(N3) M 67

Electricity and Magnetism

Electrostatic Unit (ESU) of charge is so chosen that force between Q_1 and Q_2 at distance r cm is $Q_1 Q_2/r^2$ dynes.

Unit magnetic pole is chosen to make force $m_1 m_2/r^2$.

Field E such that force is EQ dynes.

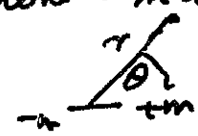
ESU unit of current = no. of ESU of charge / second.

$\operatorname{div} E = 4\pi\rho$ all in ESU.

Potential V. $E = -\operatorname{grad} V$.

Magnetic particle. Poles m at distance ds. Moment $= m\,ds = \mu$.

Potential due to magnetic particle $\Phi = \mu\cos\theta/r^2$



Magnetic shell of strength ϕ: any element $d\underset{\sim}{S}$ has effect of magnetic particle of moment $\phi\, d\underline{S}$.

If ϕ is constant, potential is $-\phi\Omega$, where Ω is solid angle subtended by shell at point in question.

B magnetic induction. H magnetic force.

$\iint \underline{B}\, d\underline{S} = 0$ so $\operatorname{div} B = 0$.

Hence, there exists vector potential $\underline{A}$: $B = \operatorname{curl} \underline{A}$.

EMU (p 425) Q A small current loop produces effect of magnetic particle.

For E.M.U. of current, a current i flowing in a loop produces magnetic field equal to that of magnetic shell, of strength i, bounded by loop.

Consequence of this: work done by unit pole in threading current i is $4\pi i$.

It can be deduced that field is $\int \frac{i\, ds \wedge r}{r^3}$

$= \int \frac{i \sin\theta\, ds}{r^2}$ ds θ r $\otimes H$

Induced currents. Induced EMF $= -\frac{dN}{dt} = -\frac{d}{dt}$ flux B.

$\int \underline{E}\, d\underline{s} = -\frac{d}{dt}\iint \underline{B}\, d\underline{S}$ $\therefore \operatorname{curl} E = -\frac{dB}{dt}$.

The EMF is the work done on unit charge as it goes round circuit. ~~[illegible]~~. pp 452-453 Jeans derive this by considering magnetic field produced by circuit. So current i is EMU.

p576 $-\frac{1}{c}\frac{dH}{dt} = \text{curl } E$ (528)

$4\pi j + \frac{1}{c}\frac{dE}{dt} = \text{curl } H$ (529)

j, H, B in EMU.

E in ESU.

N.1

M 69

Jeans p.539.

Volume density ρ at x, y, z with velocity U, V, W

Thus $(u, v, w) = (\rho U, \rho V, \rho W)$ in ESU

In eqn (529), u, v, w in EMU, so $\left(\frac{\rho U}{c}, \frac{\rho V}{c}, \frac{\rho W}{c}\right)$ EMU

Thus $$\frac{4\pi}{c}\left(\rho U + \frac{df}{dt}\right) = \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z} \quad \text{etc}$$

$$\frac{4\pi \rho U}{c} + \frac{1}{c}\frac{dE}{dt} = \text{curl } H$$

Eqn of continuity $\frac{\partial}{\partial x}(\rho U) + \frac{\partial}{\partial y}(\rho V) + \frac{\partial}{\partial z}(\rho W) + \frac{d\rho}{dt} = 0$

$H = \alpha, \beta, \gamma \qquad B = a, b, c \qquad \S 432 \qquad E = (X, Y, Z)$

Induction $= N = \iint B . dS$

Potential in circuit $= \oint E . ds = \iint \frac{dB}{dt} . dS$

$$\iint \text{curl } E\, dS = - \iint \frac{dB}{dt} . dS$$

$$\therefore \quad - \frac{dB}{dt} = \text{curl } E$$

$$- \frac{dH}{dt} = \text{curl } E \quad \underline{\text{all in EMU}}$$ p.575

$H = \text{curl } A \qquad \text{curl}\left(E + \frac{dA}{dt}\right) = 0$

$$\therefore \quad E = - \frac{dA}{dt} \cancel{+} - \text{grad } \varphi$$

$$\text{curl } H = 4\pi j$$

Klein C xix. Vol 2. Chap 1.B. § 4.
Transition to basic concepts of vector- and tensor-algebra.

Discusses concepts, in 3 and 4 dimensions, that are available to physicists from viewpoint of invariance.

1. $n=3$ (x, y, z) Restrict group to homog. orthogonal.

(10) $x^2+y^2+z^2$ is simplest invariant. Polar form

(11) $x'x+y'y+z'z$ Dot product

(11) being invariant, $x'y'z'$ is both cogredient and contragredient to xyz. For orthog. transformations the distinction disappears

(12) $\begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}$. Transformation multiplies by $+1$ or -1.

As det. $= x(y'z''-y''z')+y(z'x''-z''x')+z(x'y''-x''y')$

(13) $y'z''-y''z', z'x''-z''x', x'y''-x''y'$

is contragredient to x, y, z. Vector product.

(14) $ax^2+by^2+cz^2+2dyz+2ezx+2fxy$

Voigt calls a, b, c, d, e, f a tensor. For affine trans these may differ in rank: for real coefficients, in Trägheitsindex. If we restrict ourselves to orthog. trans $x^2+y^2+z^2$ is invariant, and this leads us to consider

(16) $\begin{vmatrix} a+\lambda & f & e \\ f & b+\lambda & d \\ e & d & c+\lambda \end{vmatrix}$ The coefficient are the

3 fundamental orthogonal invariant of the tensor.

(16') $\lambda^3+\lambda^2(a+b+c)+\lambda(bc+ca+ab-d^2-e^2-f^2)+\begin{vmatrix} a & f & e \\ f & b & d \\ e & d & c \end{vmatrix}$

We can form from them invariant

(17) $(a+b+c)^2-2(bc+ca+ab-d^2-e^2-f^2)=a^2+b^2+c^2+2d^2+2e^2+2f^2$

Forming the polar (18) $aa'+bb'+cc'+2dd'+2ee'+2ff'$ we see that $a, b, c, d\sqrt{2}, e\sqrt{2}, f\sqrt{2}$ are contragredient to themselves. From (14) we see they are contragredient to

(18') $x^2, y^2, z^2, yz\sqrt{2}, zx\sqrt{2}, xy\sqrt{2}$

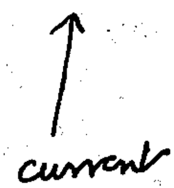
Hence cogredient to these.

E.S. unit of charge. Force $= Q_1 Q_2 / r^2$ in vacuum. This defines unit.

E.M. unit of magnetic pole Force $= m_1 m_2 / r^2$ in vacuum.

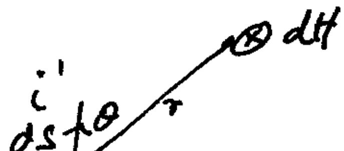
E.M unit of current.

In unit magnetic field, experiences 1 dyne on each centimetre.



Unit of electricity is that produced by unit current in unit time. EM units carry dashes.

$c = i/i' = Q/Q'$

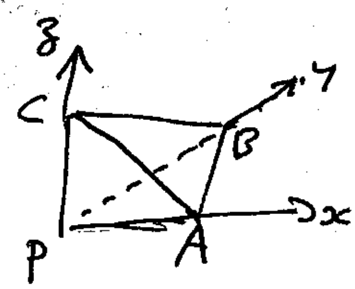


(a) Current i', length ds produce $\frac{i' ds \sin\theta}{r^2}$

(b) By integration, infinite straight wire produces $H = 2i'/a$ at distance a.

(c) Magnetic pole of strength m, at distance a, experiences force $2mi'/a$. Carried round wire, work done $= 4\pi i' m$. Independent of path.

Let circulation / area have components $X_c Y_c Z_c$ near P.



circ BCP $= X_c \, l S$

circ CAP $= Y_c \, m S$

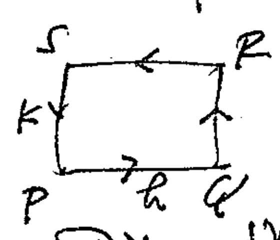
circ ABP $= Z_c \, n S$ where S is area of ABC

since projections of ABC have areas lS, mS, nS.

$$\frac{\text{circ ABC}}{\text{area ABC}} = l X_c + m Y_c + n Z_c$$

Thus circulation per unit area for infinitesimal element is component of $\overline{(X, Y, Z)}$ along normal.

Now let (X, Y, Z) be force of which we are finding circulation. Circ PQRS



$= (Y_{QR} - Y_{PS})\, k + (X_{PQ} - X_{SR})\, h$

$= k h \frac{\partial Y}{\partial x} - h k \frac{\partial X}{\partial y}$

$= k h \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)$ Thus $Z_c = \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}$

Jeans. pp 426-427

A current flowing in a loop has an equivalent magnetic shell.

An element dS behaves like a magnet of moment $\phi\, dS$, ϕ being strength of shell.
If we use E.M.U, $\phi = i'$.

Potential $\Omega_Q = \iint \varphi\, d\omega = \varphi\omega$ ω = solid angle.

Work done in carrying unit magnetic pole around a current i^* is $4\pi i^*$. (E.M.U.)

p 511 u, v, w current in E.M.U.
Displacement current $\frac{df}{dt}, \frac{dg}{dt}, \frac{dh}{dt}$ in E.S.U. *

In EMU displacement current is $\frac{1}{c}\frac{df}{dt}, \frac{1}{c}\frac{dg}{dt}, \frac{1}{c}\frac{dh}{dt}$

'Total current' in E.M.U is $u + \frac{1}{c}\frac{df}{dt}, v + \frac{1}{c}\frac{dg}{dt}, w + \frac{1}{c}\frac{dh}{dt}$

Total current is like incompressible fluid.
Work done in carrying unit magnetic pole around a circuit is 4π (total current through it)

Thus $4\pi \iint \left(u + \frac{1}{c}\frac{df}{dt}, .., ..\right) dS = \int H\, ds$

$$\therefore \quad 4\pi\left(j + \frac{1}{c}\frac{df}{dt}\right), .., .. = \text{curl } H$$

In empty space, $u=0$.
$(f, g, h) = \frac{1}{4\pi} E$

$$\therefore \quad \text{curl } H = \frac{1}{c}\frac{\partial E}{\partial t}$$

~~everything in E.M.U.~~ H in EMU.
E in E.S.U. (see * above)

In conducting material, $(u, v, w) = j$ in E.S.U.

Maxwell's Equations and Tensor K_{ij}.

$\left(\frac{\rho v_x}{c}, \frac{\rho v_y}{c}, \frac{\rho v_z}{c}, \rho\right)$ is a 4-vector for

$\xi_1 = x \quad \xi_2 = y \quad \xi_3 = z \quad \xi_4 = ct$.

If we put $x_1 = \xi_1 \quad x_2 = \xi_2 \quad x_3 = \xi_3 \quad x_4 = i\xi_4$ we must transform the vector in the same way as we have done the coordinates. The vector becomes $\left(\frac{\rho v_x}{c}, \frac{\rho v_y}{c}, \frac{\rho v_z}{c}, i\rho\right)$.

As $\Box A = \frac{-4\pi\rho v}{c}$, $\Box V = -4\pi\rho$ we must write (A_1, A_2, A_3, iV) for the potential 4-vector.

It is contravariant. (We arrived at $\rho v_x/c$ etc. by considering dx^1, dx^2, dx^3, dx^4) However, in orthogonal system the distinction disappears, so we may write it as covariant

Let $s_1 = A_1 \quad s_2 = A_2 \quad s_3 = A_3 \quad s_4 = iV$.

Let $K_{ij} = \frac{\partial s_i}{\partial x^j} - \frac{\partial s_j}{\partial x^i}$

$$K_{12} = \frac{\partial s_1}{\partial x^2} - \frac{\partial s_2}{\partial x^1} = \frac{\partial A_1}{\partial x^2} - \frac{\partial A_2}{\partial x^1} = -H_3.$$

$$K_{13} = \frac{\partial s_1}{\partial x^3} - \frac{\partial s_3}{\partial x^1} = \frac{\partial A_1}{\partial x^3} - \frac{\partial A_3}{\partial x^1} = H_2$$

$$K_{14} = \frac{\partial s_1}{\partial x^4} - \frac{\partial s_4}{\partial x^1} = \frac{\partial A_1}{ic\,\partial t} - \frac{\partial iV}{\partial x} = i\left[-\frac{1}{c}\frac{\partial A_1}{\partial t} - \frac{\partial V}{\partial x}\right]$$

$$= iE_1 = \mathcal{E}_1 .$$

for $E = -\frac{1}{c}\frac{\partial A}{\partial t} - \operatorname{grad} V$. (eqn ①, N57)

$$K_{23} = \frac{\partial s_2}{\partial x_3} - \frac{\partial s_3}{\partial x_2} = \frac{\partial A_2}{\partial z} - \frac{\partial A_3}{\partial y} = -H_1$$

$$K_{24} = \frac{\partial s_2}{\partial x_4} - \frac{\partial s_4}{\partial x_2} = \frac{\partial A_2}{ic\,\partial t} - \frac{\partial iV}{\partial y} = i\left(-\frac{1}{c}\frac{\partial A_2}{\partial t} - \frac{\partial V}{\partial y}\right) = iE_2 = \mathcal{E}_2$$

$$K_{34} = \frac{\partial s_3}{\partial x_4} - \frac{\partial s_4}{\partial x_3} = \frac{\partial A_3}{ic\,\partial t} - \frac{\partial iV}{\partial z} = i\left[-\frac{1}{c}\frac{\partial A_3}{\partial t} - \frac{\partial V}{\partial z}\right] = iE_3 = \mathcal{E}_3$$

$$\therefore K = \begin{pmatrix} \cdot & -H_3 & H_2 & \mathcal{E}_1 \\ H_3 & \cdot & -H_1 & \mathcal{E}_2 \\ -H_2 & H_1 & \cdot & \mathcal{E}_3 \\ -\mathcal{E}_1 & -\mathcal{E}_2 & -\mathcal{E}_3 & \cdot \end{pmatrix}$$

MER (1)

In these symbols the Maxwell eqns are

(I)
$$0 = \cdot + \frac{\partial K_{12}}{\partial x_2} + \frac{\partial K_{13}}{\partial x_3} + \frac{\partial K_{14}}{\partial x_4}$$
$$0 = \frac{\partial K_{21}}{\partial x_1} \quad \cdot \quad \frac{\partial K_{23}}{\partial x_3} + \frac{\partial K_{24}}{\partial x_4}$$
$$0 = \frac{\partial K_{31}}{\partial x_1} \quad \frac{\partial K_{32}}{\partial x_2} \quad \cdot \quad \frac{\partial K_{34}}{\partial x_4}$$
$$0 = \frac{\partial K_{41}}{\partial x_1} + \frac{\partial K_{42}}{\partial x_2} + \frac{\partial K_{43}}{\partial x_3} \quad \cdot$$

$$0 = \frac{\partial K_{ij}}{\partial x^j}$$

This is a tensor equation, that is accordingly valid in any system.

$$K_{ij} = \frac{\partial S_i}{\partial x_j} - \frac{\partial S_j}{\partial x_i}$$

$$\frac{\partial K_{ij}}{\partial x^k} = \frac{\partial^2 S_i}{\partial x^j \partial x^k} - \frac{\partial^2 S_j}{\partial x^k \partial x^i}$$

The second term is obtained from the first by cyclic permutation (ijk)

Hence $\frac{\partial K_{ij}}{\partial x^k} + \frac{\partial K_{jk}}{\partial x^i} + \frac{\partial K_{ki}}{\partial x^j} = 0.$

If two of i, j, k are equal, say $i = j$, LHS becomes

$$0 + \frac{\partial K_{ik}}{\partial x^i} + \frac{\partial K_{ki}}{\partial x^i} = \frac{\partial}{\partial x^i}(K_{ik} + K_{ki}) = \frac{\partial}{\partial x^i} 0.$$

We get substantial results only when i, j, k distinct.
There are 4 ways of choosing 3 distinct ~~each~~ numbers from 1, 2, 3, 4.

MER (4) M 75

$i=2 \quad j=3 \quad k=4.$

$$\frac{\partial K_{23}}{\partial x^4} + \frac{\partial K_{34}}{\partial x^2} + \frac{\partial K_{42}}{\partial x^3} = -\frac{\partial H_1}{\partial x^4} + \frac{\partial \mathcal{E}_3}{\partial x^2} - \frac{\partial \mathcal{E}_2}{\partial x^3} \qquad \text{II}(1)$$

$i=1 \quad j=3 \quad k=4$

$$\frac{\partial K_{13}}{\partial x^4} + \frac{\partial K_{34}}{\partial x^1} + \frac{\partial K_{41}}{\partial x^3} = \frac{\partial H_2}{\partial x^4} + \frac{\partial \mathcal{E}_3}{\partial x^1} - \frac{\partial \mathcal{E}_1}{\partial x^3} \qquad \text{II}(2)$$

$i=1 \quad j=2 \quad k=4$

$$\frac{\partial K_{12}}{\partial x^4} + \frac{\partial K_{24}}{\partial x^1} + \frac{\partial K_{41}}{\partial x^2} = -\frac{\partial H_3}{\partial x^4} + \frac{\partial \mathcal{E}_2}{\partial x^1} - \frac{\partial \mathcal{E}_1}{\partial x^2} \qquad \text{II}(3)$$

$i=1 \quad j=2 \quad k=3$

$$\frac{\partial K_{12}}{\partial x^3} + \frac{\partial K_{23}}{\partial x^1} + \frac{\partial K_{31}}{\partial x^2} = -\frac{\partial H_3}{\partial x^3} - \frac{\partial H_1}{\partial x^1} - \frac{\partial H_2}{\partial x^2} \qquad \text{II}(4).$$

Velocities of molecules in a gas.

Assumption of molecular chaos. Randomness. In particular, no direction singled out.

Let velocity V have components x, y, z. If the probability of first component being in $x, x+dx$ is $f(x)\,dx$, the probability for y must be given by $f(y)\,dy$ and for z, $f(z)\,dz$.

Thus probability of ~~~~ velocity lying near (x, y, z) within a region of volume $dx\,dy\,dz$ is $f(x)f(y)f(z)\,dx\,dy\,dz$. For any small region, we have $f(x)f(y)f(z)\,dV$, where dV is its volume.

If no direction is singled out, the probability of a speed v should be the same for all directions — that is to say, it should depend only on the magnitude of v. Now $v^2 = x^2+y^2+z^2$. Thus we should have

$$f(x)\,f(y)\,f(z) = \varphi(x^2+y^2+z^2) = \varphi(v^2)$$

$$\frac{\partial}{\partial x}: \quad f'(x)\,f(y)\,f(z) = \varphi'(x^2+y^2+z^2)\cdot 2x.$$

Divide

$$\frac{f'(x)}{f(x)} = \frac{2x\,\varphi'(v^2)}{\varphi(v^2)}$$

$$\therefore \quad \frac{1}{2x}\,\frac{f'(x)}{f(x)} = \frac{\varphi'(v^2)}{\varphi(v^2)}$$

Hence each side is constant.

$$\frac{1}{2x}\,\frac{f'(x)}{f(x)} = c$$

$$\therefore \quad \frac{f'(x)}{f(x)} = 2xc$$

$$\int \qquad \ln f(x) = cx^2$$

$$\therefore \quad f(x) = e^{cx^2}$$

If $c > 0$, we have large probabilities for large velocities.

