -Et + S(q)$$

$$\frac{1}{2}(q_1^2+q_2^2+q_3^2) = E - V.$$

$ds^2 = dx^2+dy^2+dz^2$ as usual.

$$\text{grad } W = \left(\frac{\partial W}{\partial x}, \frac{\partial W}{\partial y}, \frac{\partial W}{\partial z}\right)$$

$$q_1 = \frac{\partial W}{\partial x} \quad q_2 = \frac{\partial W}{\partial y} \quad q_3 = \frac{\partial W}{\partial z}$$

$$(\text{grad } W)^2 = 2(E-V)$$

$$\text{grad } W = \sqrt{2(E-V)}.$$

Along normal to $W = \text{const.}$ $dW = \sqrt{2(E-V)}\,ds$

So, to get to $W + dW$ surface we go

$$ds = \frac{dW}{\sqrt{2(E-V)}} \text{ along normal}$$

Try step fn.

$\overset{W}{\parallel}$

A particle moves in a plane under no force.

$V=0 \therefore T=E.$

$\therefore S = \int_0^t 2T\,dt = 2Tt = 2Et.$

$\frac{1}{2}mv_0^2 = E \therefore v_0 = \sqrt{\frac{2E}{m}}$

If it reaches distance r in time t

$r = v_0 t \therefore t = \frac{r}{v_0}$

$\therefore S = \frac{2Er}{v_0} = 2Er\sqrt{\frac{m}{2E}} = r\sqrt{2mE}$

If it is at (x,y) after time t,

$S = \sqrt{2mE}\sqrt{x^2+y^2}.$

$p_1 = \frac{\partial S}{\partial x} = \sqrt{2mE}\cdot\frac{x}{\sqrt{x^2+y^2}}$

$= m\sqrt{\frac{2E}{m}}\cos\theta = mv_0\cos\theta$
x-momentum

$p_2 = m\sqrt{\frac{2E}{m}}\sin\theta = \sqrt{2mE}\,\frac{y}{\sqrt{x^2+y^2}}.$

$W = S - Et = \sqrt{2mE}\sqrt{x^2+y^2} - Et.$

If we fix a value of W, surface having this value has equation

$W_0 = \sqrt{2mE}\sqrt{x^2+y^2} - Et.$

At time t, $r = \frac{W_0 + Et}{\sqrt{2mE}}$

$\frac{dr}{dt} = \frac{E}{\sqrt{2mE}} = \sqrt{\frac{E}{2m}}.$

$E = \frac{1}{2}mv_0^2$

$\frac{E}{2m} = \frac{v_0^2}{2}$ $\qquad \frac{dr}{dt} = \frac{1}{2}v_0.$

P.T.O.

W 12

A unit mass move in 2 dimensions, under a force numerically equal to the distance from the origin.

Thus $\ddot{r} = r$

$\therefore \quad r = Ae^{t} + Be^{-t}$

Initially $r_0 = 0 \quad \dot{r}_0 = v_0$

$\therefore \quad B = -A \quad r = A(e^{t} - e^{-t})$

$v_0 = A(e^{t} + e^{-t})_{t=0} = 2A.$

$\therefore \quad A = \frac{v_0}{2} \quad r = \frac{v_0}{2} \sinh t.$

$T = \frac{1}{2}\dot{r}^2 = \frac{v_0^2}{8}\cosh^2 t$

$S = \int_0^t 2T\,dt = \frac{v_0^2}{4}\int_0^T \cosh^2 t\,dt$

$\int_0^T \cosh^2 t\,dt = \int_0^T \left(\frac{e^{t} + e^{-t}}{2}\right)^2$

$= \frac{1}{4}\int_0^T e^{2t} + 2 + e^{-2t}\,dt.$

$= \frac{1}{4}\left[\frac{1}{2}e^{2t} - \frac{1}{2}e^{-2t} + 2t\right]_0$

$= \frac{1}{4}\sinh 2T + \frac{1}{2}T$ replace T by t.

$2\sinh t\cosh t = 2\left(\frac{e^{t} - e^{-t}}{2}\right)\left(\frac{e^{t} + e^{-t}}{2}\right) = \frac{1}{2}(e^{2t} - e^{-2t})$

$= \sinh 2t$

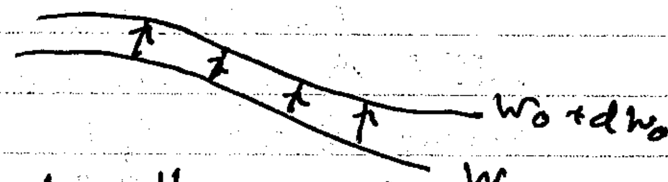
$\therefore$ If r is given, $t = \sinh^{-1}\frac{2r}{v_0}$

$S = \frac{1}{2}\frac{2r}{v_0}\sqrt{\frac{4r^2}{v_0^2} - 1} + \frac{1}{2}\sinh^{-1}\frac{2r}{v_0}$

It is easy to show the meaning of this requirement. Suppose we have a function $W = -Et + S(q)$ that satisfies this condition. We can represent this function at a definite time t by drawing the family W = constant, each surface being labelled with the value of W for it.

Given one surface and the value of W for it, we can successively construct the other surfaces with their values. Alternatively, we can choose any surface quite arbitrarily as the one from which the construction starts. The time is kept constant. This construction gives the total meaning of the P.D.E.

At each point of $W = W_0$ we go a distance



(4) $$ds = \frac{dW_0}{\sqrt{2(E-V)}}$$ along the normal. The point so reached form $W = W_0 + dW_0$. The initial direction may be either way. We get the same set of surfaces whichever we choose. The values run in opposite directions.

Now consider the simple dependence on time. We get the same set of surfaces at $t + t'$ as at t, but with a value of W on each less by Et'. So a fixed value moves from one surface to another in the direction of increasing t. We can equally well think of the surfaces moving, carrying their W values with them. The surface with W_0 at t moves, at time $t + dt$, to the surface that had $W = W_0 + Edt$ at time t. So $dW_0 = Edt$ and (5) $ds = \frac{Edt}{\sqrt{2(E-V)}}$. So surfaces have normal velocity (6) $u = \frac{ds}{dt} = \frac{E}{\sqrt{2(E-V)}}$ where E is given this is purely a function of position.

p497 W/14

§2 "Geometrical" and "Wave" Mechanics.

We assume the waves given by sine functions. We emphasize the arbitrariness of this fundamental assumption but it is the simplest and most plausible (naheliegendste). So the wave function is to involve time only in the form sin (...), the brackets containing a linear function of W. As we can only find sine of a pure number, W must have coefficient with dimension reciprocal of action. We assume this coefficient is universal, independent of E and of the mechanical system. Let it be $2\pi/h$. Then time factor reads

(10) $\sin(2\pi W/h + \text{const}) = \sin\left(-\frac{2\pi Et}{h} + \frac{2\pi S(q)}{h} + \text{const}\right)$

So frequency ν is given by

(11) $\nu = E/h$.

In classical mechanics E involves an arbitrary additive constant: here E is absolute. Wavelength, by (6) and (11) is

(12) $\lambda = u/\nu = h/\sqrt{2(E-V)}$.

(The classical added constant does not appear here, as $E - V$ = kinetic energy.)

[He shows that this wavelength is of the order of magnitude at which classical mechanics fails.]

(6') $u = h\nu/\sqrt{2(h\nu - V)}$

so we get dispersion: the velocity of phase propagation depends on the frequency. From (9), (11), (6') we find

(13) $v = d\nu/d(\nu/u)$

where v is the velocity of the representative point in (9).

p 509 The wave equation is not determined by what we know but it seems reasonable to consider a P.D.E. of second order $\nabla^2\psi - \frac{1}{u^2}\frac{\partial^2\psi}{\partial t^2} = 0$ (18)

valid for processes in which time factor is $e^{2\pi i \nu t}$.
In view of (6), (6'), (11)

(18') $\nabla^2\psi + \frac{8\pi^2}{h^2}(h\nu - V)\psi = 0$

(18'') $\nabla^2\psi + \frac{8\pi^2}{h^2}(E - V)\psi = 0.$

(He uses div grad instead of ∇^2, in metric defined by (3).)

A reference to earlier paper in which
$$U'' + (\delta_0 + \frac{\delta_1}{r})U' + (\varepsilon_0 + \frac{\varepsilon_1}{r})U = 0$$
is solved by Laplace transformation

$$U = \int_L e^{rz}(z - c_1)^{\alpha_1 - 1}(z - c_2)^{\alpha_2 - 1}dz$$

where path L in complex must make

$$\int_L \frac{d}{dz}\left[e^{rz}(z-c_1)^{\alpha_1}(z-c_2)^{\alpha_2}\right]dz = 0$$

c_1 and c_2 are roots of quadratic $z^2 + \delta_0 z + \varepsilon_0 = 0$

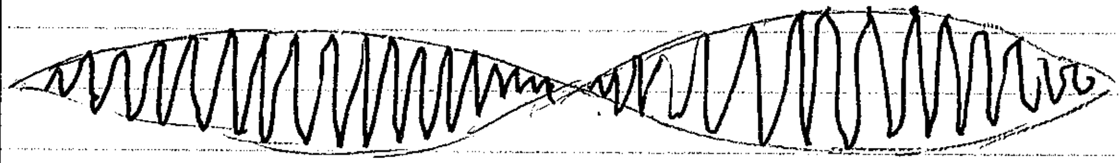
and $\alpha_1 = \frac{\varepsilon_1 + \delta_1 c_1}{c_1 - c_2}$ $\alpha_2 = \frac{\varepsilon_1 + \delta_1 c_2}{c_2 - c_1}$.

Group velocity and phase velocity.

Let $y = \cos 2\pi(x - t) + \cos 2\pi(1.02x - 1.06t)$

$= 2\cos 2\pi(1.01x - 1.03t)\cos 2\pi(.01x - .03t)$

$\cos 2\pi(1.01x - 1.03t)$ repeats when x increases by $1/1.01$; $\cos 2\pi(.01x - .03t)$ when x increases by 100. Thus the graph is much like a modulated radio signal, a slow variation of amplitude



The second factor essentially determines the appearance of the process. Its maximum occurs (for example) where $.01x - .03t = 0$ i.e. $x = 3t$, and so moves with velocity 3. (group velocity)

$\cos 2\pi(x-t)$ and $\cos 2\pi(1.02x - 1.06t)$ have peaks that move with velocity $\doteqdot 1$. (phase velocity)

It will be seen that we can arrange for phase velocity to remain 1, and have any group velocity we may fancy.

Consider now $\cos 2\pi(\alpha x - \nu t) + \cos 2\pi((\alpha + \delta\alpha)x - [\nu + \delta\nu]t)$

$= 2\cos[(\alpha + \frac{\delta\alpha}{2})x - (\nu + \frac{\delta\nu}{2})t]\cos[\frac{\delta\alpha}{2}x - \frac{\delta\nu}{2}t]$

As before, the position of the group maximum depends on the behaviour of the second factor. It has a maximum where $\frac{\delta\alpha}{2}x - \frac{\delta\nu}{2}t = 0$ i.e. $x = \frac{d\nu}{d\alpha}t$

so the group velocity is $\frac{d\nu}{d\alpha}$.

The magnitude of this will depend on the way ν is related to α.

p500 W 17

A wave group, restricted to a certain small region, goes a substitute for a particle in the classical theory. Such groups can be built up by the construction that Debye and Laue used in ordinary optics to obtain the exact analytic representation of a cone or bundle of rays. There is an interesting connection with the Jacobi-Hamilton theory by which the integrated eqns of motion can be obtained by differentiating the constants in H-J eqn. We shall see that this agrees with the wave result, the moving point coincides with the place where a certain continuum of waves are in phase.

Debye's method is to superpose plane waves, each of which fills the whole space. In fact, superpose a continuum of such waves with the normals varying inside a certain cone. The waves cancel each other by interference almost completely outside a double cone. It is possible to use an infinitesimal solid angle for the cone. v. Laue used this in his celebrated paper (1914). Tacitly we have supposed the waves monochromatic. By varying frequencies ~~suitably we ca~~ suitably within an infinitesimal range we can get a domain that is relatively small longitudinally. The portion of the energy packet is the place where all the superposed waves are exactly in phase.

p501 In q-space we choose a point P, a direction R and a prescribed mean frequency ν (energy E) at time t.

We take a soln W of H.P(1')

$$\frac{\partial W}{\partial t} + T\left(q, \frac{\partial W}{\partial q}\right) + V(q) = 0$$

Let $W = W_0$ be the surface through P with normal R at time t.

$P = (q_1 \ldots q_n)$. We vary W infinitesimally so as to give $(n-1)$ dimensional solid angle for normals, and an infinitesimal 1-dimensional domain for the frequency E/h. We must make all of them have same phase at P. We then have to show where this coincidence of phases occurs at a later time.

The soln W must be of form $W(\alpha_1 \ldots \alpha_n)$. Take $\alpha_1 = E$. We suppose $\alpha_2 \ldots \alpha_n$ so chosen that normal at P has direction R. We suppose $\alpha_1 \ldots \alpha_n$ have these values throughout

(15) $W + \frac{\partial W}{\partial \alpha_1} d\alpha_1 + \ldots \frac{\partial W}{\partial \alpha_n} d\alpha = \text{constant}$

is taken with fixed $d\alpha_1 \ldots d\alpha_n$ and constant varying gives a family of surfaces. The surface that goes through P at time t is given by

(15') $W + \frac{\partial W}{\partial \alpha_1} d\alpha_1 + \ldots \frac{\partial W}{\partial \alpha_n} d\alpha_n = W_0 + \left(\frac{\partial W}{\partial \alpha_1}\right)_0 d\alpha_1 + \ldots \left(\frac{\partial W}{\partial \alpha_n}\right)_0 d\alpha_n$

The surfaces (15'), for all possible $d\alpha_i$ form a family. At time t they all go through P, their normals fill an infinitesimal solid angle, and their E-value varies in a small domain. ~~E~~ (15') picks out one representation of each family (15), namely the one through P at time t. We suppose the phases of the wave functions in (15) agree with those of the representation (15').

At an arbitrary time, will there be a place where all ~~the phases~~ the surfaces (15') meet and so all wave fns in (15) agree in phase?

(p 503) The answer: there is a point where phases agree, but it is not the common point of (15'), for such a point never exists again. Rather the point of agreeing phases comes about by the families (15) continually change their representative in (15').