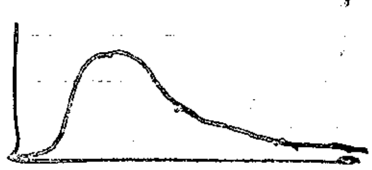
Put $c = -a$.

$$f(x) = e^{-ax^2}$$

$$f(x)f(y)f(z) = e^{-a(x^2+y^2+z^2)} = e^{-av^2}$$

~~If v lies between~~ If the velocity lies between v and $v+dv$, the end of the velocity vector must lie in a region of volume $4\pi v^2\,dv$. The probability of this is $4\pi v^2 e^{-av^2}\,dv$.

Graph



PROBLEMS RELATED TO ORTHOGONAL MATRICES.

1. Write the matrix M for rotation through θ. What is the geometrical meaning of M^T? Is there any operation, other than transposition, that will change M to M^T? Of what general theorem is this an example?

Find the eigenvalues of M, and plot them on the Argand Diagram. Why might these values be expected?

2. Find the matrix M for the rotation that sends OX to OY, OY to OZ, and OZ to OX. Find M^2 and M^3.

M has only one real eigenvector; call it **v**. Find **v** and the eigenvalue associated with it.

Choose new axes, which are not to involve any change of scale on the axes. The first axis is to be in the direction of **v**; the other two are to be perpendicular to **v** and to each other. Let X,Y,Z be co-ordinates in this new system. Find the matrix L that gives x,y,z in terms of X,Y,Z; find the matrix N that gives X,Y,Z in terns of x,y,z. Mention anything you notice when comparing L and N, and explain it.

3. Choose any 3 perpendicular unit vectors, **u,v,w.** Do not take obvious ones; none of them should be along any original axis. X,Y,Z are co-ordinates based on **u,v,w.** Write the equations giving x,y,z in terms of X,Y,Z; also the equations giving X,Y,Z in terms of x,y,z.

Choose an ellipsoid having axes in the directions of **u,v,w.** Write its equation (i) in terms of X,Y,Z, (ii) in terms of x,y,z. Write the matrix associated with its equation in the x,y,z system. Find its eigenvalues and eigenvectors. Comment on the result.

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1. Find the shape and orientation of the conic $36x^2 + 24xy + 29y^2 = 180$. (Find the eigenvalues of the associated matrix and take co-ordinate axes along them.)

2. Find the 10th power of the matrix $\begin{pmatrix} 5/2 & -1/2 \\ -1/2 & 5/2 \end{pmatrix}$.

What is the most complicated fraction that can occur in the n^{th} power of this matrix?

Exercise. The transformation $S = \begin{pmatrix}2&3\\1&2\end{pmatrix}$ and the matrix $M = \begin{pmatrix}10&-6\\2&3\end{pmatrix}$.

Find the matrices, as shown in the new axes, for the transformation and for the conic defined by $v^T M v$ = constant.

Solution. $S = \begin{pmatrix}2&3\\1&2\end{pmatrix}$ has the inverse $\begin{pmatrix}2&-3\\-1&2\end{pmatrix} = S^{-1}$.
M as a transformation appears as $S^{-1}MS$ in the new axes. $\begin{pmatrix}2&-3\\-1&2\end{pmatrix}\begin{pmatrix}10&-6\\2&3\end{pmatrix}\begin{pmatrix}2&3\\1&2\end{pmatrix} = \begin{pmatrix}14&-21\\-6&12\end{pmatrix}\begin{pmatrix}2&3\\1&2\end{pmatrix}$

$$= \begin{pmatrix}7&0\\0&6\end{pmatrix}.$$

For a conic $S^TMS = \begin{pmatrix}2&1\\3&2\end{pmatrix}\begin{pmatrix}10&-6\\2&3\end{pmatrix}\begin{pmatrix}2&3\\1&2\end{pmatrix}$

$$= \begin{pmatrix}22&-9\\34&-12\end{pmatrix}\begin{pmatrix}2&3\\1&2\end{pmatrix} = \begin{pmatrix}35&48\\56&78\end{pmatrix}$$

03

MATRICES.

The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can arise as an abbreviation for the transformation $(x,y) \to (x^*,y^*)$ where

(I) $\quad x^* = ax + by, \quad y^* = cx + dy. \qquad v^* = Mv$ for short.

This is probably the most natural way into the subject of matrices.

A matrix can also by used to specify the expression

$$ax^2 + 2hxy + by^2 \text{ as } (x\ y)\begin{pmatrix} a & h \\ h & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \qquad \text{(II)}$$

To do this we have to bring in the row vector $(x\ y)$ as well as the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$. $(x\ y)$ is known as the **transpose** of $\begin{pmatrix} x \\ y \end{pmatrix}$. We can write $(x\ y) = \begin{pmatrix} x \\ y \end{pmatrix}^T$.

Equally, the column vector is the transpose of the row vector. Transposition means that rows are changed to columns and columns to rows. This operation can be applied to a matrix as well as to a vector.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

The expression (II) has the form v^TMv.

These expressions have geometrical significance. v^TMv = constant is the equation of a conic. If P is a point on this conic, and $v_o = OP$ it can be proved that Mv_o is a vector giving the direction of the normal to the conic at P.

Transpose of a product.

The product A=BC is given by the formula $a_{ik}=\sum b_{ij}c_{jk}$.

The transpose of A, a^T has $a^T_{ik}=a_{ki}=\sum b_{kj}c_{ji}$. The formula for a matrix product must have the form $()_{ik}=\sum()_{i\,}()_{\,k}$ which is not so in the equation above. We can bring it to this form by using $b_{kj}=b^T_{jk}$ and $c_{ji}=c^T_{ij}$.

this gives $a^T_{ik}=\sum b^T_{jk}c^T_{ij}=\sum c^T_{ij}b^T_{jk}$, which indicates $A^T = C^TB^T$. The transpose of a product is the product of the transposes **in the reverse order.**

04

Exercise. Consider the case in which
$M = \begin{pmatrix} 10 & -6 \\ 2 & 3 \end{pmatrix}$ and $S = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$. Find the matrices that correspond to M in the new axis system (i) when M is used to define a transformation, (ii) when M is used to define a conic.

Transformations that preserve lengths and angles.

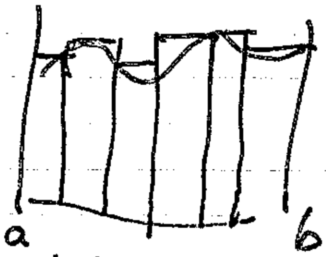
It is an advantage of matrix notation that the theory takes the same form in any number of dimensions. If a rigid body moves, all the lengths and angles are unaltered, so it is natural to consider transformations that have this property. There are some transformations that have this property but cannot be realized by moving the body - reflections in a mirror, for instance. We shall not exclude these from our considerations.

It is possible to find this set of transformations solely from the condition that all lengths are to remain unaltered, for if the lengths of the sides of a triangle are unchanged, the same must be true for the angles in it. However the algebra is slightly simpler if we ppose the condition that all scalar products (dot products) are to remain unaltered. The scalar product, u.v, can be expressed in matrix notation as $u^T v$.

An old joke is that a matrix can represent either an alibi or an alias. "Alibi" means "somewhere else" and can be used to describe a transformation that sends some object to a new position. "Alias" means a new name for the same person, and applies when we change axes, and thus specify the same vectors by a new set of numbers. The algebra is exactly the same in these two cases. You can use either you like to visualize what is happening.

We suppose then that our vectors u and v are to be replaced by U and V, with u=SU and v=SV, and that this is to make no difference when we calculate the scalar product, $u^T v$. The transpose rule shows that $u^T = U^T S^T$, so $u^T v$ becomes $U^T S^T S V$, and we demand that this be the same as $U^T V$. Clearly this will be so if $S^T S = I$, the identity transformation. In fact this is the only way it can happen.

Riemann Integral is based on diagram



function values finite values, a, b finite

Improper Riemann Integral applies to cases such as

$$\int_1^\infty \frac{1}{x^2}\,dx \quad \text{and} \quad \int_0^1 \frac{1}{\sqrt{x}}\,dx$$

in which a, b may be infinite, or $f(x)$ may take infinite value.

(1) Define $\int_1^\infty = \underset{N\to\infty}{Lt} \int_1^N \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^N = -\frac{1}{N}+1 \to 1$ as $N\to\infty$.

So $\int_1^\infty \frac{1}{x^2}\,dx = 1.$

(2) Define $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ as $\underset{a\to 0}{Lt} \int_a^1 \frac{1}{\sqrt{x}}\,dx$

$$= \left[2\sqrt{x}\right]_a^1 = 2 - 2\sqrt{a} \to 2$$

So $\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2.$

If we try to deal with $\int_{-1}^1 \frac{1}{x}\,dx$ as

$$\underset{a\to 0}{Lt}\ \underset{b\to 0}{Lt} \int_{-1}^{-a} \frac{1}{x}\,dx + \int_b^1 \frac{1}{x}\,dx \qquad a, b \text{ +ve}$$

We get $=$ $\left[\ln(-x)\right]_{-1}^{-a} + \left[\ln x\right]_b^1$

$$\ln a - 0 + 0 - \ln b = \ln a/b.$$

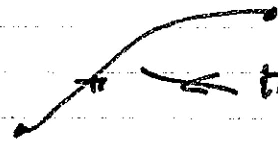
This does not have a definite limit. It depends on the way in which a and b tend to zero.

Cauchy was led to define "Principal Integral" for such a situation by work on complex variable.

The idea of principal integral.

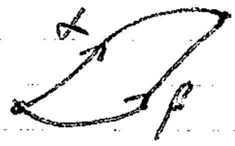
In work on complex variable $z = x + iy$, $w = f(z)$, we often have to use

$$\int_\Gamma f(z)\,dz$$


← the path Γ.

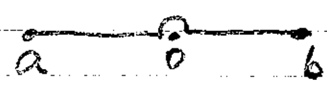
This has the property that, in a region where $f(z)$ is well behaved (i.e. analytic), we may vary the path, provided we keep the end points fixed.

$$\int_\alpha = \int_\beta$$



Suppose $f(z)$ has a singularity at $z = 0$. and we are interested in $\int_\gamma f(z)\,dz$

We may vary the path to this;-



This is OK as the function is analytic in the region bounded by the two curves.

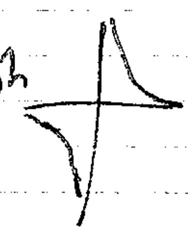
The integral now consists of 2 parts.

(1) around the semicircle.

(2) $\int_a^{-\varepsilon} + \int_\varepsilon^b$

This leads us to define the <u>principal integral from a to b</u> as $\lim_{\varepsilon \to 0} \left\{ \int_a^{-\varepsilon} f(z)\,dz + \int_\varepsilon^b f(z)\,dz \right\}$

Usually $\int_{-1}^{+1} \frac{dx}{x}$ is meaningless. Graph



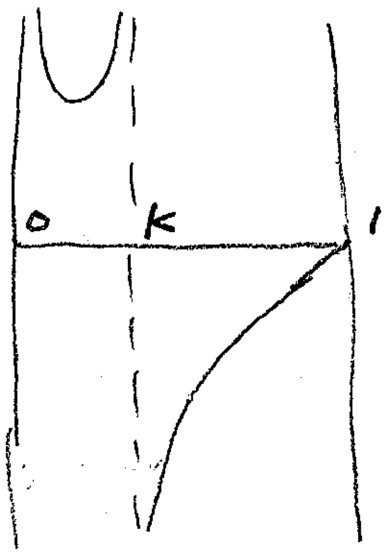
on account of the $-\infty$, $+\infty$ effect.

P 3

A problem I was working on involved $\int_0^1 \frac{-\ln t\, dt}{k-t}$.

Let $F(k,t) = \frac{-\ln t}{k-t}$. Computer gives following values

Graph of integrand.




```
K=C/N WITH N=?20
C=?7
F(0.35,0.05)=9.98577425
F(0.35,0.1)=9.21034037
F(0.35,0.15)=9.48559992
F(0.35,0.2)=10.7295861
F(0.35,0.25)=13.8629436
F(0.35,0.3)=24.0794561
F(0.35,0.4)=-18.3258146
F(0.35,0.45)=-7.98507696
F(0.35,0.5)=-4.6209812
F(0.35,0.55)=-2.989185
F(0.35,0.6)=-2.04330249
F(0.35,0.65)=-1.43594305
F(0.35,0.7)=-1.01907127
F(0.35,0.75)=-0.719205181
F(0.35,0.8)=-0.495874558
F(0.35,0.85)=-0.325037859
F(0.35,0.9)=-0.191564574
F(0.35,0.95)=-0.0854888241
F(0.35,1)=0
```

The graph near $t=k$ is approximately symmetrical, and the infinity can be made to disappear if we superpose the graph from 0 to k, reversed, on the graph of f on k to $2k$. In effect take temporary origin at k and measure away from k.

$\int_0^1 = \int_0^k + \int_k^{2k} + \int_{2k}^1$. (Consider $k \le \frac{1}{2}$).

In $(0,k)$ let $t = k - v$. In $(k, 2k)$ let $t = k + v$

$$\int_0^k \frac{-\ln t\, dt}{k-t} = \int_{v=k}^{v=0} \frac{-\ln(k-v)\,(-dv)}{v} = \int_0^k \frac{-\ln(k-v)\,dv}{v}$$

$$\int_k^{2k} \frac{-\ln t\, dt}{k-t} = \int_{v=0}^{v=k} \frac{-\ln(k+v)\,dv}{-v} = \int_0^k \frac{\ln(k+v)\,dv}{v}$$

Thus $\int_0^{2k}$ gives $\int_0^k \frac{\ln(k+v) - \ln(k-v)}{v}\,dv$

$$\ln(k+v) = \ln k + \ln\left(1+\frac{v}{k}\right)$$

$$\ln(k-v) = \ln k + \ln\left(1-\frac{v}{k}\right)$$

So $$\int_0^{2k} = \int_0^k \frac{\ln\left(1+\frac{v}{k}\right) - \ln\left(1-\frac{v}{k}\right)}{v}\,dv$$

$$= \int_0^k \frac{1}{v}\left\{ \begin{array}{l} \frac{v}{k} - \frac{v^2}{2k^2} + \frac{v^3}{3k^3} - \frac{v^4}{4k^4} \cdots \\ + \frac{v}{k} + \frac{v^2}{2k^2} + \frac{v^3}{2k^3} + \frac{v^4}{4k^4} \end{array}\right.$$

$$= 2\int_0^k \frac{1}{k} + \frac{v^2}{3k^3} + \frac{v^4}{5k^5} + \dots\,dv$$

$$= 2\left[\frac{v}{k} + \frac{v^3}{9k^3} + \frac{v^5}{25k^5} + \dots\right]_0^k$$

$$= 2\left[1 + \frac{1}{9} + \frac{1}{25} + \dots\right]$$

independent of k!

$$= 2\left(\frac{\pi^2}{8}\right) = \frac{\pi^2}{4}.$$

Independence of k. $\int_0^k \frac{\ln(k+v) - \ln(k-v)}{v}\,dv$

Let $v = ku$. $\int_{u=0}^1 \frac{\ln(k+ku) - \ln(k-ku)}{ku}\,k\,du$

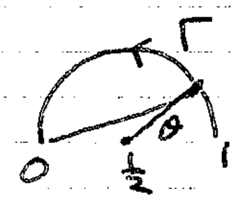
$$= \int_0^1 \frac{\ln(1+u) - \ln(1-u)}{u}\,du$$

If $k = \frac{1}{2}$, $\int_0^{2k}$ is $\int_0^1$. We have

Principal $\int_0^1 \frac{-\ln t}{\frac{1}{2} - t}\,dt = \frac{\pi^2}{4}$.

Now principal integral $\int_0^1 \frac{-\ln t}{\frac{1}{2}-t}\,dt$ is the real part of $\int_0^1 \frac{-\ln t\,dt}{\frac{1}{2}-t}$ taken by a path that avoids the singularity $t=\frac{1}{2}$. Consider integral along semicircle on $(0,1)$.

Reverse sign of integral, and direction $0,1$.



We get $\int_\Gamma \frac{-\ln t\,dt}{t-\frac{1}{2}}$.

At angle θ on this semicircle $t=\frac{1}{2}+\frac{1}{2}e^{i\theta}$

so $t-\frac{1}{2}=\frac{1}{2}e^{i\theta}$

t is at distance $2(\frac{1}{2}\cos\frac{\theta}{2})$ and angle $\frac{\theta}{2}$ from origin.

Thus $t = \cos\frac{\theta}{2}\,e^{i\theta/2}$ $\quad dt = \frac{1}{2}ie^{i\theta}$.

$$\ln t = \ln\cos\frac{\theta}{2} + \frac{i\theta}{2}.$$

$$\therefore \int_\Gamma = \int_{\theta=0}^{\pi}\left(-\ln\cos\frac{\theta}{2}-\frac{i\theta}{2}\right)\frac{\frac{1}{2}ie^{i\theta}\,d\theta}{\frac{1}{2}e^{i\theta}}$$

$$= \int_0^{\pi}\left(-\ln\cos\frac{\theta}{2}-\frac{i\theta}{2}\right)i\,d\theta$$

Real part $= \int_0^{\pi} -\frac{i^2\theta}{2}\,d\theta = \int_0^{\pi}\frac{\theta}{2}\,d\theta$

$= \left[\frac{\theta^2}{4}\right]_0^{\pi} = \frac{\pi^2}{4}$ as required.

Woman has some apples. She sells $\frac{1}{2}$ of her stock and $\frac{1}{2}$ an apple. Then $\frac{1}{3}$ of remaining stock and $\frac{1}{3}$ apple. Then $\frac{1}{4}$ and $\frac{1}{4}$ apple. Then $\frac{1}{5}$ and $\frac{1}{5}$ apple.

Suppose she begins with a apples, and has b, c, d, e at various stages.

Stage 1 Has a. Sells $\frac{1}{2}a + \frac{1}{2}$.

$$\therefore b = a - (\tfrac{1}{2}a + \tfrac{1}{2}) = \tfrac{1}{2}a - \tfrac{1}{2}$$
$$\therefore a = 2b + 1$$

Stage 2 Has b sells $\frac{1}{3}b + \frac{1}{3}$.

$$\therefore c = b - (\tfrac{1}{3}b + \tfrac{1}{3}) = \tfrac{2}{3}b - \tfrac{1}{3}$$
$$\therefore 3c = 2b - 1 \qquad \therefore 2b = 3c + 1.$$

Stage 3 Has c sells $\frac{1}{4}c + \frac{1}{4}$

$$d = c - (\tfrac{1}{4}c + \tfrac{1}{4}) = \tfrac{3}{4}c - \tfrac{1}{4}$$
$$\cancel{3c =}\ 4d = 3c - 1.$$

Stage 4 Has d sells $\frac{1}{5}d + \frac{1}{5}$

$$e = d - (\tfrac{1}{5}d + \tfrac{1}{5})$$
$$= \tfrac{4}{5}d - \tfrac{1}{5} \qquad 5e = 4d - 1.$$

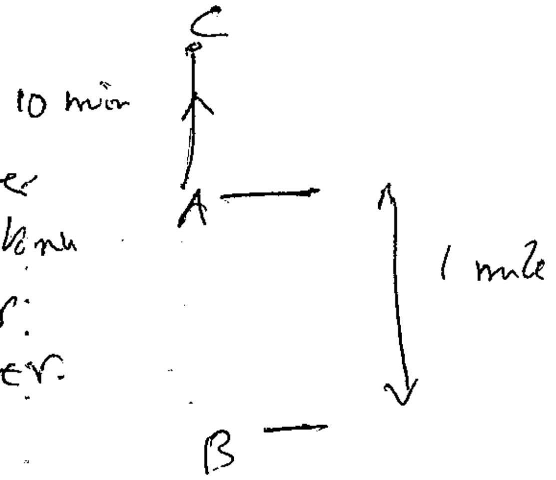
Substitute

$$a = 3c + 2$$
$$a = 4d + 3$$
$$a = 5e + 4.$$

$$\therefore a+1 = 3c+3 = 3(c+1)$$
$$a+1 = 4d+4 = 4(d+1)$$
$$a+1 = 5e+5 = 5(e+1)$$

$\therefore$ $a+1$ is divisible by 3, 4, 5.

$$\therefore a = 60k - 1 \quad k \in \mathbb{N}$$



Let ~~r~~ be speed of river

b speed of boat relative to river

Upstream velocity $b - r$.

Downstream velocity $b + r$.

10 min = $\frac{1}{6}$ hour.

So $AC = \frac{1}{6}(b-r)$

$BC = 1 + \frac{1}{6}(b-r)$

Time to row from C to B is $\dfrac{1+\frac{1}{6}(b-r)}{b+r}$

Total time from passing upstream thru A is

$$\frac{1}{6} + \frac{1+\frac{1}{6}(b-r)}{b+r}$$

Time for hat to float from A to B is $\frac{1}{r}$

$$\therefore \frac{1}{r} = \frac{1}{6} + \frac{1+\frac{1}{6}(b-r)}{b+r}$$

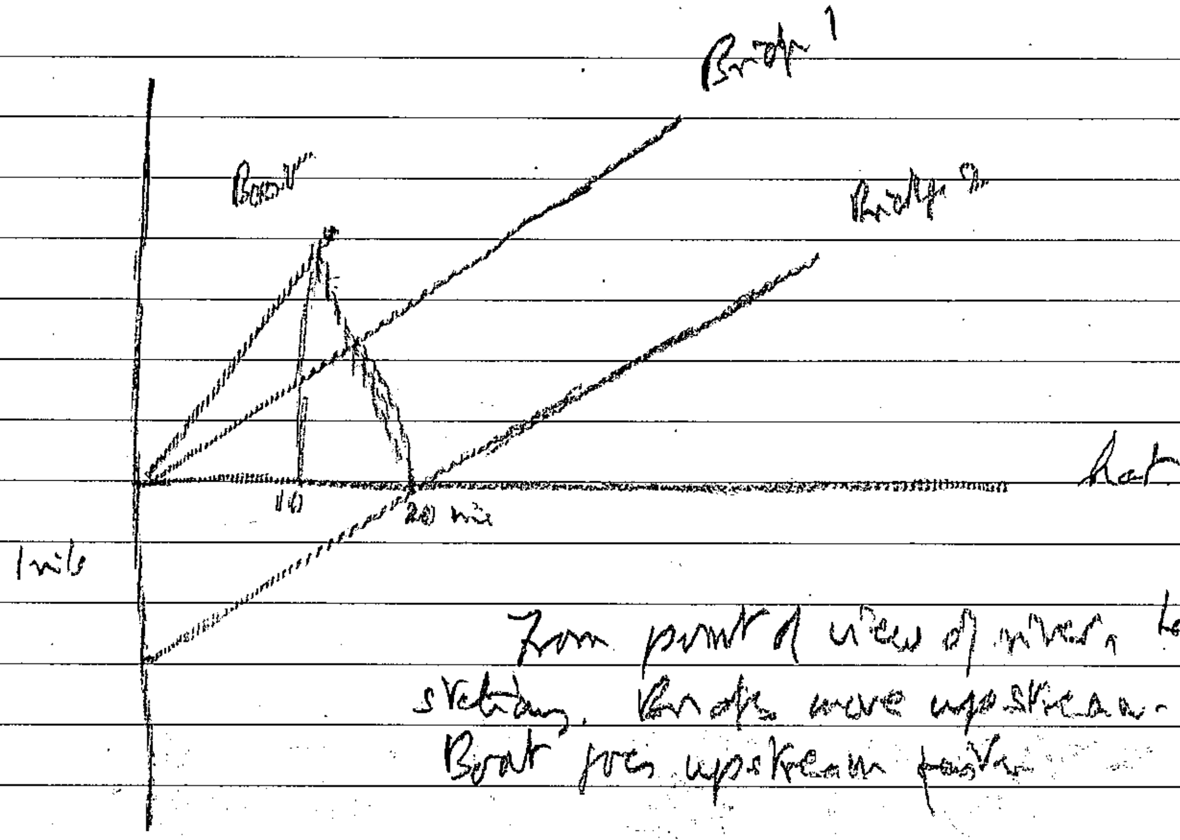
$\times(b+r)$ $\quad \therefore \dfrac{b+r}{r} = \frac{1}{6}(b+r) + 1 + \frac{1}{6}(b-r)$

$$= \frac{1}{3}b + 1$$

$$\therefore \frac{b}{r} = \frac{1}{3}b \quad \therefore \frac{1}{r} = \frac{1}{3}$$

$$r = 3$$

p 8



From point of view of river, hat is stationary. Bridges move upstream. Boat goes upstream faster

(0:25 0:35
0:15 – 0:05)

A woman has some apples.
She sells $\frac{1}{2}$ of what she has and $\frac{1}{2}$ an apple.
Then $\frac{1}{3}$ - - - - - - - - $\frac{1}{3}$ . . .
Then $\frac{1}{4}$ - - - - - - - - $\frac{1}{4}$ . . .
Then $\frac{1}{5}$ - - - - - - - - $\frac{1}{5}$ apple

Let n_0, n_1, n_2, n_3, n_4 be the number she has after 0, 1, 2, 3, 4 sales.

$n_1 = n_0 - (\frac{1}{2}n_0 + \frac{1}{2}) = \frac{1}{2}n_0 - \frac{1}{2}$ $\quad n_0 = 2n_1 + 1 \quad (1)$

$m_2 = n_1 - (\frac{1}{3}n_1 + \frac{1}{3}) = \frac{2}{3}n_1 - \frac{1}{3}$ $\quad 2n_1 = 3n_2 + 1 \quad (2)$

$n_3 = n_2 - (\frac{1}{4}n_2 + \frac{1}{4}) = \frac{3}{4}n_2 - \frac{1}{4}$ $\quad 3n_2 = 4n_3 + 1 \quad (3)$

$n_4 = n_3 - (\frac{1}{5}n_3 + \frac{1}{5}) = \frac{4}{5}n_3 - \frac{1}{5}$ $\quad 4n_3 = 5n_4 + 1 \quad (4)$

$n_0 = 2n_1 + 1 \quad (1)$

In (1) substitute for $2n_1$ from (2) $\quad n_0 = 3n_2 + 2 \quad (2^*)$

In (2^*) substitute for $3n_2$ from (3) $\quad n_0 = 4n_3 + 3 \quad (3^*)$

In (3^*) substitute for $4n_3$ from (4) $\quad n_0 = 5n_4 + 4 \quad (4^*)$

Hence
$n_0 + 1 = 2(n_1 + 1)$ from (1)
$n_0 + 1 = 3(n_2 + 1)$ from (2^*)
$n_0 + 1 = 4(n_3 + 1)$ from (3^*)
$n_0 + 1 = 5(n_4 + 1)$ from (4^*)

$\therefore$ $n_0 + 1$ is divisible by 2, 3, 4 and 5

$\therefore$ $n_0 + 1 = 60k \quad k \in \mathbb{Z}$.
60 is LCM.

If this is so, it will be possible to find n_1+1, n_2+1, n_3+1, n_4+1 as whole numbers from last 4 equations, and to deduce from these (1), (2), (3) (4).

Then $60k = 5(n_4 + 1)$
$n_4 + 1 = 12k$.

The simplest solution is $k = 1$, and 11 apples after all the sales.

PARTIAL DIFFERENTIATION.

1.) P is the point (x,y), Q is (X,Y). Find OP^2, OQ^2, PQ^2.

What is the condition for OP to be perpendicular to OQ? Simplify your result.

If **u** and **v** are the vectors (x,y) and (X,Y), how would your result be written in vector notation? What is it called?

2. Let $f(x,y) = (ax^2 + 2hxy + by^2)/2$. At what rate does f(x,y) change if (x,y) moves with velocity (dx/dt , dy/dt)?

If a=1, h=0, b=1, what is the condition for df/dt to be zero? What does this condition tell you about the vectors (x,y) and (dx/dt,dy/dt)? (Remember question 1.) Is this reasonable in the light of what you know about geometry?

The line through a point on a curve, perpendicular to the tangent is know as the **normal.** If the curve is f(x,y)=constant, with f(x,y) as given at the start of this question, find a vector that gives the direction of the normal to the curve.

P 11

PARTIAL DIFFERENTIATION.

1.) P is the point (x,y), Q is (X,Y). Find OP^2, OQ^2, PQ^2.
What is the condition for OP to be perpendicular to OQ? Simplify your result.
If **u** and **v** are the vectors (x,y) and (X,Y), how would your result be written in vector notation? What is it called?

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The line through a point on a curve, perpendicular to the tangent is know as the **normal**. If the curve is f(x,y)=constant, with f(x,y) as given at the start of this question, find a vector that gives the direction of the normal to the curve.

P
12

PARTIAL DIFFERENTIATION.

1.) P is the point (x,y), Q is (X,Y). Find OP^2, OQ^2, PQ^2.
What is the condition for OP to be perpendicular to OQ? Simplify your result.
If **u** and **v** are the vectors (x,y) and (X,Y), how would your result be written in vector notation? What is it called?

2. Let $f(x,y) = (ax^2 + 2hxy + by^2)/2$. At what rate does f(x,y) change if (x,y) moves with velocity (dx/dt , dy/dt)?
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The line through a point on a curve, perpendicular to the tangent is know as the **normal**. If the curve is f(x,y)=constant, with f(x,y) as given at the start of this question, find a vector that gives the direction of the normal to the curve.

Xrays Medical Council

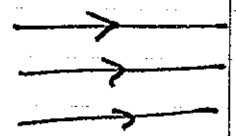
The difference between $\left.\frac{\partial f}{\partial x}\right)_y$ and $\left.\frac{\partial f}{\partial x}\right)_u$. P13

Let $u = y - x$. $f(x,y) = x^2 + y^2$

As $y = u + x$ $\quad f = x^2 + (u+x)^2$

$\left.\frac{\partial f}{\partial x}\right)_y = 2x \qquad \left.\frac{\partial f}{\partial x}\right)_u = 2x + 2(u+x)$

In a variation of (x,y) where $x = t$, $y = c$.
the point (x,y) moves along lines



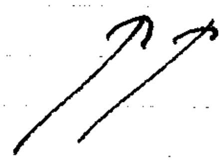
i.e. along lines y = constant.

In a variation where u is constant

$y - x = c$.

If $x = t$, $y = c + t$.

Thus point (x,y) moves along the lines.

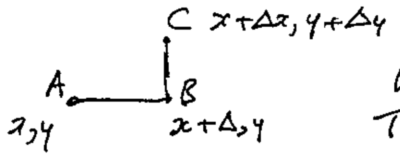


Partial Differentiation.

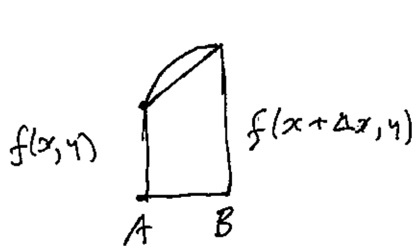
Rolle's Theorem  Mean value theorem 

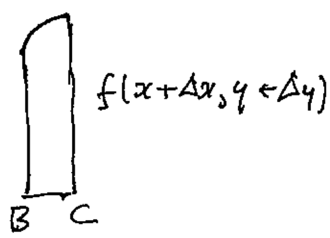
There is a point where $\frac{dy}{dx}$ = gradient of chord.

$z = f(x,y)$ given. x, y grow at rates $\frac{dx}{dt}, \frac{dy}{dt}$. What is dz/dt?



In time Δt let x increase by Δx, y by Δy.
Thus we move from A to C in diagram.





$f(x,y) = f_A \quad f(x+\Delta x, y) = f_B \quad f(x+\Delta x, y+\Delta y) = f_C$

In first diagram $\frac{f_B - f_A}{AB}$ = gradient at intermediate point $= \frac{\partial f}{\partial x}(x+\alpha\Delta x, y)$

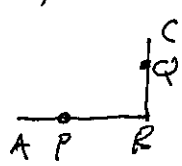
In second diagram $\frac{f_C - f_B}{BC} = \frac{\partial f}{\partial y}(x+\Delta x, y+\beta\Delta y)$

$\therefore \quad f_C - f_A = (f_C - f_B) + (f_B - f_A) = AB\frac{\partial f}{\partial x}(x+\alpha\Delta x, y) + BC\frac{\partial f}{\partial y}(x+\Delta x, y+\beta\Delta y)$

$\frac{dz}{dt}$ is limit of $\frac{f_C - f_A}{\Delta t}$, so limit of

$$\frac{\Delta x}{\Delta t}\frac{\partial f}{\partial x}(x+\alpha\Delta x, y) + \frac{\Delta y}{\Delta t}\frac{\partial f}{\partial y}(x+\Delta x, y+\beta\Delta y)$$

Thus this is related to gradients at P and Q.
All we know about P and Q is that they are somewhere on AB and BC.



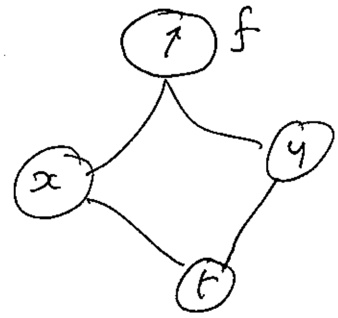
When $\Delta t \to 0$, $\Delta x \to 0$ and $\Delta y \to 0$. Thus all the points P, B, Q, C tend to A.

<u>We have to assume</u> $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ <u>continuous</u> if we <u>are to have</u> $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.

[In the counter example $z = \frac{xy}{r}$, $f(0,0) = 0$
if we are at a point $(x, 0)$, x/r is stationary as y co-ordinate begins to increase, for $x/r = \cos\theta$, with max. at $\theta = 0$.
y grows at rate 1, so $\frac{\partial f}{\partial y} = 1$ at $(x,0)$ where $x \neq 0$.
As $z = 0$ for $x = 0$ and for $y = 0$, $\frac{\partial f}{\partial y} = 0$ at $(0,0)$ $x \to 0$ does not give this]

① $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$ is often written

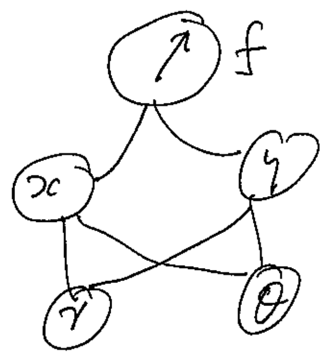
② $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ and this form is often very convenient.



The situation first considered has this scheme. The meter reading, f, depends on the position of the dials. These dials are driven by a clock, t.

The equation (2) corresponds to the fact that small changes in f are caused by small changes in x, y. The rate at which these changes occur is irrelevant.

<u>Change of variable.</u>



x, y can be expressed in terms of polar co-ordinates. r, θ. We have

$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta$

$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta$

So $df = \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta\right) + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta\right)$

$= dr\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}\right) + d\theta\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}\right)$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

Corresponds to chain rule. We have $\frac{\partial f}{\cdot}\frac{\cdot}{\partial r}$ and in

• spaces, variables corresponding to the different routes by which r can influence f.

Note: we cannot cancel ∂x and ∂y: this would give $\frac{\partial f}{\partial r} = 2\frac{\partial f}{\partial r}$.

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta)$$

[Set this]. Example. Consider $f = x^2 + y^2$ $\frac{\partial f}{\partial x} = 2x$ $\frac{\partial f}{\partial y} = 2y$.

$$\frac{\partial f}{\partial r} = 2x\cos\theta + 2y\sin\theta = 2r\cos^2\theta + 2r\sin^2\theta = 2r.$$

$$\frac{\partial f}{\partial \theta} = 2x(-r\sin\theta) + 2y(r\cos\theta)$$

$$= 2r\cos\theta(-r\sin\theta) + 2r\sin\theta(r\cos\theta) = 0$$

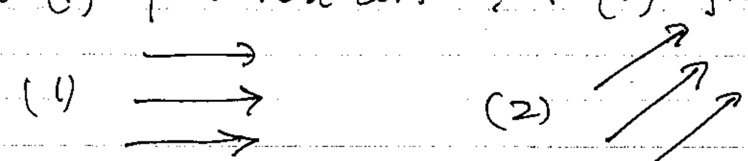
as it should.

Partial Differentiation.

$f(x,y)$. $\frac{\partial f}{\partial x}$ is rate of increase when x grows at unit rate, y held constant. $\frac{\partial f}{\partial y}$ is rate of increase when y grows at unit rate, x held constant.

These are sometimes denoted by $\left.\frac{\partial f}{\partial x}\right)_y$ and $\left.\frac{\partial f}{\partial y}\right)_x$, the $)_x$ and $)_y$ showing what is held constant.

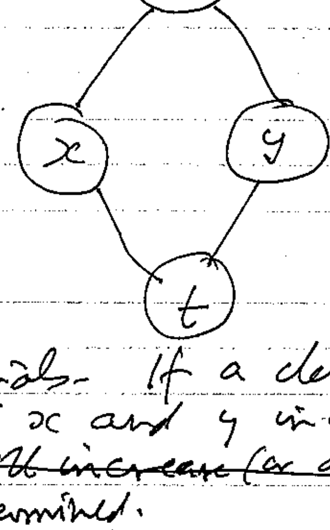
In both the following situations x is growing, but in (1) y is held constant, in (2) $y-x$ is held constant.