We see this as follows.
For the common point of intersection of all members of (15'') at any time we must have simultaneously

$$(16)\quad \begin{cases} W = W_0 \quad \partial W/\partial \alpha_1 = (\partial W/\partial \alpha_1)_0 \;\cdots\quad \dfrac{\partial W}{\partial \alpha_n} = \left(\dfrac{\partial W}{\partial \alpha_n}\right)_0 \end{cases}$$

as $d\alpha_i$ are arbitrary within a small domain.
RHS are constants, LHS functions of $n+1$ quantities t $q_1 \ldots q_n$. The eqns are satisfied for the initial conditions, coords of P and time t.
Proceed as follows. Leave $W = W_0$ aside and find $q_1 \ldots q_n$ as fns of time from n other eqns. Let this give Q. For it, naturally, the first eqn will not be satisfied, but RHS and LHS will differ by a certain amount. If we go back from (16) to its origin from (15') we see ~~that RHS differ from LHS of (15') change by a number that is the same for all families~~ Q is not a common point of (15'), but is for (15'') which arises by changing RHS of 15' by a fixed number, the same for all surfaces of the family. This comes by changing the constant in (15). by the same amount for all representatives. Thus the phase angle of all representatives is changed by the same amount. As the old representatives agreed in phase, the new must do the same. This means, the point given by the n eqns.

$$\frac{\partial W}{\partial \alpha_1} = \left(\frac{\partial W}{\partial \alpha_1}\right)_0 \qquad \frac{\partial W}{\partial \alpha_n} = \left(\frac{\partial W}{\partial \alpha_n}\right)_0 \qquad (17)$$

as a fn of time is always a point where phases agree.
Of the n surfaces, giving Q as point of intersection for (17), only the first is movable. (only the first eqn involves the time.) The other n-1 eqns determine the orbit of Q. It is easy to see that this orbit is orthogonal to the family W = constant. By the assumption, W satisfies the H. P (1') $2T(q, \partial W/\partial q) = 2(E-V)$ identically in $\alpha_1 \ldots \alpha_n$. Differentiating H.P wrt α_k ($k = 2 \ldots n$) we get the result that the normal of a surface $\partial W/\partial \alpha_k = $ const at each point of the surface is $\perp$ to the normal of the surface W = const. through this point: i.e. that each surface contains the normal to the other. Hence intersection is orthogonal trajectory to surfaces W = const.
(17) is known to give the orbits for mechanical problems

Ruark & Urey.

Hamilton's Principle. $\delta \int L\,dt = 0$

$$S = \int 2T\,dt \qquad 2T = \sum p_k \dot{q}_k$$

$$S = \int p_k\,dq_k$$

$$p_k = \frac{\partial S}{\partial q_k}$$

$$W = -Et + S$$
$$= -\int E\,dt + \int 2T\,dt$$
$$= \int 2T - E \,.\, dt$$
$$= \int T - (E - T)\,dt$$
$$= \int T - V\,dt = \int L\,dt.$$

Hamilton Principal Function.

W is a function of final position and time.

$W = \int L\,dt.$ in terms of final position and time taken.

Lagrange's Equations. $L = T - V$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

express that $\delta \int_{t_0}^{t_1} L\,dt = 0$.

Thus, if $W = \int_{t_0}^{t_1} L\,dt$, the orbit goes from initial t_0 ~~finale~~ final position by the path that makes W a minimum. Thus 'least W' replaces Fermat's Least Time.

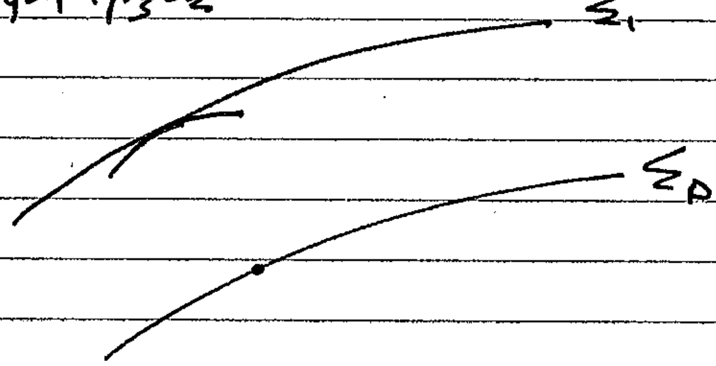
Action $S = \int_{0}^{t_2} 2T\,dt$.

$$W = \int_{t_0}^{t_2} (T - V)\,dt = \int_{t_0}^{t_2} T - (E - T)\,dt$$

$$= \int_{0}^{t} 2T - E\,.\,dt = \int_0^t 2T\,dt - Et$$

$$= S - Et.$$

Action $= \int 2T\,dt = \int m\dot{x}^2 + m\dot{y}^2 + m\dot{z}^2\,dt = \int m\dot{x}\,dx + m\dot{y}\,dy + m\dot{z}\,dz$
$= \int p_x dx + p_y dy + p_z dz$



$(x_0\ y_0\ z_0)$ on surface Σ_0
$S(x_1\ y_1\ z_1\ x_0\ y_0\ z_0) = k$
is equation of constant action, touching Σ_1 at $x_1 y_1 z_1$
~~the~~ Let $l_1\ m_1\ n_1$ be normal to Σ_1 at x_1, y_1, z_1. As surfaces touch there

$$\frac{\partial S}{\partial x_1} = \kappa l_1 \quad \frac{\partial S}{\partial y_1} = \kappa m_1 \quad \frac{\partial S}{\partial z_1} = \kappa n_1 \text{ for some } \kappa.$$

Now suppose we consider path of moving object from $(x_1\ y_1\ z_1)$ to $(x_1 + \delta x_1,\ y_1 + \delta y_1,\ z_1 + \delta z_1)$. This is along normal at $(x_1\ y_1\ z_1)$ so $\delta x_1 = l_1 ds$ $\delta y_1 = m_1 ds$ $\delta z_1 = n_1 ds$. The change in action is

$$p_x \cdot l_1 ds + p_y m ds + p_z \cdot n ds$$

This must equal
$$\delta S = \frac{\partial S}{\partial x_1}\delta x_1 + \frac{\partial S}{\partial x_2}\delta x_2 + \frac{\partial S}{\partial x_3}\delta x_3$$
$$= \kappa l_1^2 ds + \kappa m_1^2 ds + \kappa n_1^2 ds$$
$$= \kappa ds.$$

$$\therefore\ \kappa = p_x l_1 + p_y m_1 + p_z n_1$$

Now $p_x = p l_1$ $p_y = p m_1$ $p_z = p n_1$ where $p = \sqrt{p_x^2 + p_y^2 + p_z^2}$

$$\therefore\ \kappa = p(l_1^2 + m_1^2 + n_1^2) = p.$$

$$\therefore\ \frac{\partial S}{\partial x_1} = p l_1 = p_x, \text{ similarly } \frac{\partial S}{\partial y_1} = p_y \quad \frac{\partial S}{\partial z_1} = p_z.$$

We can show in the same way $\frac{\partial S}{\partial x_0} = -p_{x0}$ etc.
The minus sign is due to the fact that path of integration is reduced when its starting point moves forward.

Whittaker. Analytical Dynamics. Chap XI.

Fermat's principle $\int \mu(x,y,z)\,ds$ stationary

Principle of Least Action asserts $\int \{E - \varphi(x,y,z)\}^{1/2} ds$ stationary

$\varphi(x,y,z)$ potential energy, E energy.

The trajectories of the particle in the dynamical problem coincide with the rays in the optical problem.

Huygens 1690. $V(x\,y\,z\;x'y'z')$ the time taken by light to travel from (x,y,z) to $(x'y'z')$

Secondary waves.