(1) would arise naturally if we have $z = f(x,y)$
(2) if ~~$z = f(x,y)$~~ $z = f(x,u)$ with $u = y-x$.

In physics this arises for a gas, temperature rising with constant pressure has different effects from temperature rising at constant volume (gas held in solid container.)

In the first case, heat supplied has two parts — that used to raise the temperature, and that doing work to cause expansion. In the second case, all the heat is used to raise temperature.



If $f = f(x,y)$ and x and y are increasing at rates $\frac{dx}{dt}$, $\frac{dy}{dt}$ we may picture f as given by a meter reading, the value of which depends on the settings of 2 dials. If a clock is geared to these so that x and y increase in a specified way, ~~f will increase (or decrease)~~ the behaviour of f is determined.

If x increases at rate $\frac{dx}{dt}$, $f \uparrow$ at rate $\frac{\partial f}{\partial x}\frac{dx}{dt}$

y - - - - - - $\frac{dy}{dt}$ $\qquad \frac{\partial f}{\partial y}\frac{dy}{dt}$

In many situations (not in all), if x and y vary simultaneously

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \qquad (1)$$

[This result is equivalent to $z = f(x,y)$ having a tangent plane at the point (x, y, z).]

It is often convenient to write eqn (1) as

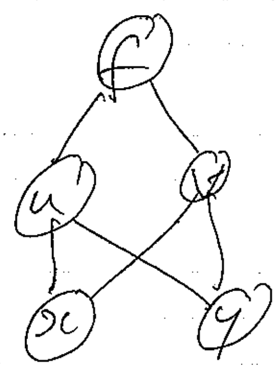
$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \qquad (2)$$

The chain rule $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$ looks as if it is a result in algebra. Note that (1) does not. We cannot cancel ∂x and dx, etc.

$\frac{\partial f}{\partial x}\frac{dx}{dt}$ is that part of $\frac{df}{dt}$ due to the change in x; $\frac{\partial f}{\partial y}\frac{dy}{dt}$ that part due to y varying.

Change of variable

Let $f = f(u, v)$ where u and v are functions of x and y



$$df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

Substituting gives

$$df = \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)$$

$$= \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}\right)dy$$

So $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$; $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$

When x changes, part of the effect is via u, part via v.

Edwards. p 125 No 4.

If $x = r\cos\theta$ and $y = r\sin\theta$ prove

$dx = \cos\theta\, dr - r\sin\theta\, d\theta$

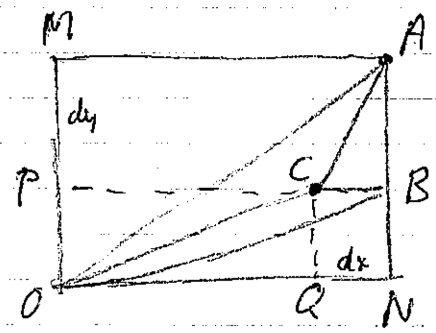
$dy = \sin\theta\, dr + r\cos\theta\, d\theta$.

Deduce $dx^2 + dy^2 = dr^2 + r^2 d\theta^2$

$x\,dy - y\,dx = r^2 d\theta$.

But why? ds^2. $\frac{1}{2}r^2 d\theta$ is area swept out.

C, A points on curve.
$OC = r$ $\angle AOC = d\theta$
$MN = x$ $AB = dy$
$x\,dy = \square$ PBAM
$= 2\triangle OAB$
$y\,dx = \square$ CBNQ
$= 2\triangle OBC$



$x\,dy - y\,dx = 2(\triangle OAB - \triangle OBC) = 2(\triangle OAC + \triangle ACB)$

$2\triangle ACB = dx\,dy$. 2nd order.

Note We have taken MA as x. Perhaps it should be $x+dx$.
This would account for an extra $dx\,dy$.

5) If $u = \ln(x^2+y^2+z^2)$ prove

$$x\frac{\partial^2 u}{\partial y\partial z} = y\frac{\partial^2 u}{\partial z\partial x} = z\frac{\partial^2 u}{\partial x\partial y}.$$

(5) $u = \ln(x^2+y^2+z^2)$

$$du = \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2}$$

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2+z^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \cancel{\frac{2(x^2+y^2+z^2) - 2x\,2y}{(x^2+y^2+z^2)^2}}$$

$$= \frac{-2x(2y)}{(x^2+y^2+z^2)^2} = -\frac{4xy}{(x^2+y^2+z^2)^2}$$

$$\therefore z\frac{\partial^2 u}{\partial x \partial y} = -\frac{4xyz}{(x^2+y^2+z^2)^2} \quad \text{Sym QED}$$

(6) If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ prove $\frac{dy}{dx} = -\frac{b^2x}{a^2y}$ $\quad \frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$

If $f(x,y) = \text{const}$ $\quad dx\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}dy = 0 \quad \therefore \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$

For $f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ $\quad \frac{\partial f}{\partial x} = \frac{2x}{a^2}$ $\quad \frac{\partial f}{\partial y} = \frac{2y}{b^2}$ Hence (i)

$$b^2x + a^2yy' = 0$$

$$\frac{d}{dx} \qquad b^2 + a^2(yy'' + y'^2) = 0$$

$$\therefore a^2yy'' = -b^2 - a^2y'^2$$

$$\therefore y'' = -\frac{b^2}{a^2y} - \frac{y'^2}{y}$$

$$= -\frac{b^2}{a^2y} - \frac{b^4x^2}{a^4y^3} = -\frac{a^2b^2y^2 + b^4x^2}{a^4y^3}$$

$$= -\frac{a^2b^4\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)}{b^4\,a^4y^3}$$

$$= -\frac{b^4}{a^2y^3}$$

---

or $\quad y' = -\frac{b^2x}{a^2y}$

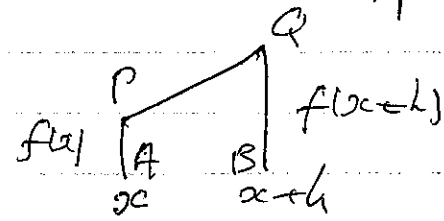
$$y'' = -\frac{b^2\cancel{a^2}y - \cancel{a^2}y'b^2x}{a^4y^2} \qquad y' = -\frac{b^2x}{a^2y}$$

Approximations to $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. P21

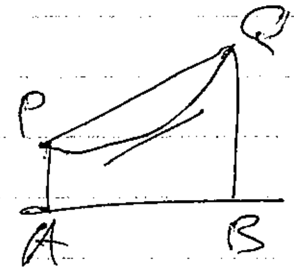
A computer may give us a table for $y = f(x)$ but not the analytic formula. If we wish to do some work involving $f'(x)$ and $f''(a)$ we have to use approximations and hope these will work well.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

If we have a table of $f(x)$ at intervals h, we naturally consider $\frac{f(x+h) - f(x)}{h}$ as a suitable approximation.

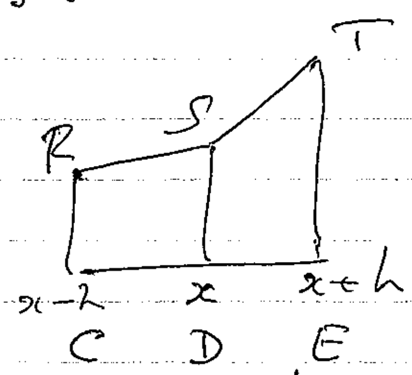


This is the gradient of PQ. It may very well give the gradient at some point between A and B.



If h is extremely small, and we have reason to believe that the gradient varies gradually we might even take it as giving $f'(a)$ at A.

$f''(x)$



The approximation here is slightly less arbitrary. $f''(x)$ is the rate of change of $f'(x)$.

$\frac{f(x) - f(x-h)}{h}$ is an estimate for $\frac{dy}{dx}$ in CD,

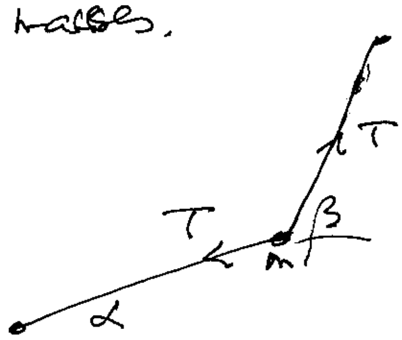
$\frac{f(x+h) - f(x)}{h}$ a ST. Now ST is h to the right of RS. This leads to the estimate

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

Note. This is $\frac{\Delta^2 f}{(\Delta x)^2}$.

We can use this approximation to make plausible – not to prove – various results in physics.

Vibrating wire, loaded with masses.
Tension T unaltered.
Force on mass m is
$T\sin\beta - T\sin\alpha$
$\doteq T\tan\beta - T\tan\alpha$



$$= T\,\frac{f(x+h)-f(x)}{h} - T\,\frac{f(x)-f(x-h)}{h}$$

$$= T\,\frac{f(x+h)-2f(x)+f(x-h)}{h}$$

If masses are at intervals h the mass per unit length $= m\cdot\frac{1}{h} =$

Acceleration given by

$$m\frac{\partial^2 y}{\partial t^2} = T\,\frac{f(x+h)-2f(x)+f(x-h)}{h}$$

$$\therefore \rho\frac{\partial^2 y}{\partial t^2} = \frac{m}{h}\frac{\partial^2 y}{\partial t^2} = T\,\frac{f(x+h)-2f(x)+f(x-h)}{h^2}$$

$$\doteq T\frac{\partial^2 y}{\partial x^2}$$

~~If $c = \sqrt{T/\rho}$, ∂^2~~ If $c^2 = \frac{T}{\rho}$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}.$$

Dimensions. $\rho = M/L$. T = force = mass × accel^n
$= MLT^{-2}$

$$\therefore c^2 = \frac{T}{\rho} = \frac{MLT^{-2}}{M/L} = L^2T^{-2}$$

LT^{-1} is velocity. c has the dimensions of a velocity.

Other applications left for the moment.

Change of variable in $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$.

Let $u = x + ct$.
$v = x - ct$.

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}.$$

$$\frac{\partial}{\partial t} = \frac{\partial u}{\partial t}\frac{\partial}{\partial u} + \frac{\partial v}{\partial t}\frac{\partial}{\partial v} = c\frac{\partial}{\partial u} - c\frac{\partial}{\partial v}$$

$$\therefore \frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)^2 - \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)^2$$

$$= 4\frac{\partial^2}{\partial u \partial v}$$

$$\therefore \underline{\frac{\partial^2 y}{\partial u \partial v} = 0}$$

What is meaning of such an equation?

Consider 1) $\frac{\partial f}{\partial x} = 0$

2) $\frac{\partial f}{\partial y} = 0$

3) $\frac{\partial^2 f}{\partial x \partial y} = 0$.

Paul Phyllocontor
Soak. Cream Freedom

$(2x-5)(3x+4) = 6x^2-7x-20$

$6x^2-7x-20 = 0$

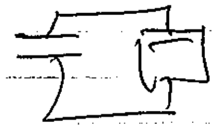
$6x^2-7x = 20$

$x^2-\frac{7}{6}x = \frac{20}{6} = \frac{10}{3}$

$\left(x-\frac{7}{12}\right)^2 = \frac{10}{3} + \frac{49}{144} = \frac{480+49}{144} = \frac{9}{144}$

$3\overline{)1440}$
48

Very thought Factors — not seen before. Dinner the

Perel's sense Predicts Vy 

$u = x + ct$

$v = x - ct$

$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$

$\frac{\partial}{\partial t} = c\frac{\partial}{\partial u} - c\frac{\partial}{\partial v}$

$\left(\frac{\partial}{\partial x}\right)^2 - \frac{1}{c^2}\left(\frac{\partial}{\partial u}\right)^2$

Q1 p 25

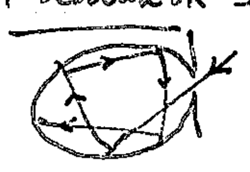
Paradoxes of Classical Physics.

1. The existence of atoms.
The stability of chemical compounds led chemists to believe in a static model of the atom. When it became accepted that electrons moved in orbits, a difficulty arose. ~~A moving charge radi~~ An accelerated charge ~~radiates~~ ~~energy~~ causes radiation and thus loses energy. An electron in orbit around a nucleus should lose energy and spiral inward, until it collided with the nucleus.

2. Line spectra
Very often the spectrum of an atom contains lines. Each line corresponds to a particular frequency. Classical theory had no way of explaining why certain frequencies should be picked out as special.

3 The photoelectric effect.
When light (or other radiation) falls on a solid it may cause electrons to be emitted. The way in which this happened was puzzling. If the frequency of the light was below a certain value, no electrons came however intense the illumination. When the frequency is sufficient to cause the emission of electrons, the maximum kinetic energy of the emerging electrons cannot be increased by more intense illumination. The effect is that more electrons emerge per second. The kinetic energy does increase if the ~~higher~~ frequency of the light increases.

4. Black body radiation
A body is "black" when it absorbs all radiation falling on it. A heated body with a cavity is used. Radiation enters through a tiny hole. It is partly absorbed, partly reflected many times. The chance of it emerging is very small.



If the cavity is heated to, say, red heat, a small amount of radiation will emerge from the hole. This is studied as "black-body radiation".

Q 2
p26

Radiation in a heated cavity can be studied by the methods of statistical mechanics. Rather as for a vibrating string, various simple modes are possible

We can have n parts such as ⌒ or ⌣. On classical theory, each such vibration has the same energy.

How many oscillations have wavelength in $\lambda-\varepsilon$ to λ?

If $\lambda = \frac{L}{n}$ and $\lambda - \varepsilon = \frac{L}{m}$, $m = \frac{L}{\lambda-\varepsilon}$, $n = \frac{L}{\lambda}$

$m - n = \frac{\varepsilon}{\lambda(\lambda-\varepsilon)}$. If ε is small, this is approximately ε/λ^2. Thus the number of modes lying between n and m is much larger for small λ.

In fact, there are an infinity of modes, and only a finite number of these have $\lambda \geq a$, for any a.

Thus energy is concentrated in the very short waves.

The Rayleigh-Jeans catastrophe showed that it fails ~~the energy~~ $e_\lambda\, d\lambda = cT\lambda^{-4}\, d\lambda$. This fits the experimental data only for longer wavelengths.

Correspondence principle — classical theory works all right for large or slow phenomena. Any new theory should agree with old in such situations.

Planck seems to have found it mathematically convenient to use an argument in which the energy of any vibrator had to be $0, \varepsilon, 2\varepsilon, 3\varepsilon \ldots$. with the intention of letting $\varepsilon \to 0$ at the end of the calculation. However he found he could get a better result by not taking this limit.

He found $$e_\lambda\, d\lambda = 8\pi\lambda^{-4}\frac{\varepsilon}{e^{\varepsilon/kT}-1}\, d\lambda \quad (1)$$

If ε is small, $e^{\varepsilon/kT} \doteq 1 + \varepsilon/kT$ and the limit is $8\pi kT\lambda^{-4}$, the Rayleigh Jeans catastrophe.

Thermodynamic reasoning had shown that the formula ought to have the form $C\lambda^{-5} f(\lambda T)\, d\lambda$ (2) for SOME FUNCTION f.

Two differences: (1) has λ^{-4}, (2) has λ^{-5}. Both differences disappear if we take $\varepsilon = \alpha/\lambda$. Then we have

$$8\pi\lambda^{-5}\frac{\alpha}{e^{\alpha/k\lambda T}-1}\, d\lambda. \quad (3)$$

If we have a long wavelength, λ is large, $\alpha/k\lambda T$ is small and $e^{\alpha/k\lambda T} \doteqdot 1 + \frac{\alpha}{k\lambda T}$

$$\text{Thus } (3) \rightarrow 8\pi\lambda^{-5} \cdot \frac{\alpha\, d\lambda}{\left(\frac{\alpha}{k\lambda T}\right)} = 8\pi k \lambda^{-4}\, d\lambda,$$

the Rayleigh Jeans result. So the Correspondence Principle is satisfied: the classical result checks for long wavelengths.

Wave length λ now occurs with packets of energy $\varepsilon = \alpha/\lambda$. As $\lambda = c/\nu$, where ν is the frequency, $\varepsilon = \frac{\alpha}{c}\nu$. Putting $h = \alpha/c$, the packet is $h\nu$, where h is Planck's constant.

We have a paradox: light behaves like waves: it also behaves like a hail of bullets.

This explains the photo electric effect. Light of frequency ν corresponds to bullets of energy $h\nu$. If this energy is not enough to knock an electron out of a metal, no photo-electrons will appear, however intense the light. Greater intensity means more bullets, but as each bullet is ineffective, this makes no difference.

If the energy is enough to knock the electron out, the maximum velocity of electrons emerging is not increased by increasing the light intensity, - only the frequency with which electrons emerge.

This explanation was given by Einstein in 1905.

The Bohr model of the atom.

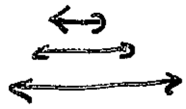
The next step forward after Planck was a very remarkable piece of thinking and quite as revolutionary.

The following considerations may prepare you for it.

Planck's basic assumption was that a simple oscillator with frequency ν would have energy $0, h\nu, 2h\nu, \ldots$. If $x = a\cos(\nu t/2\pi)$ describes its motion the energy is $\frac{1}{2}m\left(\frac{\nu a}{2\pi}\right)^2$, thus a^2 can have the values

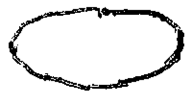
$$\frac{8\pi^2}{m\nu^2}\cdot nh\nu = n\,\frac{8\pi^2 h}{m\nu}.$$

Thus the possible vibrations can have only certain amplitudes



$n=1$
$n=2$
$n=3$

Suppose we have a particle that can oscillate in the x-direction and in the y-direction. If $n=3$ for the x oscillation and $n=2$ for the y oscillation it will describe a path



There will be other possible paths singled out by the conditions.

Bohr postulated that, for the motion of an electron about a proton, certain orbits were singled out. On classical theory, an electron moving in an orbit would radiate energy and move in towards the proton. Bohr maintained that this was not so. The orbits were stable. An electron could remain in an orbit for a considerable time.

However it was possible for an electron to cease to be in an orbit, and appear instead in a ~~lower~~ orbit with less energy. Radiation would be emitted. The frequency of the radiation would not bear any relationship to the velocity or acceleration of the electron in either orbit. It would be determined by the conservation of energy. A photon (pulse of light) with frequency ν has energy $h\nu$. ~~The the~~ If the electron falls from an orbit of energy E_1 to an orbit of energy E_2 then $E_1 - E_2 = h\nu$.

If the electron is in the orbit with least energy it cannot fall further. Thus atoms can survive.

The theory explains why the spectrum contains separate lines. On classical theory it would be continuous.

When radiation is absorbed, it can kick an electron to a higher level

Q8. P29

The quantum conditions

A very simple case is that of a mass m describing a circle of radius r with constant velocity, $v = r\dot{\theta}$

The angular momentum $rv = r^2\dot{\theta}$. The mass returns to the same position when θ changes by 2π.

The quantum condition is $2\pi r^2 \dot{\theta} m = nh$. ①

From dynamics $mr\dot{\theta}^2 = e^2/r^2$ ②

From ①, $\dot{\theta}^2 = \dfrac{n^2h^2}{4\pi^2 m^2 r^4}$

Substitute in ②

$$\frac{mr \cdot n^2h^2}{4\pi^2 m^2 r^4} = \frac{e^2}{r^2}$$

$$\therefore \quad r = \frac{n^2h^2}{4\pi^2 me^2} \qquad ③$$

$$E = \frac{1}{2}mr^2\dot{\theta}^2 - e^2/r$$

$$= \frac{1}{2}mr^2 \cdot \frac{e^2}{mr^3} - \frac{e^2}{r} \quad \text{from ②}$$

$$= \frac{e^2}{2r} - \frac{e^2}{r} = -\frac{e^2}{2r}$$

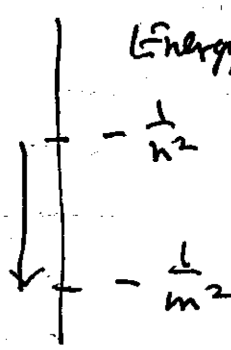
So $$E = -\frac{4\pi^2 me^4}{2n^2h^2} = -\frac{2\pi^2 me^4}{n^2h^2}.$$

The lines in the spectrum fit

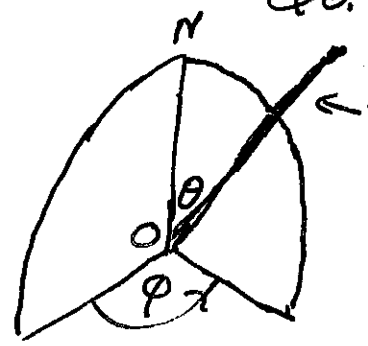
$$\text{frequency} = c\left(\frac{1}{m^2} - \frac{1}{n^2}\right) \quad \text{where } m < n.$$

This is a fall in energy when radiation is emitted, a rise when radiation is absorbed.

Energy levels.

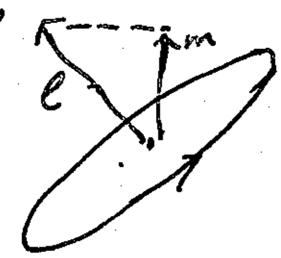


In a hydrogen atom, the position of an electron requires 3 numbers to specify it. Q6.



r the distance from the origin.
θ the co-latitude
φ the longitude.

If the electron describes an orbit ~~ellipse~~ in a plane not perpendicular to ON, all three variables, r, θ, φ, are periodic; they return to their initial values when the electron has returned to its initial position.



Bohr found conditions for the orbits to be permissible, which gave 3 quantum numbers

n' associated with r
l associated with θ
m associated with φ.

The total quantum number, $n = n' + l + 1$, is important. The energy is proportional to $-1/n^2$.

n can be $1, 2, 3, \dots$
l $0, 1, \dots\dots n-1$.
m $-l, -l+1, \dots 0, \dots l-1, l$.

In an atom with several electrons it is no longer possible to find exact expressions for the orbits or for the energy, but it is assumed that each electron has numbers n, l, m and also a spin number $\pm 1/2$.

Pauli exclusion principle: two electrons cannot have all their numbers identical.

The fact that s can be $+\frac{1}{2}$ or $-\frac{1}{2}$ means that if n, l, m have been fixed, there are just 2 possibilities.

Thus, we can forget about s if we allow every number system n, l, m to be taken twice.

Lowest order If $n=1$, $l=0$, $m=0$. Two atoms can be accomodated at this level.

	$n=1$, $l=0$
H	1
He	2

Choices for m. $-l \le m \le l$. For given l, there are $2l+1$ choices for m.
Doubling this, to allow for $s = \pm\frac{1}{2}$, we have $2(2l+1)$

Thus, with $l=0$ we have niches for 2 atoms.
$l=1$ 6
$l=2$ 10

To begin with, these niches are filled in a quite systematic way.

n	1	2		3			4	
l	0	0	1	0	1	2	0	1
H	1							
He	2							
Li	2	1						
Be	2	2						
B	2	2	1					
C	2	2	2					
N	2	2	3					
O	2	2	4					
F	2	2	5					
Ne	2	2	6					
Na				1				
Mg				2				
Al				2	1			
Si				2	2			
P				2	3			
S				2	4			
Cl				2	5			
Ar				2	6			

notice the similarity in the outside shells.

This part
[s]tays the same

Q8 p 32

At the next stage we would expect to find $n=3$, $l=2$ filling up. But it does not happen like that.

n	1	2		3			4	
l	0	0	1	0	1	2	0	1
K	2	2	6	2	6		1	
Ca							2	
Sc						1	2	
Ti						2	2	
V						3	2	
Cr						5	1	
Mn						5	2	
Fe						6	2	
Co						7	2	
Ni						8	2	
Cu						10	1	
Zn						10	2	
Ga						10	2	1
Ge						10	2	2
As						10	2	3
Se						10	2	4
Br						10	2	5
Kr						10	2	6

The transition elements are the most industrially important metals and this is mainly due to the presence of strong interatomic bonding ~~forces~~ which results in them having high melting points and good mechanical properties.

Liptrot. p 350 Modern Inorganic Chemistry.

See also Durrant General and Inorganic Chemistry p 157

D P33

A sequence $a_0 a_1 \ldots a_n$ is to have the property that 0 occurs in it a_0 times, 1 occurs a_1 times and so, n occurring a_n times. An example is

r	0	1	2	3	4	5	6	7	8
a_r	5	2	1	0	0	1	0	0	0

The sum of the numbers in the sequence, found by considering how many times each of 0, 1 .. n occurs, is $a_0(0) + a_1(1) + a_2(2) + \ldots + a_n(n)$.
But the sum is $a_0 + a_1 + a_2 + \ldots + a_n$.
Equating these we have $a_0 = a_2 + 2a_3 + 3a_4 + \ldots (n-1)a_n$. (I)
It follows that $a_3, a_4 \ldots a_n$ are all less than a_0. Also $a_2 < a_0$ unless $a_3 \ldots a_n$ are all 0, and then $a_2 = a_0$. We first consider this exceptional case.

The sequence $a_0, a_1, a_2, 0 \ldots 0$.
Let $a_2 = a_0 = k$, $a_1 = m$. The numbers in the sequence are selected from 0, 1, 2. The number of non-zero numbers in the sequence is $k + m$. ~~This cannot be 3, for then~~ This cannot exceed 3, since only a_0, a_1, a_2 can be other than 0. As $a_3 = 0$, $a_4 = 0 \ldots$ no number larger than 2 can occur in the sequence. $k = 0$ is ruled out, since this would say no 0 occurs, and at the same time say $a_0 = 0$. These conditions allow 5 possible ~~There are 5 possible sequences~~ starts ~~sequences~~. (a) 101.. (b) 111.. (c) 121 (d) 202, (e) ~~201~~ 212

(a) and (e) must be ruled out; they say no 1 occurs, but it does. (b) falsely claims the presence of 2.
(c) gives 1210 and (d) gives 2020
both of which are solutions

The General Case.

The general case ~~occurs~~ is that when $a_3 \ldots a_n$ are not all zero. Equation (I) shows that a_0 is at least 2, so $a_0 \neq 1$. We have shown that a_0 is larger than any number in the sequence except possibly a_1. P.34

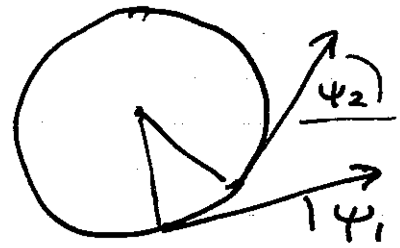
We proceed to eliminate certain possibilities. As before, let $a_0 = k$, $a_1 = m$.

We show $a_0 \neq a_1$. If $m = k$, all the other numbers in the sequence being less than k we have $a_k = 2$. ~~Thus neither a_0 nor a_k can be 1. The sites where 1 can occur are We know $a_0 \neq 1$, a~~ Also $a_1 = m$, and $m = k > 1$. So a_0, a_1 and a_k are not places available for 1. This leaves $a_2 \ldots a_{k-1}$, only $k-2$ possible sites, so a_1 cannot exceed $k-2$. There is a contradiction in supposing $a_1 = k$.

We also show a_1 is not greater than k. If it were we would have $a_k = 1$, $a_m = 1$. ~~We try to fit $m-2$ "1"s~~ ~~Thus $m-2$ entries of 1 must occur for a_r with $r < k$.~~ As all the other numbers are less than k, we cannot find $a_r = 1$ with $r > k$ except for $r = m$. The only places where 1 can ~~occur are a~~. $a_1 > k > 1$ excludes the possibility that $a_1 = 1$. Thus the possible ~~sites~~ possible for $a_r = 1$ are $a_2 \ldots a_{k-1}$. There are $k-2$ in number. 1 must occur m times in all, so it must occur $m-2$ times apart from a_k and a_m. So $m-2$ cannot exceed $k-2$, ~~the~~ which means m cannot exceed k. This contradicts the initial hypothesis $m > k$.

Accordingly $a_1 < a_0$.

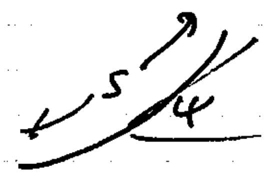
Object in circle
radius R
ψ angle between tangent
and fixed direction



Angle at centre is $\psi_2 - \psi_1$

$$\text{arc} = R(\psi_2 - \psi_1)$$

$$ds = R\, d\psi \qquad R = \frac{ds}{d\psi}$$

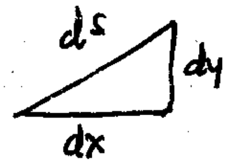


For a circle $\frac{ds}{d\psi}$ is constant.

For any curve $\frac{ds}{d\psi}$ gives a measure of the curvature at that point, the radius of the circle that this part of the curve resembles.

We define $\rho = \frac{ds}{d\psi}$, the "radius of curvature".

<u>Formula for ρ in Cartesian co-ordinates.</u>

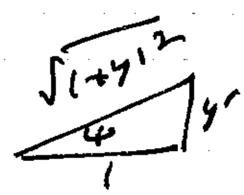


$$ds^2 = dx^2 + dy^2$$

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\frac{ds}{dx} = \sqrt{1 + y'^2}$$

$$y' = \tan\psi \qquad \psi = \tan^{-1} y'$$

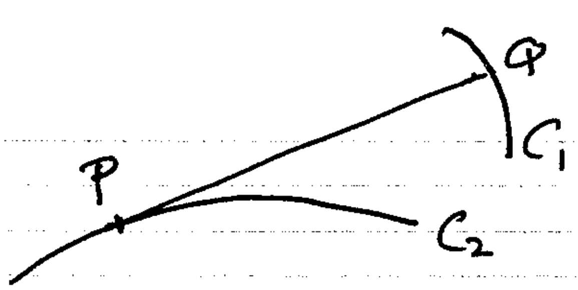


$$\frac{d}{dx} \qquad y'' = \sec^2\psi \cdot \frac{d\psi}{dx}$$

$$\frac{d\psi}{dx} = \frac{y''}{\sec^2\psi} = \cos^2\psi \cdot y'' = \frac{y''}{1+y'^2}$$

$$\rho = \frac{ds}{d\psi} = \frac{ds/dx}{d\psi/dx} = \frac{\sqrt{1+y'^2}}{\left(\frac{y''}{1+y'^2}\right)} = \frac{(1+y'^2)^{3/2}}{y''}$$

2
R_2



If we get curve C_1 by unwinding a thread from curve C_2 it looks as if PQ is normal to curve C_1 at Q, PQ is tangent to curve C_2 at P and P is the centre of circle resembling the curve C_1 at Q.

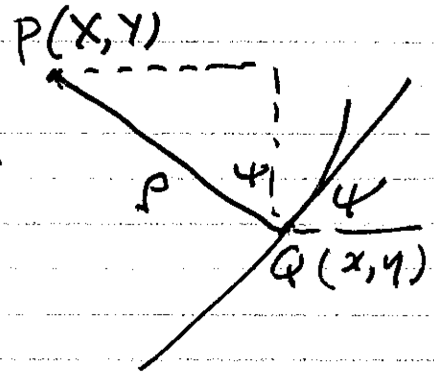
It is good to verify intuitive ideas analytically.

We need to show that, if a little later we have P′ and Q′, then in the limit

P′P = the increase in ρ

P′P should be in line with PQ.

—



Centre of curvature is distance ρ along the normal from Q.

Curve C_2 is to be the locus of centre of curvature.

Let P be (X, Y)

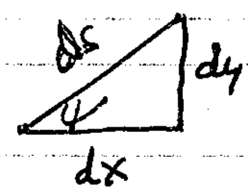
$$X = x - \rho \sin\psi$$
$$Y = y + \rho\cos\psi$$

We want to show that in an infinitesimal change, centre of curvature moves in direction QP, and a distance $d\rho$.

$$dX = dx - d\rho\sin\psi - \rho\cos\psi\, d\psi$$

$$dY = dy + d\rho\cos\psi - \rho\sin\psi\, d\psi$$

$$\rho = \frac{ds}{d\psi} \qquad \rho\, d\psi = ds$$



$$-\rho\cos\psi\, d\psi = -ds\cos\psi$$
$$= -dx$$

$$\therefore \underline{dX = -d\rho \sin\psi}$$

$$-\rho \sin\psi \, d\psi = -\frac{ds}{d\psi} \sin\psi \, d\psi$$

$$= -ds \sin\psi = -dy$$

$$\therefore \underline{dY = d\rho \cos\psi}$$

So (dX, dY) represents a distance $d\rho$ along normal. Q.E.D.

A possible construction for the involute (locus of centres of curvature)

Radius of curvature.

If a particle is moving in a circle 
of radius R, and ψ is angle between tangent and a fixed direction, $R\,d\psi = ds$

so $R = \frac{ds}{d\psi}$.

For a curve that is not a circle we take $\rho = \frac{ds}{d\psi}$ as the instantaneous radius of curvature. This would, for example, allow us to estimate the speed at which a car could be driven at that point, by comparing it to driving round a circle with that radius.

ρ is called "radius of curvature".

ρ in terms of x, y.

$$\tan\psi = \frac{dy}{dx} \qquad \psi = \tan^{-1} y'$$

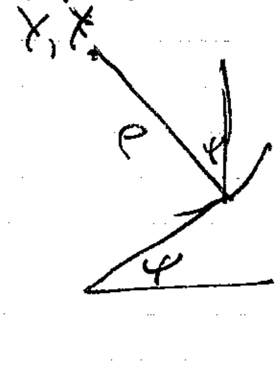
$$\therefore \frac{d\psi}{dx} = \frac{1}{1+y'^2}\cdot y''$$

$$\frac{ds}{dx} = \sqrt{1+y'^2}$$

$$\therefore \frac{ds}{d\psi} = \frac{(1+y'^2)^{3/2}}{y''} = \rho.$$

Centre of curvature is reached by going a distance ρ along the normal.

$$X = x - \rho\sin\psi$$
$$Y = y + \rho\cos\psi.$$

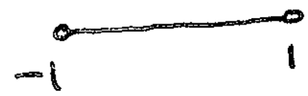


Set. **Sets, Open & Closed**

Neighbourhood supposed defined.
Neighbourhood of a point $P = \{Q : \|PQ\| < k\}$ for some $k > 0$.
Limit point of set S. P, such that any neighbourhood of P contains points of S other than P. It does not matter whether $P \in S$ or not.

Example $S_1 = \{x : |x| < 1\}$ $x = 1$ is a limit point.

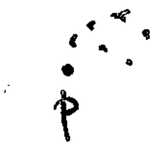
1 is still a limit point if $S_2 = \{x : |x| \leq 1\}$



In fact every point of S_1 is a limit point of S_1. Same for S_2.

There must be ∞ points in a neighbourhood of a limit point.

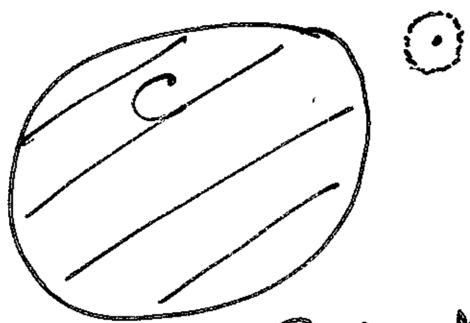
Proof Suppose that a neighbourhood contains only $Q_1, Q_2 \ldots Q_n$.
Then P cannot be a limit point: consider
$k <$ minimum ~~$\{Q_1, Q_2 \ldots Q_n\}$~~ $\{\|PQ_1\| \ldots \|PQ_n\|\}$



Closed set. S is closed if it contains all its limit points.
Example (1) In $\mathbb{R}$, S_2 above or $[a,b] = \{x : a \leq x \leq b\}$
(2) In $\mathbb{R}^2$, $\{Q : \|PQ\| \leq r\}$

Interior point of S. Point P such that some neighbourhood of P consists entirely of points of S.

Open set A set in which every point is an interior point.



C is closed. P is not in C, i.e. $P \in \overline{C}$

Is it possible to find a neighbourhood P consisting entirely of points in $\overline{C}$?

Suppose this is not possible. Then, in any neighbourhood of P, there is a point not in $\overline{C}$, i.e. a point in C.

Then P is a limit point of C.

But a closed set contains all its limit points.

$\therefore$ $P \in C$.

Contradiction.

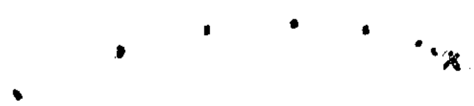
$\therefore$ Every point of $\overline{C}$ ~~can be~~ has a neighbourhood of points of $\overline{C}$.

$\therefore$ Every point is an interior point.

$\therefore$ $\overline{C}$ is an open set.

Complement of an open set.

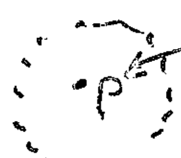
O open set. We want to show that any limit point of $\bar{O}$ is in $\bar{O}$. (R.T.P.)



If P is a limit point of $\bar{O}$, any neighbourhood of P contains points of $\bar{O}$, not P.
We want to show that P must be in $\bar{O}$.

If this is not true $P \in O$.

So some neighbourhood of P consists entirely of points of O.
i.e. it contains no points of $\bar{O}$.



O only inside this.

So if $P \in O$, it cannot be a limit point of $\bar{O}$.

$\therefore$ All limit points of $\bar{O}$ are in $\bar{O}$.

$\therefore$ Complement, $\bar{O}$, of an open set is closed.

Union and intersection of open sets and of closed sets.