Disturbance occurring at $(x\,y\,z)$ at time t, at t' will extend over surface with eqn $\int$ in co-ords $x'y'z'$?

is $V(x\,y\,z\;x'y'z') = t'-t$. (1)

Wave front σ at time t, Σ at t'.

Normal to σ at (x,y,z) (l,m,n)

to Σ at $x'y'z'$ $(l'm'n')$

As Σ is envelope of surfaces V corresponding to point on σ

$$\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = 0$$

must be satisfied by all those values of dx, dy, dz that correspond to direction in tangent plane to σ i.e. for which $l\,dx + m\,dy + n\,dz = 0$

$$\therefore \frac{1}{l}\frac{\partial V}{\partial x} = \frac{1}{m}\frac{\partial V}{\partial y} = \frac{1}{n}\frac{\partial V}{\partial z} \qquad (2)$$

$(l'm'n')$ being normal to surface V at $(x'y'z')$ we have

$$\frac{1}{l'}\frac{\partial V}{\partial x'} = \frac{1}{m'}\frac{\partial V}{\partial y'} = \frac{1}{n'}\frac{\partial V}{\partial z'} \qquad (3)$$

A ray of light passing through $(x\,y\,z)$ in direction (l,m,n) at t will pass through $(x'y'z')$ in direction $(l'm'n')$ at time t'

$$l'^2 + m'^2 + n'^2 = 1 \qquad (4)$$

Eqns (1), (2), (3), (4) are 6 eqns, suff to determine $x'y'z'l'm'n'$ in terms of $x\,y\,z\,l\,m\,n$. So the single function $V(x\,y\,z\;x'y'z')$ completely determines the behavior of light (from obvious from geometrical aspect)

These are not differential eqns. They explicitly give the quantities in question.

$V(x\,y\,z\,x'y'z')$ defines a contact transformation

Action function $S = \int_{t_0}^{t_1} 2T\,dt.$

Hamilton's Principal function $W = -Et + S$

$$W = \int 2T\,dt - \int E\,dt$$

$$= \int 2T - E \,.\, dt = \int 2T - (T+V) \,.\, dt$$

$$= \int T - V\,dt = \int L\,dt.$$

Hamilton's Principle. $\int L\,dt$ is stationary for actual orbit when small deviations considered

Schrödinger emphasizes that S is to be expressed in terms of the co-ordinates.

Simplest example Particle in plane. $V=0$. Then $T=E$

If particle starts at origin at time $t=0$, at time t_1

$$S = \int_0^{t_1} 2T\,dt = 2Et_1.$$

To reach (x,y) at time t_1 particle must have $v_0 t_1 = \sqrt{x^2+y^2}$ $\quad E = \frac{1}{2} m v_0^2 \;\therefore\; v_0 = \sqrt{\frac{2E}{m}}$

$$\therefore\; t_1 = \sqrt{\frac{m}{2E}}\sqrt{x^2+y^2} \;\therefore\; S = 2Et_1 = \sqrt{2mE}\sqrt{x^2+y^2}$$

$$W = -Et + \sqrt{2mE}\sqrt{x^2+y^2}$$

The surface $W=0$ at time t has

$$\sqrt{2mE}\sqrt{x^2+y^2} = Et$$

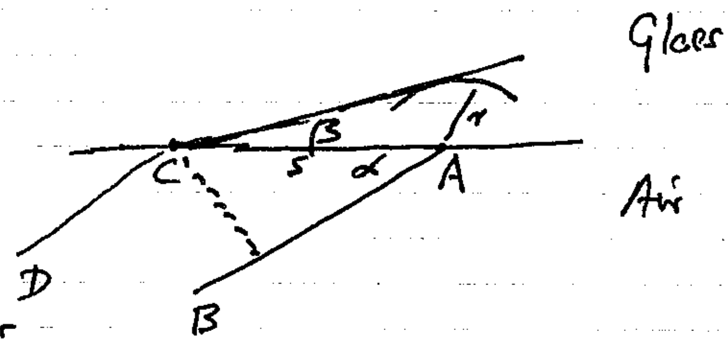
$$\therefore\; \sqrt{x^2+y^2} = \sqrt{\frac{E}{2m}}\,t$$

$$S = \sqrt{2mE}\sqrt{x^2+y^2} = mv_0\sqrt{x^2+y^2}$$

$$\frac{\partial S}{\partial x} = \frac{mv_0 x}{\sqrt{x^2+y^2}} \qquad \frac{\partial S}{\partial y} = \frac{mv_0 y}{\sqrt{x^2+y^2}}$$

These are the components of the momentum parallel to the axes.

Optics: refraction.



AB is wavefront at a certain time.
CD later.
u velocity of light in air
v - - - - - glass

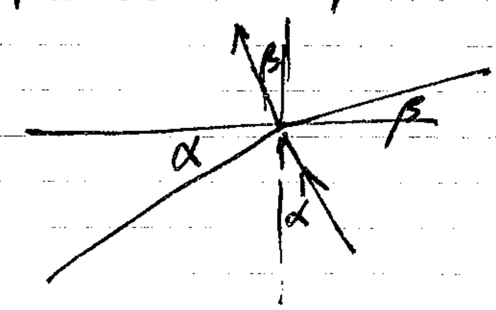
If $AC = s$, the wave has moved through a distance $s \sin\alpha$ as it goes from AB to CD
Time $= \dfrac{s \sin\alpha}{u}$.
When it has just reached C, the disturbance has had this time to spread out from A.
Moving with velocity v it can go $\dfrac{s \sin\alpha}{u} v = r$

$$\therefore \quad \frac{r}{s} = \frac{v}{u}\sin\alpha \quad \text{which is constant}$$

Thus $\quad \sin\beta = \dfrac{r}{s} = \dfrac{v}{u}\sin\alpha$

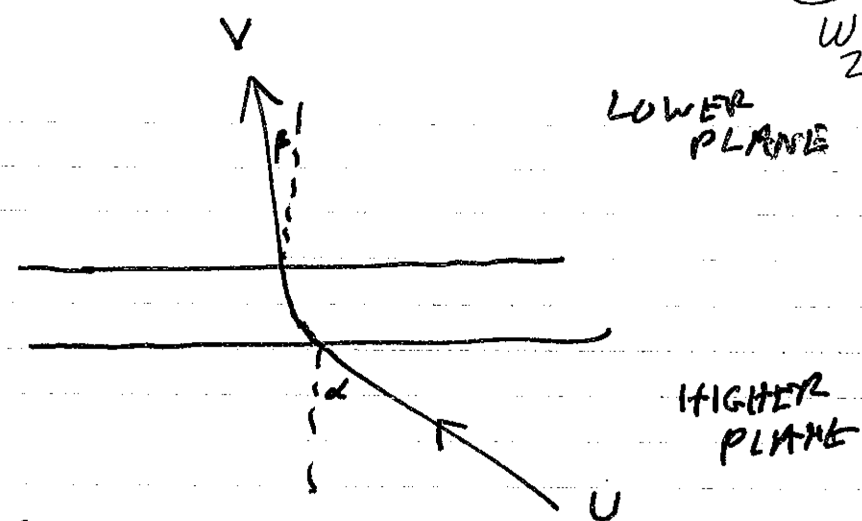
So $\quad \dfrac{\sin\beta}{\sin\alpha} = \dfrac{v}{u}$.

Rays are perpendicular to plane wave fronts



Thus α, β are angles between original ray, and refracted ray and the normal to interface.

Note that, on wave theory, light travels more slowly in glass.



Particle Theory

Imagine a green on a golf course that has two level parts with a slope between.

We suppose the slope has the same cross section everywhere, so that it has no tendency to divert the ball to left or right. Let h be the difference in height of the parts.

Momentum is conserved in x-direction

$$U \sin\alpha = V \sin\beta$$

From energy $\frac{1}{2} m V^2 = \frac{1}{2} m U^2 + mgh$

$$\therefore \quad V^2 = U^2 + 2gh.$$

$$\frac{\sin\beta}{\sin\alpha} = \frac{U}{V}$$

Now $\frac{V^2}{U^2} = 1 + \frac{2gh}{U^2}$ so $\frac{V}{U}$ will differ from a constant unless U is constant. If U is constant, V will be. To get the known law, we have to assume that all rays of light travel at a rate, which is characteristic of the medium.

The velocity in glass would have to be higher than in air.

Cabbar dos 45
we want 34 I=2-2

③

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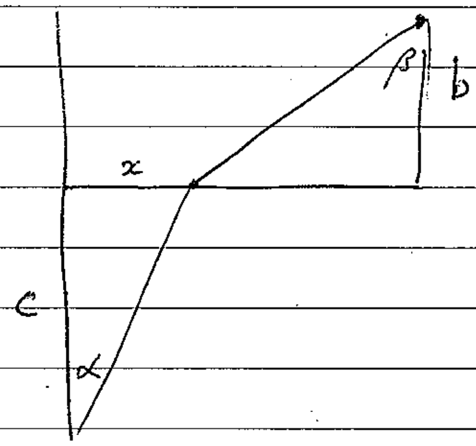
28



Time

$$= \frac{\sqrt{c^2+x^2}}{u} + \frac{\sqrt{(a-x)^2+b^2}}{v}$$

$$0 = \frac{dt}{dx}$$

$$0 = \frac{x}{u\sqrt{c^2+x^2}} - \frac{a-x}{\sqrt{(a-x)^2+b^2}}$$

$$\frac{x}{\sqrt{c^2+x^2}} = \sin\alpha \qquad \frac{a-x}{\sqrt{(a-x)^2+b^2}} = \sin\beta$$

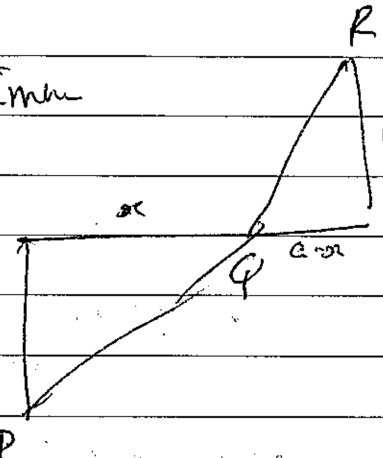
$$\therefore \; 0 = \frac{\sin\alpha}{u} - \frac{\sin\beta}{v} \text{ so } \frac{\sin\beta}{\sin\alpha} = \frac{v}{u}$$ as required.

(6)

W 29

Least Action $\int T\,dt$ is a minimum

$$\int \tfrac{1}{2} U^2\,dt = \int \tfrac{1}{2} U\,ds$$



so $\int U\,ds + \int V\,ds$ is

to be a minimum

i.e. $U.PQ + V.QR$ a minimum

$$U\sqrt{x^2+c^2} + V\sqrt{(a-x)^2+b^2}$$

This is the same as before, but with

U for $\frac{1}{u}$, V for $\frac{1}{v}$.

so gives $\dfrac{\sin\beta}{\sin\alpha} = \dfrac{1/V}{1/U} = \dfrac{U}{V}$.

Hamilton. page A3 in Saturday file.

The role of phase is played by W

$$W = \int_0^t (T-V)\,dt = \int_0^t T-(E-T)\,dt = \int_0^t 2T-E.\,dt$$

$$= \int_0^t 2T\,dt - Et = S - Et.$$

$$\delta \int_{t_0}^{t_1} L(q, \dot{q}, t)\,dt$$

$$= \int_{t_0}^{t_1} \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} + \cancel{\frac{\partial L}{\partial t}}\delta t$$

$$= \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_{t_0}^{t_1}$$

$$\frac{\partial L}{\partial \dot{q}_{s_1}}\delta q_{s_1} - \frac{\partial L}{\partial \dot{q}_{s_0}}\delta q_{s_0}$$

$$\frac{\partial S}{\partial q_s} = \frac{\partial L}{\partial \dot{q}_s} = p_s$$

Q1
W 31

Early quantum theory.

1900 Planck Radiation of frequency ν associated with packets of energy. $h\nu$.

1905 Einstein shows this in agreement with observed facts for photoelectric effect.

Classically an accelerated electron radiates and thus loses energy. Electrons would spiral down into nucleus and disappear. Impossible on this basis to explain discrete lines in the spectrum.

Bohr 1913 made revolutionary proposal.
(i) most of the time electrons do not radiate energy, but remain in stable orbits. (ii) Radiation occurs when an electron suddenly passes from one such orbit to another.
(iii) If the orbits have energies E_n and E_m, the frequency of the radiation is given by $\nu = \frac{E_n - E_m}{h}$.

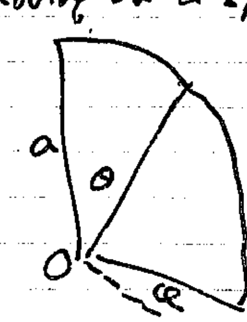
Conjugate quantities.

A theory known as general dynamics gives results that are not confined to a particular problem, but apply (with certain restrictions) to all dynamical systems.

The uncertainty principle relates two quantities known as conjugate: the more precisely one is measured, the less information there can be about the other. The most familiar example is position, x, and momentum $m\dot{x}$. Unfamiliar examples constitute an immense family: the conjugate quantities are a generalized co-ordinate and its generalized momentum.

Examples of generalized co-ordinates:
r, θ in polar co-ordinates
θ, φ for double pendulum
Latitude and longitude for particle moving on a sphere.

For a particle constrained to move on a sphere (say by a thread joining it to point O), it is often convenient to use the longitude, φ, and θ, the latitude subtracted from a right angle, so $\theta = 0$ gives North Pole, $\theta = \frac{\pi}{2}$ the equator.



The kinetic energy is given by T,

$$T = \frac{1}{2} m \left(a^2 \dot{\theta}^2 + a^2 \sin^2\theta \, \dot{\varphi}^2 \right)$$

The momentum conjugate to θ is given by

$$\frac{\partial T}{\partial \dot{\theta}} = m a^2 \dot{\theta}$$

The momentum conjugate to φ is

$$\frac{\partial T}{\partial \dot{\varphi}} = m a^2 \sin^2\theta \, \dot{\varphi}$$

which we can recognize as the angular momentum about ON, where N is the North pole.

The simplest example is for a single mass.

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial T}{\partial \dot{y}} = m\dot{y}, \quad \frac{\partial T}{\partial \dot{z}} = m\dot{z}$$

the three components of momentum.

For a rotating flywheel, $T = \frac{1}{2} I \dot{\theta}^2$ and the angular momentum is $\frac{\partial T}{\partial \dot{\theta}} = I\dot{\theta}$.

There is a perfect analogy here with a mass moving in 1 dimension. $T = \frac{1}{2} m\dot{x}^2$, $\partial T/\partial \dot{x} = m\dot{x}$.