Union of open sets $\bigcup O_r$

We expect this to be open. i.e every point of $\bigcup O_r$ should be an interior point of $\bigcup O_r$.

P is in $\bigcup O_r$ if $P \in O_i$ for some i in family of i.

$\therefore$ P is an interior point of O_i,

i.e there is a neighbourhood P entirely in O_i.

$\therefore$ neighbourhood of P entirely in $\bigcup O_r$.

$\therefore$ $\bigcup O_r$ is open

Intersection $\bigcap O_r$

Example. $O_n = \{x : -\frac{1}{n} < x < \frac{1}{n}\}$.

$\bigcap_{n=1}^{\infty} O_n = \{0\}$ 0 is not interior point.

It could be open, e.g. if all O_r identical.

Union of closed sets.

$\bigcup C_r$.

$\{x : a \leq x \leq b\}$. Let $C_n = \{x : \frac{1}{n} \leq x \leq 1 - \frac{1}{n}\}$

$\bigcup_{n=2}^{\infty} C = \{x : 0 < x < 1\}$ open

If all C_r were identical, $\bigcup C_r$ closed

Intersection of closed sets $\bigcap C_r$.

Suppose P is a limit point of $\bigcap C_r$.

Then any neighbourhood of P contains a point of $\bigcap C_r$.

A point is in $\bigcap C_r$ if it is in every C_r.

$\therefore$ Any neighbourhood P contains a point of C_r.

$\therefore$ P is a limit point of C_r $\therefore$ $P \in C_r$.

~~Similarly for every~~ other This is true for all C_r

$\therefore$ $P \in \bigcap C_r$ $\therefore$ $\bigcap C_r$ is closed

Complement of $\bigcup S_r$.

If P is not in $\bigcup S_r$.

$P \notin S_r$ for any r.

$\therefore$ $P \in \bar{S_r}$ for each r.

$\therefore$ $P \in \bigcap \bar{S_r}$.

Complement of $\bigcap S_r$

P is not in $\bigcap S_r$

P is not in every S_r

$\therefore$ P is not in some S_r

$\therefore$ $P \in \bar{S_r}$ for some r.

$\therefore$ $P \in \bigcup \bar{S_r}$

$\therefore$

<u>Intersection of closed sets.</u>

$\bigcap C_r$.

Complement of this is $\bigcup \bar{C_r}$

$\bar{C_r}$ is open

We showed union of open is open

$\therefore$ $\bigcup \bar{C_r}$ is open.

$\therefore$ its complement is closed.

$\therefore$ $\bigcap C_r$ is closed.

STIRLING APPROX TO $n!$ E ① 56

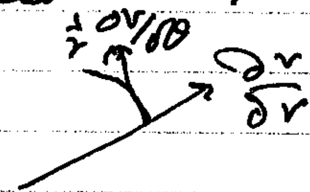
$\nabla^2 V$.

$\nabla^2 V =$ div grad V.

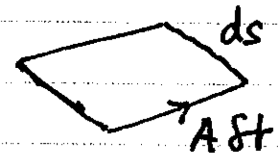
If $V = V(x,y,z)$ grad $V = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)$

The component of grad V in any direction gives the rate of change of V in that direction. This is the space rate of change dV/ds where ds is the distance between the points in question.

It is important to notice this when co-ordinates other than (x,y,z) are used. eg in 2 dimensions, with polar co-ordinate the distances in r and θ direction are dr, $r d\theta$ so grad V has components $\partial V/\partial r$, $\frac{1}{r}\frac{\partial V}{\partial \theta}$



flux A. A is defined at each point, and we interpret it as a velocity



In time δt, any point within the parallelogram shown will cross the line element ds. The area of this parallelogram is $A_n\, ds\, \delta t$, where A_n is the component of A along normal to ds. In 3-D, it will be a volume $A_n\, dS\, \delta t$, dS being an element of surface area. The rate at which areas or volumes are carried across a boundary is known as flux A. In 3-D, the vector $d\underline{S}$ is usually taken to mean a vector of magnitude dS in the direction of the normal. Thus rate at which volume crosses is $\underline{A}\cdot d\underline{S}$. For a surface it is

$\int \underline{A}\cdot d\underline{S}$, $d\underline{S}$ being outward normal in direction div A is the ratio of flux A out of some small region to the volume of that region, as dimensions $\to 0$.

If $A = (X, Y, Z)$

the flux across ~~and~~ a face $\perp$ OX is $X\,dy\,dz$. This flow

will be into the box through left end, and out of box through right end. Thus net rate of loss is the difference between $X\,dy\,dz$ taken at $x + dx$ and taken at x.

The change in $f(x,y,z)$ for such a displacement is $dx \frac{\partial f}{\partial x}$.

Thus faces $\perp$ Ox contribute $dx \frac{\partial}{\partial x}(X\,dy\,dz)$.

The dimensions dy, dz do not change, so we have $\frac{\partial X}{\partial x}\,dx\,dy\,dz$.

Applying same argument to faces $\perp$ OY and faces $\perp$ OZ we get

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) dx\,dy\,dz \quad \text{as flux.}$$

Flux per unit volume is $\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = \operatorname{div}(X Y Z)$

If $(X\,Y\,Z) = \operatorname{grad} V$, $X = \frac{\partial V}{\partial x}$, $Y = \frac{\partial V}{\partial y}$, $Z = \frac{\partial V}{\partial z}$

$$\operatorname{div} \operatorname{grad} V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}.$$

Note. When dealing with coordinates other than x, y, z care must be taken to consider areas of the faces.

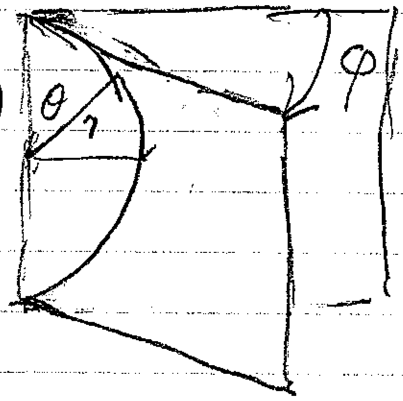
Note however that for any co-ordinate u, $\frac{\partial f}{\partial u} du$ gives $f(u+du) - f(u)$.

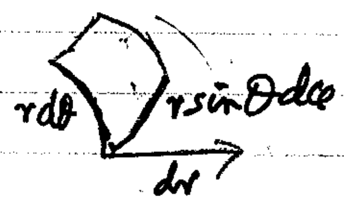
The question whether du is the same as ds does not arise. Only the definition of differentiation is involved.

Spherical Polars.

To $dr, d\theta, d\varphi$ correspond distances $dr, r\,d\theta, r\sin\theta\, d\varphi$.

$$\text{grad } V = \left(\frac{\partial V}{\partial r}, \frac{1}{r}\frac{\partial V}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\right)$$





For (A, B, C) flux through face $\perp dr$ is $Ar^2\sin\theta\, d\theta\, d\varphi$

Net flux for faces $\perp dr$ is $dr\frac{\partial}{\partial r} Ar^2\sin\theta\, d\theta\, d\varphi$

$$= dr\, d\theta\, d\varphi \frac{\partial}{\partial r}(Ar^2\sin\theta).$$

If $A = \frac{\partial V}{\partial r}$ this is $dr\, d\theta\, d\varphi \frac{\partial}{\partial r}\left(r^2\sin\theta \frac{\partial V}{\partial r}\right)$.

Flux through face $\perp d\theta$ is $Br\sin\theta\, d\varphi\, dr$.

Net flux is $d\theta \frac{\partial}{\partial \theta}(Br\sin\theta\, d\varphi\, dr)$

$$= dr\, d\theta\, d\varphi \frac{\partial}{\partial \theta}(Br\sin\theta)$$

If $B = \frac{1}{r}\frac{\partial V}{\partial \theta}$ this is $dr\, d\theta\, d\varphi \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right)$

Flux through face $\perp d\varphi$ is $C\, r\, d\theta\, dr$

Net flux is $d\varphi \frac{\partial}{\partial \varphi}(C\, r\, d\theta\, dr) = dr\, d\theta\, d\varphi\, r\frac{\partial C}{\partial \varphi}$.

If $C = \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}$ this $= dr\, d\theta\, d\varphi \frac{1}{\sin\theta}\frac{\partial^2 V}{\partial \varphi^2}$.

Volume of box is $r^2\sin\theta\, dr\, d\theta\, d\varphi$. Hence flux/volume

$$= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2}$$

So $$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cancel{\sin\theta}\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2}$$

We now consider $\nabla^2 V = FV$ where F may be zero, a constant, or a function of r alone. These cases arise in various situations. In all cases $F = F(r)$.

Separation of Variables. We look for solutions of the form $R(r)S(\theta,\varphi)$. We have

$$\frac{S}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) + \frac{R}{r^2\sin^2\theta}\frac{\partial^2 S}{\partial\varphi^2} = F(r).RS$$

Multiply by $\frac{r^2}{RS}$ and rearrange. This gives

$$\frac{1}{S\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) + \frac{1}{S\sin^2\theta}\frac{\partial^2 S}{\partial\varphi^2} = r^2F(r) - \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)$$

L.H.S is $f(\theta,\varphi)$: RHS is $f(r)$. They must both be constant, say k. Then, from L.H.S

$$\frac{1}{S\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) + \frac{1}{S\sin^2\theta}\frac{\partial^2 S}{\partial\varphi^2} = k.$$

Multiply by $\sin^2\theta$ and look for $S = \Theta(\theta)\Phi(\varphi)$. We find

$$\frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = k\sin^2\theta$$

As before, this means $\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}$ must be constant.

To give a satisfactory function this constant must be $-m^2$. $d^2\Phi/d\varphi^2 = -m^2\Phi$.

Then $$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - (m^2 + k\sin^2\theta)\Theta = 0.$$

Let $\mu = \cos\theta$ $\quad \frac{d\Theta}{d\mu} = \frac{d\Theta}{d\theta}\Big/\frac{d\mu}{d\theta} = -\frac{1}{\sin\theta}\frac{d\Theta}{d\theta}$

As $\sin^2\theta = 1-\mu^2$ we have

$$(1-\mu^2)\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta}{d\mu}\right] - \left(m^2 + k(1-\mu^2)\right)\Theta = 0.$$

The singularities are at $1, -1, \infty$.

Let $\mu = 1 + t$ $\quad 1-\mu^2 = -(2t+t^2)$

$$(2t+t^2)\frac{d}{dt}(2t+t^2)\frac{d\Theta}{dt} - [m^2 + k(2t+t^2)]\Theta = 0$$

The lowest power terms are $2t\frac{d}{dt}2t\frac{d\Theta}{dt} - m^2\theta = 0$.

If $\Theta = t^\rho$ $\quad 2t\frac{d}{dt}\Theta = \rho t^\rho$ $\quad \therefore \quad 4\rho^2 - m^2 = 0$

$\rho = \pm\frac{m}{2}$. To stay finite we need $\rho = \frac{m}{2}$

Similarly, or by symmetry, $\rho = m/2$ for $\mu = -1$.

Thus we have $(1-x^2)^{m/2}$. Let $\Theta = (1-x^2)^{m/2} z$.

x for μ]

$$\frac{d\Theta}{dx} = (1-x^2)^{m/2} z' - mx(1-x^2)^{\frac{m}{2}-1} z$$

$$\cancel{1-x^2}\ (1-x^2)\frac{d\Theta}{dx} = (1-x^2)^{\frac{m}{2}+1} z' - mx(1-x^2)^{\frac{m}{2}} z$$

$$\frac{d}{dx}\left\{(1-x^2)\frac{d\Theta}{dx}\right\} = (1-x^2)^{\frac{m}{2}+1} z'' - (m+2)x(1-x^2)^{\frac{m}{2}} z'$$

$$- mx(1-x^2)^{m/2} z' - m(1-x^2)^{\frac{m}{2}} z$$

$$+ m^2x^2(1-x^2)^{\frac{m}{2}-1} z$$

$$= (1-x^2)^{\frac{m}{2}+1} z'' - \cancel{(2m+2)x}\ (2m+2)x(1-x^2)^{\frac{m}{2}} z' -$$

$$- m(1-x^2)^{\frac{m}{2}} z + m^2x^2(1-x^2)^{\frac{m}{2}-1} z$$

$$(1-x^2)\frac{d}{dx}(1-x^2)\frac{d\theta}{dx} = (1-x^2)^{\frac{m}{2}+2} z'' - (2m+2)x(1-x^2)^{\frac{m}{2}+1} z'$$

$$- m(1-x^2)^{\frac{m}{2}+1} z + \underline{m^2x^2(1-x^2)^{\frac{m}{2}} z}.$$

To this we add

$$\underline{-m^2(1-x^2)^{\frac{m}{2}} z} + k(1-x^2)^{\frac{m}{2}+1} z.$$

Underlined terms combine to give

$$m^2(1-x^2)^{\frac{m}{2}} z\,[x^2-1] = -m^2(1-x^2)^{\frac{m}{2}+1}$$

We may divide by $(1-x)^{\frac{m}{2}+1}$ and obtain

$$(1-x^2)z'' - 2(m+1)xz' + z[k - m^2 - m] = 0.$$

The solution has index 0 at $x=1$ and $x=-1$. $\therefore$ unbranched.

From highest powers, if $z \sim x^N$ for large x

$$-N(N-1) - 2(m+1)N + k - m^2 - m = 0$$

$$N^2 + N(2m+1) + m(m+1) - k = 0.$$

$$k = (N+m)(N+m+1)$$

N must be natural number.

A special case is $m=0$. Then we can divide by $1-\mu^2$. Replacing μ by x we have (L $4\frac{1}{2}$)

S

$$\frac{d}{dx}(1-x^2)\frac{dy}{dx} - ky = 0.$$

Singularities $-1, 1, \infty$.

Divergence method. Let $x = 1+t$ $\frac{d}{dx} = \frac{d}{dt}$

$$\frac{d}{dt}(2t+t^2)\frac{dy}{dt} + ky = 0.$$

$$(t^2+2t)y'' + (2+2t)y' + ky = 0.$$

Indicial equation, from $2ty''+2y'$ $\rho(\rho-1)+\rho=0$ $\rho^2=0$.

One soln has $\ln t$, and one is analytic at $t=0$.

Let $y = \sum a_n t^n$ Coeff of t^{n-1} gives

$$0 = \cancel{[2(n+1)n\ldots]} \left[2n(n-1)+2n\right]a_n + a_{n-1}\left[(n-1)(n-2)+2(n-1)+k\right]$$

$$0 = 2a_n n^2 + a_{n-1}[n^2-n+k)$$

$$\frac{a_n}{a_{n-1}} = -\frac{1}{2}\left[1-\frac{1}{n}+\frac{k}{n^2}\right]$$

If $t=-2$ $\quad \frac{a_n t^n}{a_{n-1}t^{n-1}} = \frac{a_n}{a_{n-1}}t = 1-\frac{1}{n}+\frac{k}{n^2}$.

This diverges. $\therefore$ Only way to get finite soln at 1 and -1 is for the series to terminate.

If $a_n t^n$ is highest power that occurs, $a_{n+1}=0$

$\therefore$ $(n+1)^2-(n+1)+k=0$. $\quad k = -n(n+1)$

Beckett's method. An acceptable soln must be log-free at 1 and -1, $\therefore$ exponent 0 at each. $\therefore$ Unbranched for circuit around infinity. From highest powers in original equation,

if $y \sim cx^n$ in $(1-x^2)y'' - 2xy' - ky = 0$

From $x^2y''+2xy'+ky$ $\quad n(n-1)+2n+k=0$ $\quad k=-n(n+1)$.

This shows n must be integer.

It must in fact be positive.

1

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For central attraction in H atom $V = -\frac{e^2}{r}$

$T = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$ $\quad T + V = E$

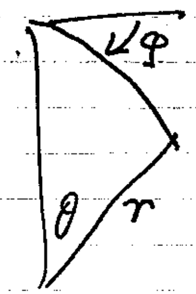
$p_x \to i\hbar \frac{\partial}{\partial x}$

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) - \frac{e^2}{r}\psi = E\psi$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{2m}{\hbar^2}\left(E + \frac{e^2}{r}\right)\psi = 0$$

Thus $\quad \nabla^2\psi + \left(a + \frac{b}{r}\right)\psi = 0.$

Spherical polar co-ordinates



Distances $dr, r\,d\theta, r\sin\theta\,d\varphi$

$\therefore$ grad $\psi = \left(\frac{\partial\psi}{\partial r}, \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\right)$

The areas of faces are

$\perp r \quad r^2\sin\theta\,d\theta\,d\varphi$

$\perp\theta \quad r\sin\theta\,dr\,d\varphi$

$\perp\varphi \quad r\,dr\,d\theta$

For vector $\underline{A} = (A_r, A_\theta, A_\varphi)$

$$\text{div}.A = \frac{dr\frac{\partial}{\partial r}(A_r r^2\sin\theta\,d\theta\,d\varphi) + d\theta\frac{\partial}{\partial\theta}(A_\theta r\sin\theta\,dr\,d\varphi) + d\varphi\frac{\partial}{\partial\varphi}(A_\varphi r\,dr\,d\theta)}{r^2\sin\theta\,dr\,d\theta\,d\varphi}$$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta A_r) + \frac{\partial}{\partial\theta}(A_\theta r\sin\theta) + \frac{\partial}{\partial\varphi}(A_\varphi r)\right]$$

Putting A = grad ψ

$$\nabla^2\psi = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial\varphi}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\varphi}\right)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}$$

So we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} + \left(a + \frac{b}{r}\right)\psi = 0$$

Let $\psi = RS \quad R(r)\, S(\theta, \varphi)$

$$\frac{S}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) + \frac{R}{r^2\sin^2\theta}\frac{\partial^2 S}{\partial\varphi^2} + \qquad S \; 13$$

$$\left(a + \frac{b}{r}\right)RS = 0.$$

$\div RS \quad \times r^2$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left(a + \frac{b}{r}\right)r^2 +$$

$$+ \frac{1}{S\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) + \frac{1}{S\sin^2\theta}\frac{\partial^2 S}{\partial\varphi^2} = 0.$$

Each part must be constant.