In 1788 Lagrange showed that what happened in a system was completely determined by the form of the expression for T, the kinetic energy and V, the potential energy.

If, for example, the position of a system is determined by 3 generalized co-ordinates, q_1, q_2, q_3, the equations of motion for the system are, with $L = T - V$,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) = \frac{\partial L}{\partial q_1}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) = \frac{\partial L}{\partial q_2}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_3}\right) = \frac{\partial L}{\partial q_3}.$$

There is an unbroken chain from Lagrange's work in 1788, through Hamilton's (around 1835) work, showing a method equally applicable to optics and mechanics, to

23.4.98

Q3

W
33

Schrödinger's wave mechanics, 1926

In Hamilton's work it was found convenient to work with $p_r\ q_r$ rather than with $\dot{q}_r\ q_r$.

The energy equation for a mass in 3-D is

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2 + V(x,y,z) = E$$

As $p_1 = m\dot{x}$ $p_2 = m\dot{y}$ $p_3 = m\dot{z}$, the energy equation takes the form

$$\frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{p_3^2}{2m} + V = E.$$

Schrödinger's rule for writing the wave equation is to replace p_1 by $+\frac{\hbar}{i}\frac{\partial}{\partial x}$ etc.

Then we have

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) + V\psi = E\psi.$$

[There is a more general method, depending on the Hamilton-Jacobi equation, that can be used when kinetic energy is not simply a sum of squares. See Rojansky & Corey, p548.]

For simple harmonic oscillator

$T = \frac{1}{2} m \dot{x}^2 \qquad V = \frac{1}{2} k x^2.$

This leads to $\frac{p^2}{2m} + \frac{1}{2} k x^2 = E$

and so to $-\frac{\hbar^2}{2m} \psi'' + \frac{1}{2} k x^2 = E$

$$\text{so} \quad \psi'' + \frac{2m}{\hbar^2}\left(E - \frac{1}{2} k x^2\right)\psi = 0.$$

Let $x = av$ and choose a so that eqn. takes form $\psi_{vv} + (b - v^2)\psi = 0.$

With $x = av$, $\psi'' = \frac{1}{a^2}\psi_{vv}$

Coeff of ψ is $-\frac{2m}{\hbar^2} \cdot \frac{1}{2} k a^2$

So we want $\frac{1}{a^2} = \frac{mk}{\hbar^2} a^2$

$$a^4 = \frac{\hbar^2}{mk}$$

The equation becomes

$$\psi_{vv} + \left(\frac{2ma^2}{\hbar^2} E - v^2\right)\psi = 0$$

$$\text{i.e.} \quad \psi_{vv} + \left(\frac{2E}{\hbar}\sqrt{\frac{m}{k}} - v^2\right)\psi = 0.$$

Now, for SHM, $x = \sin \omega t$ and so, from $m\ddot{x} + kx = 0$ $\quad \omega^2 = k/m$

Thus the constant in the bracket is $\frac{4\pi E}{h\omega}$

$\sin \omega t$ has the ~~peri~~ frequency $\omega / 2\pi$

so $\omega = 2\pi \nu_0$ where ν_0 is the frequency associated with the system.

Thus the equation takes the form

$$\psi_{vv} + \left(\frac{2E}{h\nu_0} - v^2\right)\psi = 0.$$

SHM2
W
35

We consider the equation $\varphi'' + (A - x^2)\varphi = 0$.

~~For large x, this resembles $\varphi'' - x^2\varphi = 0$.~~

~~If $\varphi = e^{\alpha x} u$ we get~~

Let $\varphi = e^{\alpha x^2} u$. Then $\varphi' = e^{\alpha x^2} u' + 2\alpha x . e^{\alpha x^2} u$

$\varphi'' = e^{\alpha x^2} u'' + 4\alpha x e^{\alpha x^2} u' + u(2\alpha e^{\alpha x^2} + 4\alpha^2 x^2 e^{\alpha x^2})$

So, removing the factor $e^{\alpha x^2}$ we have

$$0 = u'' + 4\alpha x u' + u(A + 2\alpha + (4\alpha^2 - 1)x^2)$$

So we take $4\alpha^2 = 1$, $\alpha = \pm\frac{1}{2}$.

We require φ to be bounded for all values of x, which will not be the case if it involves $e^{\frac{1}{2}x^2}$. So we take $\alpha = -\frac{1}{2}$.

Then $0 = u'' - 2xu' + u(A - 1)$

If $u \sim cx^\rho$ for small x, we find RHS $\sim cx^{\rho-2}[\rho(\rho-1)]$. There is no other term to cancel this, so $c\rho(\rho-1) = 0$

If $c \neq 0$ we must have $\rho = 0$ or $\rho = 1$.

Let $u = \sum_{n=0}^{\infty} a_n x^n$ ($a_0 = 0$ is allowed

Substitute in the differential eqn.

$$0 = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} 2na_n x^n + (A-1)\sum_0^{\infty} a_n x^n$$

From the coefficient of x^n

$$0 = (n+2)(n+1)a_{n+2} + a_n(A - 1 - 2n).$$

$$\therefore \quad a_{n+2} = \frac{2n - A + 1}{(n+1)(n+2)} a_n.$$

If $a_0 = 1$, only even terms will occur, so $n = 2r$ an $a_{2r+2}x^{2r+2}$ is $(r+1)^{th}$ term, c_{r+1}

So $$c_{r+1} = \frac{4r - A + 1}{(2r+1)(2r+2)} c_r x^2$$

$$\frac{c_{t+1}}{c_t} = \frac{4r - A + 1}{(2r+1)(2r+2)} x^2 \sim \frac{x^2}{t}$$

Now e^{x^2} has $c_t = \frac{x^{2t}}{t!}$ $c_{t+1} = \frac{x^{2t+2}}{(t+1)!}$

so $\frac{c_{t+1}}{c_t} = \frac{x^2}{t+1} \sim \frac{x^2}{t}$

This suggests that $u \sim e^{x^2}$, so $\psi = e^{-\frac{1}{2}x^2} u$ behaves like $e^{-\frac{1}{2}x^2} e^{x^2} = e^{\frac{1}{2}x^2}$, which $\to \infty$ as $x \to +\infty$ or $x \to -\infty$, so is not acceptable.

The only way out of this is if the series for u is a polynomial, not an infinite series.

If $a_{n+2} = 0$ but $a_n \neq 0$ we have $2n - A + 1 = 0$ i.e. $A = 2n + 1$.

Now $A = \frac{2E}{h\nu_0}$. $\frac{2E}{h\nu_0} = 2n + 1$

means $E = (n + \frac{1}{2}) h\nu_0$.

Thus particular discrete energies are picked out as being possible.

Note that $E = 0$ is not possible. This is a difference from early quantum theory. The changes in E are still multiples of $h\nu_0$!

Schrodinger's Equation in One Dimension *

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi.$$

Let $\Psi = X(x)T(t)$ — $X(x)$ function of x, $T(t)$ function of t (not KE).

Then $i\hbar X T' = -\frac{\hbar^2}{2m} X'' T + V X T$

Divide by XT

$$i\hbar \frac{T'}{T} = -\frac{\hbar^2}{2m}\frac{X''}{X} + V$$

We take $T = e^{+Et/\hbar i}$

Then $i\hbar\, T'/T = i\hbar \frac{d}{dt} \ln T = i\hbar \frac{d}{dt}\left(+\frac{Et}{\hbar i}\right)$

$$= +E$$

Then $E = -\frac{\hbar^2}{2m}\frac{X''}{X} + V$

and $\frac{\hbar^2}{2m} X'' + (E-V)X = 0$

We usually take ψ for X

$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (E-V)\psi = 0.$$

* Schrödinger used $\frac{\partial^2 \Psi}{\partial t^2}$, not $\frac{\partial \Psi}{\partial t}$. Both accounts lead to $T = e^{Et/\hbar i}$.

Force free field. $V=0$. $E = K.E.$ positive.

$$\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0$$

Let $\frac{2mE}{\hbar^2} = k^2$

Then $\frac{d^2\psi}{dx^2} + k^2\psi = 0.$

Solutions e^{ikx} and e^{-ikx}.

Then $\Psi_1 = e^{ikx - iEt/\hbar}$ or $\Psi_2 = e^{-ikx - iEt/\hbar}$

Ψ_1 takes a constant value if $ikx - iEt/\hbar$ is constant. If the value is zero $x = \frac{E}{k\hbar}t$

i.e the wave front moves to the right and we think of this as representing an electron moving to the right. $|\Psi_1| = 1$

So the electron is equally likely to be anywhere.

In the same way, Ψ_2 represents an electron moving to the left, equally likely to be anywhere.

The momentum operator is $p = \frac{\hbar}{i}\frac{\partial}{\partial x}$

For Ψ_1, $\frac{\hbar}{i}\frac{\partial \Psi_1}{\partial x} = \frac{\hbar}{i} ik\, e^{ikx - iEt/\hbar}$

$= \hbar k \Psi_1$

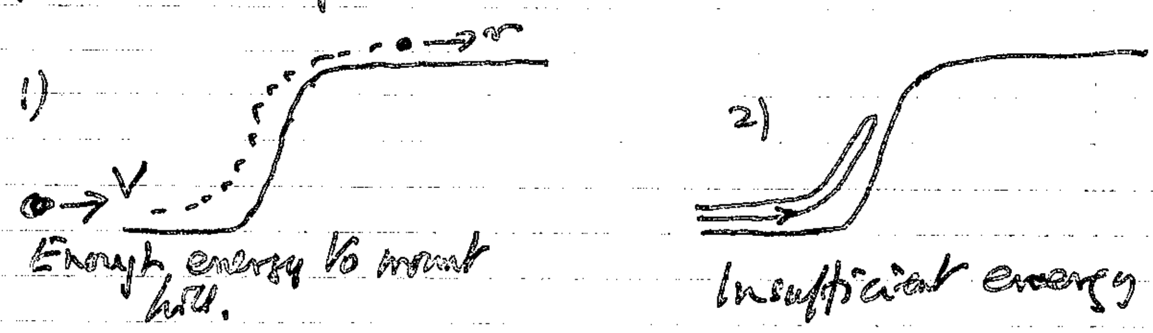
Ψ_1 is an eigenvector for $\frac{\hbar}{i}\frac{\partial}{\partial x}$.

This is interpreted to mean that the momentum has a definite value.

Thus for Ψ_1 the momentum is exactly known, the position absolutely unknown.

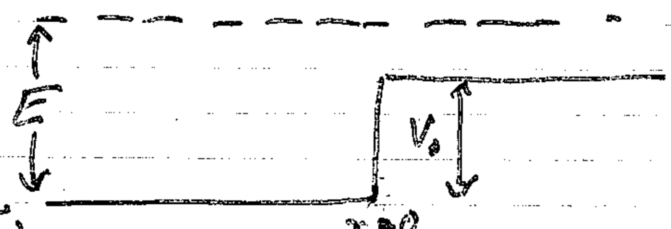
The same is true for Ψ_2.

Potential Step.



What happens with wave mechanics differs in both cases.

We are going to neglect the transition period, so our energy diagram, for case (1) will be



$\boxed{\text{Case 1}}$ $\psi'' + \frac{2m}{\hbar^2}[E - V(x)]\psi = 0.$

Write $\frac{2mE}{\hbar^2} = k^2$ $\frac{2m(E-V_0)}{\hbar^2} = q^2.$

For $x<0$, $V(x)=0$, eqn is $\psi'' + k^2\psi = 0$

with general solution $\psi(x) = Pe^{ikx} + Re^{-ikx}$

If V_0 were zero, R would be zero, and $\psi = e^{ikx}$ would represent a wave proceeding unhindered.

The wave does in fact proceed unhindered in the region $x>0$. Here $\psi'' + q^2\psi = 0$. e^{-iqx} would represent a wave moving left. As there is no source of electrons to the right, we do not use this solution. Thus for $x>0$

$$\psi = Ae^{iqx}.$$

We require ψ and ψ' to be continuous.

$x < 0$	$x > 0$
$\psi = Pe^{ikx} + Re^{-ikx}$	Ae^{iqx}
$\psi' = ik(Pe^{ikx} - Re^{-ikx})$	$iqAe^{iqx}$

For $x=0$ we have

$$P + R = A$$

$$P - R = \frac{q}{k}A.$$

$$\therefore P = A\left(\frac{q+k}{2k}\right) \quad R = A\frac{k-q}{2k}$$

As $q < k$ in case (1) it is impossible for $R=0$.
There is bound to be a reflected wave.

The incoming wave is Pe^{ikx}. If we assume $P=1$, so e^{ikx} coming in, $A = \frac{2k}{q+k}$ and accordingly $R = \frac{k-q}{k+q}$. The transmitted wave is given by $\psi = \frac{2k}{q+k}e^{iqx}$

$\boxed{\text{Case 2}}$ Insufficient energy to reach top of hill.

$E - V_0$ is negative

Let $\frac{2m(E-V_0)}{\hbar^2} = -b^2$. $(b>0)$

Then, for $x>0$, $\psi'' - b^2\psi = 0$.

Solution, e^{bx} and e^{-bx}. $e^{bx} \to +\infty$, no good.

So $\psi = Ce^{-bx}$ for $x>0$.

	$x < 0$	
ψ:	$Pe^{ikx} + Qe^{-ikx}$	Ce^{-bx}
ψ'	$ik(Pe^{ikx} - Qe^{-ikx})$	$-bCe^{-bx}$

For $x=0$, continuity of ψ, ψ'.
$$P+Q = C$$
$$ik(P-Q) = -bC$$
so $$P-Q = i\frac{b}{k}C.$$

$$\therefore P = \tfrac{1}{2}\left(1+\frac{ib}{k}\right)C$$

$$Q = \tfrac{1}{2}\left(1-\frac{ib}{k}\right)C$$

If we want $P=1$, then $Q = \frac{1-ib/k}{1+ib/k}$ Note $|Q|=1$ when $P=1$.

In a sense, no loss.

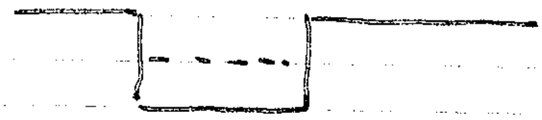
Tunnel Effect.

In 1 dimension the wave eqn is $\frac{\partial^2\psi}{\partial x^2} + \frac{2m}{\hbar^2}(E-V)\psi = 0$

Classically, $E-V$ represents kinetic energy, which must be positive or zero. It is impossible for the mass to reach (or pass through) any region in which $E-V$ is negative.