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + (ar^2 + br + \alpha)R = 0$$

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NUMBER	APPROXIMATION	FACTORIAL	RATIO
1	0.922137009	1	1.08443755
2	1.91900435	2	1.04220712
3	5.83620959	6	1.02806452
4	23.5061751	24	1.0210083
5	118.019168	120	1.01678399
6	710.078185	720	1.01397285
7	4980.39583	5040	1.01196776
8	39902.3955	40320	1.01046565
9	359536.873	362880	1.00929843
10	3598695.62	3628800	1.00836536
11	39615625	39916800	1.00760243
12	475687486	479001600	1.006967
13	6.18723948E9	6.2270208E9	1.00642958
14	8.66610018E10	8.71782912E10	1.00596911
15	1.30043072E12	1.30767437E12	1.00557019
16	2.08141144E13	2.09227899E13	1.00522124
17	3.53948329E14	3.55687428E14	1.00491343
18	6.37280462E15	6.40237371E15	1.00463989
19	1.21112787E17	1.216451E17	1.00439519
20	2.42278685E18	2.43290201E18	1.00417501
21	5.08886173E19	5.10909422E19	1.00397584
22	1.11975149E21	1.12400073E21	1.0037948
23	2.57585254E22	2.58520167E22	1.00362953
24	6.18297927E23	6.20448402E23	1.00347806
25	1.54595948E25	1.55112101E25	1.00333872
26	4.02000993E26	4.03291461E26	1.00321011

TIME
75.44
>

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```
BERNOUILLI NUMBERS
B(0)=1
B(1)=-0.5
B(2)=0.166666667
B(3)=0
B(4)=-0.0333333334
B(5)=0
B(6)=0.0238095245
B(7)=0
B(8)=-0.0333333379
B(9)=0
B(10)=0.0757576189
B(11)=0
B(12)=-0.253114136
B(13)=0
B(14)=1.16667745
B(15)=0
B(16)=-7.09241926
B(17)=0
B(18)=54.979315
B(19)=0
B(20)=-529.43755
B(21)=0
B(22)=6206.78938
B(23)=0
B(24)=-87400.5253
B(25)=0
B(26)=1479539.33
B(27)=0
B(28)=-31436265.9
B(29)=0
B(30)=966346303
B(31)=0
>
```

Bernouilli's numbers.

$$\frac{x}{e^x-1} = B_0 + \frac{B_1}{1!}x + \frac{B_2}{2!}x^2 + \dots \frac{B_n}{n!}x^n + \dots$$

$$LHS = \frac{1}{\frac{1}{1!} + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} \dots}$$

so
$$1 = \left(\frac{1}{1!} + \frac{x}{2!} + \frac{x^2}{3!} + \dots\right)\left(\frac{B_0}{0!} + \frac{B_1}{1!}x + \frac{B_2}{2!}x^2 + \dots\right)$$

$$B_0 = 1$$
$$0 = \frac{1}{2!}\frac{B_0}{0!} + \frac{1}{1!}\frac{B_1}{1!}$$

In general

Coeff x^{n-1} $\quad \frac{1}{n!}\frac{B_0}{0!} + \frac{1}{(n-1)!}\frac{B_1}{1!} + \frac{1}{(n-2)!}\frac{B_2}{2!} + \dots \frac{1}{1!}\frac{B_{n-1}}{(n-1)!} = 0$

Multiply by $n!$

$$\binom{n}{0}B_0 + \binom{n}{1}B_1 + \binom{n}{2}B_2 + \dots \binom{n}{n-1}B_{n-1} = 0$$

$(B+1)^n - B^n$: change

(Interesting special symbolism. indices to subscripts.)

Thus

$$B_0 = 1 \quad (0)$$
$$2B_1 + 1 = 0 \quad (1)$$
$$3B_2 + 3B_1 + 1 = 0 \quad (2)$$
$$4B_3 + 6B_2 + 4B_1 + 1 = 0 \quad (3)$$
$$5B_4 + 10B_3 + 10B_2 + 5B_1 + 1 = 0 \quad (4) \text{ etc.}$$

It turns out that $B_1 = -\frac{1}{2}$ $\quad B_3 = B_5 = B_7 = \dots = 0$

This suggests that L.H.S $+ \frac{1}{2}x$ is an even function.

$$LHS + \tfrac{1}{2}x = \frac{x}{e^x-1} + \tfrac{1}{2}x = x\left\{\frac{1}{e^x-1} + \frac{1}{2}\right\}$$

$$= x\left\{\frac{1+\frac{1}{2}(e^x-1)}{e^x-1}\right\} = \frac{x}{2}\,\frac{e^x+1}{e^x-1}$$

Replacing x by $-x$ gives $\quad -\frac{x}{2}\,\frac{e^{-x}+1}{e^{-x}-1} = -\frac{x}{2}\,\frac{1+e^x}{1-e^x}$

$$= \frac{x}{2}\,\frac{e^x+1}{e^x-1}$$ So all odd coeffs, after B_1, are zero.

B_2 B_4 $B_6 \dots$ can be calculated by recursion from equations above. B_2 from (2), B_4 from 4 and so on.