Suppose quantities so chosen ~~that~~ $\frac{2m}{\hbar^2}(E-V)$ is $+1$ for $-k < x < k$ and -1 for $|x| > k$.

The potential energy diagram is shown here. The particle has just half the energy needed (classically) to escape from the interval $(-k, k)$.



$-k$ $\quad$ k

For $x > k$ we have $\frac{d^2\psi}{dx^2} - \psi = 0$

$\therefore \psi = Ae^{x} + Be^{-x}$

We must take $A = 0$ for ψ to remain finite and bounded everywhere, so $\psi = Be^{-x}$.

In the same way, for $x < -k$, $\psi = Ce^{x}$.

Between $-k$ and k $\frac{d^2\psi}{dx^2} + \psi = 0$.

So $\psi = Pe^{ix} + Qe^{-ix}$.

We require ψ and $\frac{d\psi}{dx}$ continuous.

In $x < -k$ $\quad \psi = Ce^{x}$ $\quad \frac{d\psi}{dx} = Ce^{x}$

In $-k < x < k$ $\quad \psi = Pe^{ix} + Qe^{-ix}$, $\quad \frac{d\psi}{dx} = iPe^{ix} - iQe^{-ix}$

For $x > k$ $\quad \psi = Be^{-x}$ $\quad d\psi/dx = -Be^{-x}$

Hence

(1) $Ce^{-k} = Pe^{-ik} + Qe^{ik}$ $\qquad Ce^{-k} = iPe^{-ik} - iQe^{ik}$ (2)

(3) $Be^{-k} = Pe^{ik} + Qe^{-ik}$ $\qquad -Be^{-k} = iPe^{ik} - iQe^{-ik}$ (4)

From (1) & (2) $\quad Pe^{-ik} + Qe^{ik} = iPe^{-ik} - iQe^{-ik}$

so $\quad P(1-i)e^{-ik} + Q(1+i)e^{ik} = 0$ $\qquad$ (5)

From (3) & (4) $\quad P(1+i)e^{ik} + Q(1-i)e^{-ik} = 0$ $\qquad$ (6)

A solution other than $P=0$, $Q=0$ is only possible if determinant $=0$.

$$0 = \begin{vmatrix} (1-i)e^{-ik} & (1+i)e^{ik} \\ (1+i)e^{ik} & (1-i)e^{-ik} \end{vmatrix}$$

That is $(1-i)^2 e^{-2ik} = (1+i)^2 e^{2ik}$

i.e. $\quad -2i\, e^{-2ik} = 2i e^{2ik}$

i.e. $\quad e^{4ik} = -1$

$e^{i\theta}$ is the point on unit circle at angle θ. -1 is at π.

$$\therefore\ 4k = \pi + 2n\pi.$$

$$k = \frac{\pi}{4} + n\frac{\pi}{2}.$$

Only such values of k allow a non-zero solution.

In practice, the situation would be reversed. k would be given, and only discrete values of E would be possible.

Consider the value $k = \pi/4$.

$$P + Q\,\frac{1+i}{1-i}e^{2ik} = 0.$$

$1+i = i(1-i)$ so $P + ie^{2ik}Q = 0$

$$e^{2ik} = e^{i\pi/2} = i \quad \therefore\ P - Q = 0.$$

Take $P = Q = 1$. $\qquad e^{ix} + e^{-ix} = 2\cos x.$

Then $\quad Ce^{-\pi/4} = e^{-ik} + e^{ik} = 2\cos k$

$$= 2\cos\frac{\pi}{4} = \sqrt{2}$$

$$C = \sqrt{2}\,e^{\pi/4}.$$

$$Be^{-k} = e^{ik} + e^{-ik} = 2\cos k = \sqrt{2}$$

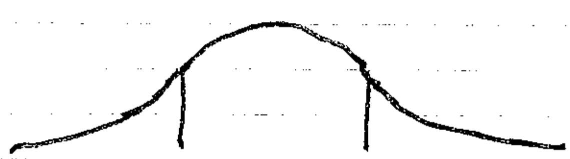
$$B = \sqrt{2}\,e^{\pi/4}$$

Thus

$x < -\frac{\pi}{4} \qquad \psi = \sqrt{2}\,e^{\pi/4}e^{+x}$

$-\frac{\pi}{4} < x < \frac{\pi}{4} \qquad \psi = 2\cos x$

$x > \pi/4 \qquad \psi = \sqrt{2}\,e^{\pi/4}e^{-x}$



More interesting, potential well for negative E.

Let $\frac{2mE}{\hbar^2} = -\kappa^2$

The solutions outside well that are acceptable are

$\psi(x) = C_1 e^{\kappa x}$ for $x < -a$
$\psi(x) = C_2 e^{-\kappa x}$ for $x > a$.

These are real, and it is convenient to write solutions inside well as
$u(x) = A \cos qx + B \sin qx$

$q^2 = \frac{2m}{\hbar^2}(E + V_0)$. We suppose $V_0 > 0$

and E negative but $E > -V_0$.

At $x = -a$ $\quad \psi = C_1 e^{-\kappa a}$ and $A \cos qa - B \sin qa$ (1)
$x = +a$ $\quad \psi = C_2 e^{-\kappa a}$ $\quad A \cos qa + B \sin qa$. (2)

To left $\psi' = \kappa C_1 e^{\kappa x}$ and $-qA \sin qx + qB \cos qx$.
So at $x = -a$
$\kappa C_1 e^{-\kappa a} = q(A \sin qa + B \cos qa)$ (3)

To right $\psi' = -\kappa C_2 e^{-\kappa x}$ and $-qA \sin qx + qB \cos qx$

$\therefore$ $\quad -\kappa C_2 e^{-\kappa a} = -qA \sin qa + qB \cos qa$ (4)

Substitute in (3) from (1)
$\kappa(A \cos qa - B \sin qa) = q(A \sin qa + B \cos qa)$ (5)

In (4) from (2),
$-\kappa(A \cos qa + B \sin qa) = q(A \sin qa - B \cos qa)$

Potential well.

$$V(x) = \begin{cases} 0 & x < -a \\ -V_0 & -a < x < a \\ 0 & x > a. \end{cases}$$

$$\psi'' + \frac{2m}{\hbar^2}(E - V(x))\psi = 0.$$

Let $K^2 = \frac{2mE}{\hbar^2}$ $\quad q^2 = \frac{2m}{\hbar^2}(E+V_0)$

In $x<-a$, $\psi'' + k^2\psi = 0.$ (1)

$$\psi = e^{ikx} + Re^{-ikx}$$ R because reflection can occur.

In $-a<x<a$ $\psi'' + q^2\psi = 0$ (2)

$$\psi = Ae^{iqx} + Be^{-iqx}$$

For $x>a$ eqn (1) again, but nothing coming from right

$$\psi = Ce^{ikx}$$

Continuity of ψ at $x=-a$ $\quad e^{-ika} + Re^{ika} = Ae^{-iqa} + Be^{iqa}$ (i)

of ψ at $x = +a$ $\quad Ce^{ika} = Ae^{iqa} + Be^{-iqa}$ (ii)

of ψ' at $x=-a$

$$ik(e^{-ika} - Re^{ika}) = iq(Ae^{-iqa} - Be^{iqa}) \quad \text{(iii)}$$

at $x=+a$ $\quad ikCe^{ika} = iq(Ae^{iqa} - Be^{-iqa})$ (iv)

These can be solved to give R, for the amount of reflected wave, and C, the amount of transmitted.