```
BERNOUILLI NUMBERS
B(0)=1
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B(22)=6206.78938
B(23)=0
B(24)=-87400.5253
B(25)=0
B(26)=1479539.33
B(27)=0
B(28)=-31436265.9
B(29)=0
B(30)=966346303
B(31)=0
>
```

N	$\sqrt{2\pi n}\, n^n e^{-n}$	n!	Ratio
1	.922 137009	1	1.0844
2	1.919 004 351	2	1.0422
3	5.836 209 590	6	1.028 064 578
4	23.506 17514	24	1.021 008 303
5	118.019 1679	120	1.016 783 986
6	710.078 1847	720	1.013 972 849
7	4980.395 833	5040	1.011 967 757
8	39,902.395 44	40,320	1.010 465 651
9	359,536.8729	362,880	1.009 298 426
10	3,598,695.619	3,628,800	1.008 365 359
11	39,615,625.06	39,916,800	1.007 602 428
12	475,687,486.4	479,001,600	1.006 966 558
13	6,187,239,477	6,227,020,800	1.006 429 575
20	$2.422\,786.8 \times 10^{18}$	2.4329020×10^{18}	1.004 175 010

Asymptotic series.

Kropp (p 521) gives example: if we can prove

(I) $f(x) = 1 - \frac{1!}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \cdots + (-1)^n \frac{n!}{x^n} + (-1)^{n+1}\vartheta\frac{(n+1)!}{x^{n+1}}$

$0 < \vartheta < 1$.

The infinite series $\sum_0^\infty (-1)^n \frac{n!}{x^n}$ is divergent for all x, since terms increase in magnitude for $n > x$.

However, since the <u>form of the remainder</u> is <u>known</u> we can calculate $f(1000)$ to about 10 decimal places with ease.

For $n = 3$ $\quad \frac{(n+1)!}{1000^{n+1}} = \frac{4!}{10^{12}} = .000\,000\,000\,024$

The error due to taking $f(x) = 1 - \frac{1}{x} + \frac{2}{x^2} - \frac{6}{x^3}$ and $x = 1000$ is less than $.000\,000\,000\,1$.

For any x, the terms decrease in magnitude for $n < x$, increase for $n > x$.

The error above is ϑT_{n+1}, where T_{n+1} is term in $\frac{1}{x^{n+1}}$.

There is a minimum term for each x.

The error cannot be made certainly less than this minimum by using the series.

We can always say, if we consider only terms up to T_n, that result is in error less than $|T_{n+1}|$ the first term omitted.

This property is possessed by a series like

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$

Series (I) has the same property, but of course for quite different reasons.

Stirling's asymptotic series for $\ln n!$

$$\ln n! \sim (n+\tfrac{1}{2})\ln n - n + \log\sqrt{2\pi} +$$
$$+ \frac{B_2}{1.2}\frac{1}{n} + \frac{B_4}{3.4}\frac{1}{n^3} + \frac{B_6}{5.6}\frac{1}{n^5} + \dots \frac{B_{2k}}{(2k-1)2k}\frac{1}{n^{2k-1}} \dots$$

This series does not converge for any value of n; this is reasonable in view of the way the coefficients B_{2k} increase.

Yet it has the property that the error made in stopping at any number of terms is numerically less than the first term omitted, and of same sign.

$B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$.

Thus error in stopping at $\log\sqrt{2\pi}$ is less than $\frac{1}{12n}$; the estimate is too small.

If we stopped at $\frac{B_2}{1.2}\frac{1}{n}$ the error is less than

$$\left|\frac{B_4}{3.4}\frac{1}{n^3}\right| = \frac{1}{360n^3}$$ As B_4 is negative,

the estimate is now too large.

So we should add less than $\frac{1}{12n}$ but more than $\frac{1}{12n} - \frac{1}{360n^3}$.

For $n=1$, $n! = 1$, $n!$ estimated $= 0.922137009$

Logarithms, 0 and $-0.08106\dots$

$\frac{1}{12} = 0.083$ $\quad \frac{1}{12} - \frac{1}{360} = 0.080\dot{5}$

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Jeffreys and Jeffreys. § 9.08.

Euler-Maclaurin formula. f differentiable. Integrating by parts, gives

$$\int_0^1 f(x)\,dx = \left[x f(x)\right]_0^1 - \int_0^1 x f'(x)\,dx$$
$$= f(1) - \int_0^1 x f'(x)\,dx$$
$$= f(1) - \int_0^1 (x-\tfrac{1}{2}) f'(x)\,dx - \int_0^1 \tfrac{1}{2} f'(x)\,dx$$
$$= f(1) - \tfrac{1}{2} f(1) + \tfrac{1}{2} f(0) - \int_0^1 (x-\tfrac{1}{2}) f'(x)\,dx$$
$$= \tfrac{1}{2} f(1) + \tfrac{1}{2} f(0) - \int_0^1 (x-\tfrac{1}{2}) f'(x)\,dx.$$

Trapezoidal rule with integral as correction.

For $0 \le x < 1$ and $r \geq 2$ define

$$\forall r \quad P_r(0) = 0$$
$$P_2'(x) = x - \tfrac{1}{2} \quad (2) \qquad (2)$$
$$P_3'(x) = b_2 + P_2(x) \quad (3)$$
$$P_4'(x) = b_3 + P_3(x) \quad (4)$$
$$- \; - \; -$$
$$P_r'(x) = b_{r-1} + P_{r-1}(x)$$

for all $r \geq 2$.

Choose b_r so that $P_r(1) = 0$

Then $\quad P_r(x) = \int_0^x b_{r-1} + P_{r-1}(\xi)\,.\,d\xi$

$P_r(1) = 0$ means $\quad 0 = \int_0^1 b_{r-1} + P_{r-1}(\xi)\,d\xi$

i.e. $\quad b_{r-1} = -\int_0^1 P_{r-1}(\xi)\,d\xi.$

$r \to r+1 \qquad b_r = -\int_0^1 P_r(\xi)\,d\xi.$

Keep integrating by parts

$$\int_0^1 P_2'(x) f'(x)\,dx = \left[P_2(x) f''(x)\right]_0^1 - \int_0^1 P_2(x) f''(x)\,dx$$
$$= -\int_0^1 P_2(x) f''(x)\,dx$$
$$= -\int_0^1 \{P_3'(x) - b_2\} f''(x) \quad \text{by } (2)(3)$$
$$= -b_2 \left[f'(x)\right]_0^1 - \left[P_3(x) f''(x)\right]_0^1 + \int_0^1 P_3(x) f'''(x)\,dx$$

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S_{22}

$$= b_2\{f'(1) - f'(0)\} + \int_0^1 P_3(x) f'''(x)\,dx$$

$$= b_2\{f'(1) - f'(0)\} + \int_0^1 (P_4'(x) - b_3) f'''(x)\,dx \quad \text{by } (2)(4)$$

$$= b_2\{f'(1) - f'(0)\} - b_3\{f''(1) - f''(0)\} + \int_0^1 P_4'(x) f'''(x)\,dx$$

$$= b_2\{f'(1) - f'(0)\} - b_3\{f''(1) - f''(0)\} + \Big[P_4(x) f'''(x)\Big]_0^1 - \int_0^1 P_4(x) f''''(x)\,dx$$

$$-\int_0^1 P_4(x) f^{iv}(x)\,dx = -\int_0^1 (P_5'(x) - b_4) f^{iv}(x)\,dx \quad \text{by } (2)(5)$$

$$\int_0^1 b_4 f^{iv}(x)\,dx = b_4\{f'''(1) - f'''(0)\}$$

$$-\int_0^1 P_5'(x) f^{iv}(x)\,dx = -\Big[P_5(x) f^{iv}(x)\Big]_0^1 + \int_0^1 P_5(x) f^{v}(x)\,dx$$

$$= \int_0^1 P_5(x) f^{v}(x)\,dx.$$

Thus

$$\int_0^1 P_2'(x) f'(x)\,dx = \sum_{s=2}^{r} (-1)^s \{f^{(s-1)}(1) - f^{(s-1)}(0)\} - (-1)^r \int_0^1 P_{r+1}'(x) f^{(r)}(x)\,dx$$

Stages.

$r=2$ $\int_0^1 P_2'(x) f'(x)\,dx = b_2\{f'(1)-f'(0)\} - \int_0^1 P_3'(x) f''(x)\,dx$

$r=3.$ $\int_0^1 P_2'(x) f'(x)\,dx = b_2\{f'(1)-f'(0)\} - b_3\{f''(1)-f''(0)\} + \int_0^1 P_4'(x) f'''(x)\,dx$

$r=4$ $\int_0^1 P_2'(x) f'(x)\,dx = b_2\{f'(1)-f'(0)\} - b_3\{f''(1)-f''(0)\} + b_4\{f'''(1) - f'''(0)\} - \int_0^1 P_5'(x) f^{(iv)}(x)\,dx.$

in agreement with Jeffreys & Jeffreys.

Now consider the series $\dfrac{a}{e^a-1} + \frac{1}{2}a = \sum_{r=0}^{\infty} b_r a^r$ (6)

$$a\,\frac{e^{at}-1}{e^a-1} = \sum_{r=0}^{\infty} P_r(t) a^r. \quad (7)$$

The $\{b_r\}$, $\{P_r(t)\}$ so defined are identical with those used above.

$$\frac{\partial}{\partial t}\, a\,\frac{e^{at}-1}{e^a-1} = a\,\cancel{\frac{ae^{at}(e^a-1) - e^a(e^{at}-1)}{(e^a-1)^2}}$$

$$= \cancel{\frac{a^2(e^{at}-1)}{e^a-1}} \quad \text{a constant!}$$

$$= \frac{a^2e^{at}}{e^a-1} = \frac{a^2(e^{at}-1)}{e^a-1} + \frac{a^2}{e^a-1}$$

$$= \sum_{r=0}^{\infty} P_r(t) a^{r+1} + a\{\sum_0^{\infty} b_r a^r - \tfrac{1}{2}a\}$$

This $= \sum_0^{\infty} P_r'(t) a^r$. equating coeffs of a^r

$P_r'(t) = P_{r-1}(t) + b_{r-1}$ exactly as (2)(i).

See over

All conditions are satisfied.

Notes on J & J derivation of Euler - Maclaurin. 9

f differentiable (means "f analytic"?) S24

$\int_0^1 f(x)\,dx = [x f(x)]_0^1 - \int_0^1 x f'(x)\,dx$

$= \frac{1}{2} f(1) + \frac{1}{2} f(0) - \int_0^1 (x-\frac{1}{2}) f'(x)\,dx$ [1]

correction of trapezoidal rule.

Sequence of polynomials defined by

$\forall r\ P_r(0)=0$. $\forall r\ P_r(1) = 1$ fixed b

$P_2'(x) = x - \frac{1}{2}$ (2)

$P_r'(x) = b_{r-1} + P_{r-1}(x)$ (r)

$P_2'(x) = x - \frac{1}{2}$ $P_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x + c.$

and $c = 0$ so $P_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x$

$P_3'(x) = b_2 + P_2(x)$

$P_3(1) - P_3(0) = \int_0^1 P_3'(x)\,dx = b_2 + \int_0^1 P_2(x)\,dx$

$\therefore\ b_2 = -\int_0^1 P_2(x)\,dx = -\left[\frac{1}{6}x^3 - \frac{1}{4}x^2\right]_0^1 =$

$-\left[\frac{1}{6} - \frac{1}{4}\right] = \frac{1}{12}$

$B_2 = \frac{1}{6} = 2!\, b_2.$

Generally $\underline{b_r = -\int_0^1 P_r(\xi)\,d\xi.}$

We have $\int_0^1 (x-\frac{1}{2}) f'(x)\,dx$ as starting point

$= \int_0^1 P_2'(x) f'(x)\,dx = [P_2(x) f'(x)]_0^1 - \int_0^1 P_2(x) f''(x)\,dx$

$P_2(x)$ zero at both end of [ ].

So $+\int_0^1 P_2(x) f''(x)\,dx =$ last term in [1]

$= \int_0^1 P_3'(x) - b_2 \cdot \cancel{dx}\ f''(x)\,dx$

$= \left[-b_2 f'(x)\right]_0^1 + \underbrace{\int_0^1 P_3(x) f''(x)\,dx}_{\text{zero.}} - \int_0^1 P_3(x) f'''\,dx$

$= -b_2 (f'(1) - f'(0)) - \int_0^1 P_3(x) f'''\,dx$

We have a term $-\int_0^1 P_3(x) f''' dx$ 10

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$= \int_0^1 \{b_3 - P_4'(x)\} \cancel{dx}\, f''' dx$

$= b_3 \{f''(1) - f''(0)\} - \int P_4'(x) f''' dx$

b_3 term has + sign, but $b_3 = 0$

$$-\int_0^1 P_4'(x) f''' dx = -\big[\underbrace{P_4(x) f'''(x)}_{=0}\big]_0^1 + \int_0^1 P_4(x) f^{iv}(x) dx$$

$$+\int_0^1 P_4(x) f^{iv}(x) dx = \int_0^1 (P_5'(x) - b_4) f^{iv} dx$$

$$= -b_4 [f'''(1) - f'''(0)] \text{ with } \int_0^1 P_5'(x) f^{iv} dx$$

$$\text{Integral} = \big[\underbrace{P_5(x) f^{iv}}_{=0}\big]_0^1 - \int_0^1 P_5(x) f^{v} dx$$

$$= -\int_0^1 P_5(x) f^{v} dx$$

Compare this with $-\int_0^1 P_3(x) f''' dx$ above.

Also compare $+\int_0^1 P_4(x) f^{iv}(x) dx$

with $+\int_0^1 P_2(x) f''(x)$ earlier.

J & J. p465 11

$$z! = \lim_{n\to\infty} \frac{n!\, n^z}{(z+1)(z+2)\cdots(z+n)} \qquad \mathcal{R}(z) > -1.$$ S 26

$$\log z! = \lim \left\{ \log n! + z \log n - \sum_{i=1}^{n} \log(z+i) \right\}$$

$$F(z) = \frac{d}{dz} \log z! = \lim \left\{ \log n - \sum \frac{1}{z+i} \right\}$$

$$F'(z) = \frac{d^2}{dz^2} \log z! = \lim \sum_{1}^{n} \frac{1}{(z+i)^2}$$

$$= \sum_{i=1}^{\infty} \frac{1}{(z+i)^2}.$$

p 466 § 15·05

$$\int_0^\infty \frac{dx}{(z+x)^2} = \left[-\frac{1}{z+x} \right]_{0=x}^{\infty} = \frac{1}{z}.$$

Euler Maclaurin.

$$\int_0^n f(x)\,dx = \tfrac{1}{2} f(0) + f(1) + f(2) + \cdots + \tfrac{1}{2} f(n)$$

$$- b_2 \{ f'(n) - f'(0) \} - b_4 \{ f'''(n) - f'''(0) \} - \cdots$$

$$- b_{2r} \{ f^{(2r-1)}(n) - f^{(2r-1)}(0) \} -$$

$$- \sum_{m=0}^{n-1} \int_m^{m+1} P_{2r+1}(x-m) f^{(2r+1)}(x)\,dx.$$

Applying this to $f(x) = 1/(z+x)^2$, $n \to \infty$.

$$F'(z) = \frac{1}{z} - \frac{1}{2z^2} + \frac{B_2}{z^3} + \frac{B_4}{z^5} + \cdot \frac{B_{2r}}{z^{2r+1}} +$$

$$+ \sum_{m=0}^{\infty} \int_m^{m+1} \frac{(2r+2)\,\phi_{2r+1}(x-m)}{(z+x)^{2r+3}}\,dx$$

where $\phi_r(x) = r!\, P_r(x)$ (p. 280) ? remark under eqn (2) p 466.

$$F(z) = \log z + \frac{1}{2z} - \frac{B_2}{2z^2} - \frac{B_4}{4z^4} \cdots - \frac{B_{2r}}{2r z^{2r}} - \sum_{m=0}^{\infty} \int_m^{m+1} \frac{\phi_{2r+1}(x-m)}{(z+x)^{2r+2}}\,dx$$

$\int$: So $\log z! = C + (z+\tfrac{1}{2}) \log z - z + \dfrac{B_2}{2z} + \dfrac{B_4}{3.4z^3} + \cdots$ (4)

in the sense that $\log z!$ lie between sum of r terms and $(r+1)$ terms of the series.

C has to be found. Take z large, and use $\quad$ 12 5 27

$$z!\,(z-\tfrac{1}{2})! = 2^{-2z}\sqrt{\pi}\,(2z)!$$ in the form

$$\log z! + \log(z-\tfrac{1}{2})! = -2z\log 2 + \tfrac{1}{2}\log\pi + \log(2z)!$$

Substitute from (4). The terms that do not tend to zero give

$$C + (z+\tfrac{1}{2})\log z^{-z}_{\wedge} + C + \cancel{\log}\, z\log(z-\tfrac{1}{2}) - (z-\tfrac{1}{2})$$
$$= -2z\log 2 + \tfrac{1}{2}\log\pi + C + (2z+1)\log 2z - (2z)$$

$$C + \underline{(z+\tfrac{1}{2})\log z} + z\log(z-\tfrac{1}{2}) \quad -2z + \tfrac{1}{2}$$
$$= \underline{-2z\log 2} + \tfrac{1}{2}\log\pi + \underline{(2z+1)\log z} + \underline{(2z+1)\log 2}.$$
$$+\log 2$$

LHS: $C - (z+\tfrac{1}{2})\log z + z\log(z-\tfrac{1}{2}) - 2z + \tfrac{1}{2}$

$$z\log(z-\tfrac{1}{2}) = z\left(\log z + \log(1-\tfrac{1}{2z}\right)$$
$$\sim z\log z - \tfrac{1}{2}$$

B_{2n} alternate in sign.

$\varphi_n(0) = 0 \quad \varphi_n(1) = 0$
by definition.

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$$\varphi_2(x) = x^2 - x$$

$$\varphi_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$$

$$= x(x-1)(x-\tfrac{1}{2})$$

$$\varphi_4(x) = x^4 - 2x^3 + x^2$$

$$= x^2(x-1)^2$$

$$\varphi_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$$

Structure of J & J Euler Maclaurin. Theorem. $\frac{S}{29}$

p 278 By successive integration by parts

$$\int_0^1 P_2'(x) f'(x)\,dx = b_2\{f'(1) - f'(0)\} + \ldots - b_3\{\quad\} + \ldots + (-1)^r b_r\{f^{(r-1)}(1) - f^{(r-1)}(0)\} - (-1)^r \int_0^1 P_{r+1}'(x) f^{(r)}(x)\,dx.$$

$$\int_0^1 P_2'(x) f'(x)\,dx = \int_0^1 (x - \tfrac{1}{2}) f'(x)\,dx$$
$$= \tfrac{1}{2} f(0) + \tfrac{1}{2} f(1) - \int_0^1 f(x)\,dx$$

Apply this result to intervals $(0,1), (1,2) .. (n-1, n)$ in turn and add

We get $\int_0^n f(x)\,dx = \frac{1}{2} f(0) + f(1) + .. + f(n-1) + \frac{1}{2} f(n)$
$- b_2\{f'(n) - f'(0)\} - b_4\{f'''(n) - f'''(0)\}$
$\ldots - b_{2r}\{f^{(2r-1)}(n) - f^{(2r-1)}(0)\}$
$- \sum_{m=0}^{n-1} \int_m^{m+1} P_{2r+1}(x-m) f^{(2r+1)}(x)\,dx$

Top of p 281. Why error less than neglected term.

S
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$$x(x-1)$$
$$(\tfrac{1}{2}+t)(-\tfrac{1}{2}+t) = t^2 - \tfrac{1}{4}$$

$\varphi_2(x) = x^2 - x$

$\varphi_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$

$\varphi_4(x) = x^4 - 2x^3 + x^2$

$x = t + \frac{1}{2}$ $\quad \varphi_2(x) = t^2 + \cancel{2}t + \frac{1}{4} = t^2 - \frac{1}{4}$
$\qquad\qquad -t - \frac{1}{2}$

$$\varphi_3(x) = t^3 + \frac{3}{2}t^2 + \frac{3}{4}t + \frac{1}{8}$$
$$-\frac{3}{2}t^2 - \frac{3}{2}t - \frac{3}{8}$$
$$+\frac{1}{2}t + \frac{1}{4}$$

$P_2 = \frac{1}{2}t^2 - \frac{1}{8}$

$$= \underline{t^3 - \tfrac{1}{4}t}$$

$P_2' = t$

$\varphi_2' = 2t$

$\varphi_3' = 3t^2 - \frac{1}{4}$ $\qquad P_3' = \frac{\varphi_3'}{6} = \frac{t^2}{2} - \frac{1}{24}$

$$= P_2 + \frac{1}{8} - \frac{1}{24}$$
$$= P_2 + \frac{3-1}{24} = P_2 + \frac{1}{12}$$

$P_3' = b_2 + \frac{1}{2}t^2 - \frac{1}{8}$ $\qquad P_3 = \frac{1}{6}(t^3 - t)$

$P_3 = \frac{1}{6}t^3 + t(b_2 - \frac{1}{8}) + c$

$P_3(0) = 0 \quad \therefore c = 0. \qquad b_2 = \frac{1}{8} - \frac{1}{6}$

$\therefore \quad \frac{1}{6} + b_2 - \frac{1}{8} = 0 \qquad = \underline{\quad}$

$P_3(1) = 0 \quad \therefore$

$P_4' = b_3 + \frac{1}{6}(t^3 - t)$

P_4 nec^y even.

$P_n(x) = \varphi_n(x)/n!$

$\cancel{P_2(x)} = \varphi_2(x) = x^2 - x \qquad \varphi_2(\tfrac{1}{2}) = -\tfrac{1}{4}$

1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

$\varphi_4(x) = [x(x-1)]^2 \qquad \varphi_4(\tfrac{1}{2}) = [\tfrac{1}{2}(-\tfrac{1}{2})]^2 = \tfrac{1}{16}$

$\varphi_6(x) = x^6 - 3x^5 + \tfrac{5}{2}x^4 - \tfrac{1}{2}x^2$

$\varphi_6(\tfrac{1}{2}) = \frac{1}{64} - \frac{3}{32} + \frac{5}{32} - \frac{1}{8} = \frac{1}{64} + \frac{1}{16} - \frac{1}{8} = \frac{1+4-8}{64}$
$= -\frac{3}{64}.$

$\varphi_6(1-t) = (1-t)^6 - 3(1-t)^5 + \tfrac{5}{2}(1-t)^4 - \tfrac{1}{2}(1-t)^2$

$= 1 - 6t + 15t^2 - 20t^3 + 15t^4 - 6t^5 + t^6$
$-3 + 15t - 30t^2 + 30t^3 - 15t^4 + 3t^5$
$+\tfrac{5}{2} - 10t + 15t^2 - 10t^3 + \tfrac{5}{2}t^4$
$-\tfrac{1}{2} + t - \tfrac{1}{2}t^2$

$-\tfrac{1}{2}t^2 + \tfrac{5}{2}t^4 - 3t^5 + t^6.$

$\varphi_6(1-t) = \varphi_6(t)$

$P_2'(1-t) = 1 - t - \tfrac{1}{2} = \tfrac{1}{2} - t = -P_2'(t)$

$P_3'(x) = b_2 + P_2(x)$

$\frac{d}{dt} P_3(x) = b_2 + P_2(x)$

$\frac{d}{dt} P_3(1-t) = -P_3'(1-t) = b_2 + P_2(1-t)$

$b_2 + P_2(x)$ is sym abt $\tfrac{1}{2}$.

$\frac{d}{dt} P_3(x) = \text{sym}$

$\therefore\ P_3(x) = \text{sym} + \text{const.}$

So $\quad P_3(1-x) = P_3(x) + \text{const}$

Proof sketch.

STIRLING'S ASYMPTOTIC SERIES.

Gauss used the definition of ln z! , valid for any z with real part > -1,

$$z! = \lim_{n \to \infty} \frac{n!\, n^z}{(z+1)(z+2)\cdots\cdots(z+n)} \qquad (1)$$

Let L(z) = ln (z!). Then

$$L(z) = \lim \left[\ln(n!) + z \ln(n) - \sum_{r=1}^{n} \ln(z+r) \right] \qquad (2)$$

Differentiate. $L'(z) = \lim [\ln(n) - \sum 1/(z+r)]$ (3)

Differentiate $L''(z) = \lim \sum_1^n 1/(z+r)^2$

$$= \sum_1^{\infty} 1/(z+r)^2 \qquad (4)$$

Now $\int_0^{\infty} (z+x)^{-2}\, dx = [-1/(z+x)]_{x=0}^{\infty} = 1/z$ (5)

Euler-Maclaurin gives for any analytic function f(x)

$$\begin{aligned} \int_0^n f(x)dx &= f(0)/2 + f(1)+f(2)\ldots+f(n-1)+f(n)/2 - \\ &- b_2[f'(n)-f'(0)] - b_4[f'''(n)-f'''(0)] - \ldots\ldots - \\ &- b_{2r}[f^{(2r-1)}(n)-f^{(2r-1)}(0)] - \\ &- \sum_{m=0}^{n-1} \int_m^{m+1} P_{2r+1}(x-m) f^{(2r+1)}(x)\, dx \end{aligned} \qquad (6)$$

← dash

Here $b_n = B_n/n!$.

We now take $f(x)= (z+x)^{-2}$ and let n --> oo, so the integral in (6) makes the left-hand side of (6) equal to 1/z.

Then $f'(x) = -2\,(x+z)^{-3}$, $f'''(x) = -2.3.4\,(x+z)^{-5}$ and so on. The factorials that appear cause b_n to be replaced by B_n. Also all the derivatives tend to zero as n -->oo. Only those involving x=0 survive, and these give negative powers of z. Thus equation (6) gives

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T1

TENSOR CALCULUS.

Tensor calculus is concerned with objects that are specified in many different systems. As an analogy, suppose our information about a building is based on a number of photographs taken from various viewpoints. All the photographs are different but they all cause us to imagine the same building.

Usually we are concerned with geometrical or physical objects. Suppose an origin has been chosen on a sheet of paper and another point marked. With one choice of axes it may be recorded as (5,0), with another it may be (4,3). We can find the transformation needed to change from one system to the other. We call (a,b) in the first system the same vector as (c,d) in the second if they correspond under the appropriate transformation. The vector is regarded as an objective thing described in many different ways in different systems. The transformations that convert one system to another are always supposed known.

We can express by a formal test what we mean by saying a vector is "an objective thing". We shall regard **v** as a vector if (i) it is specified in each system by a row of numbers, (ii) the numbers in any pair of systems are related by the transformation appropriate to these systems.

We shall be concerned with objects other than vectors. In each case, something is accepted as a tensor if all the users of different systems can agree on what it is. Again the condition for this will be that suitable transformations will enable us to change from one system to another.

Tensors play a great part in general relativity, the basic idea of which is that, amomg the representations considered, none is to have a privileged position. Different equations will of course appear in different representations, but these must differ only because of the different co-ordinate framework they use; in real terms they must all be saying the same thing .

General relativity accepts the use of curvilinear co-ordinate systems. We shall not consider these here, but restrict our work to linear systems, which are sufficient for special relativity.

Vectors are a particular example of tensors; they have objective meanings, as, for example, displacement, velocity, acceleration.They are known as tensors of order 1.

Tensors or order zero are the simplest of all. They are single numbers that can be transferred from one system to another without any change, and are known as scalars. In Newtonian mechanics mass is a scalar. It does not matter in what directions you choose your axes, masses remain the same.

Covariant and contravariant.

With oblique axes, there are two ways of specifying a vector. In (i) we resolve OP into components parallel to the axes. Then OP is recorded as $x^{(1)}$, $x^{(2)}$. Here 1 and 2 label components; they have nothing to do with powers.

In (ii) we drop perpendiculars. OF is then recorded as F_1 , F_2 .

Now let $\begin{pmatrix} 1 & \cos A \\ \cos A & 1 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$

Then $x_1 = g_{11}x^{(1)} + g_{12}x^{(2)}$

$x_2 = g_{21}x^{(1)} + g_{22}x^{(2)}$.

This means $x_i = g_{ij}x^j$ ...(1)

Now $OP^2 = \underline{x}.\underline{x} = x^i x_i = x^i g_{ij} x^j = g_{ij}x^i x^j$ ...(2)

Many treatments start from equation (2); it is supposed that the length of every vector is defined and that g_{ij} are the coefficients that occur in the expression for the square of the length. It is understood that $g_{ij} = g_{ji}$.

Equations (1) and (2) are used for work in any (finite) number of dimensions.

We were led to these equations by considering the particular case of 2 dimensions. The theory for n dimensions starts by assuming that length is defined by equation (2). It is then shown that this definition implies the existence of a scalar product, and that equation (1) fits nicely into this. This will be shown in the following section.

The Polar Form.

Consider any two vectors $\underline{x}$ and $\underline{y}$. Let $|\underline{x}|$ and $|\underline{y}|$ denote their lengths. We have

$$|x+y|^2 = g_{ij}(x^i+y^i)(x^j+y^j)$$
$$= g_{ij}x^i x^j + g_{ij}y^i x^j + g_{ij}x^i y^j + g_{ij}y^i y^j$$
$$= |x|^2 + 2g_{ij}x^i y^j + |y|^2 \quad \text{as } g_{ij} \text{ is symmetric}$$

Therefore $2g_{ij}x^i y^j = |x+y|^2 - |x|^2 - |y|^2$

so $g_{ij}x^i y^j$ is invariant, i.e. it is a scalar, the same in every system; it is in fact the scalar product of $\underline{x}$ and $\underline{y}$.

We can write the abbreviation y_i for $g_{ij}y^j$, so the scalar product appears in the form $x^i y_i$, which we met earlier.

Equation (1) may be regarded as defining y_i, a symbol that has not been used previously in the general case.

To show that there exists a tridiagonal matrix that commutes with H, where $H_{rs} = 1/(r+s+1)$, r, s from 0 to n.

Let M be the matrix $\begin{pmatrix} b_0 & a_0 & \cdot & \cdot & \cdot & \cdot \\ a_0 & b_1 & a_1 & \cdot & \cdot & \cdot \\ \cdot & a_1 & b_2 & a_2 & \cdot & \cdot \\ & & & & & \\ & & & \cdot & a_{n-1} & b_n \end{pmatrix}$

M and H commute
$\Longleftrightarrow$ MH is symmetrical.

$$(MH)_{ik} = \sum_j M_{ij} H_{jk}$$

$$= M_{i,i-1}H_{i-1,k} + M_{ii}H_{ik} + M_{i,i+1}H_{i+1,k}$$

$$= \frac{a_{i-1}}{i+k} + \frac{b_i}{i+k+1} + \frac{a_i}{i+k+2}.$$

This is to equal $\dfrac{a_{k-1}}{i+k} + \dfrac{b_k}{i+k+1} + \dfrac{a_k}{i+k+2}$.

It is understood that $a_{-1}=0$ and $a_n = 0$. We require

$$\frac{a_{i-1}-a_{k-1}}{i+k} + \frac{b_i - b_k}{i+k+1} + \frac{a_i - a_k}{i+k+2} = 0 \quad (1)$$

Let $\varphi(i) = a_{i-1}$. $\varphi(i) - \varphi(k)$ automatically has $i-k$ as a factor. It will have $i+k$ as a factor (to cancel) if $\varphi(i) - \varphi(-i) = 0$, Remainder theorem with k as variable.

Thus we choose an even polynomial for φ.

Try $a_{i-1} = \alpha i^4 + \beta i^2 + \gamma$. $a_{-1}=0$ means $\gamma = 0$.

$a_n = 0$ means $\alpha(n+1)^4 + \beta(n+1)^2 = 0$ Take $\alpha = 1$, $\beta = -(n+1)^2$

$a_{i-1} = i^4 - (n+1)^2 i^2$, $a_i = (i+1)^4 - (n+1)^2(i+1)^2$

This automatically gives cancelling in $(a_i - a_k)/(i+k+2)$, since this comes from $(a_{i-1} - a_{k-1})/(i+k)$ by $i \to i+1$, $k \to k+1$.

We would like $b_i - b_k$ to have the factor $i+k+1$.

Let $b_i = \psi(i)$ $\psi(i) - \psi(k)$ is zero for $i = -k-1$ if $\psi(-k-1) - \psi(k) = 0$ This will happen if $\psi(k)$ is an even function of $k+\frac{1}{2}$.

Let $b_k = p(k+\frac{1}{2})^4 + q(k+\frac{1}{2})^2$

A routine calculation gives

$$\frac{a_{i-1}-a_{k-1}}{i+k}+\frac{a_i-a_k}{i+k+2}=(i-k)\left[2i^2+2k^2+2i+2k+2-2(n+1)^2\right]$$

$$\frac{b_i-b_k}{i+k+1}=(i-k)\left[p\left(i^2+k^2+i+k+\tfrac{1}{2}\right)+q\right]$$

The terms involving i, k in the sum of all terms in eqn. these will disappear if $p=-2$. There will remain
$q+1-2(n+1)^2$ which will be zero if $q=2n^2+4n+1$.

Thus $a_i=(i+1)^4-(n+1)^2(i+1)^2$

$b_i=-2\left(i+\tfrac{1}{2}\right)^4+(2n^2+4n+1)\left(i+\tfrac{1}{2}\right)^2$

gives a tridiagonal matrix, $n+1$ by $n+1$, commuting with the Hilbert matrix with $n+1$ rows and columns.

$$\frac{a_{i-1}-a_{k-1}}{i+k}+\frac{b_i-b_k}{i+k+1}+\frac{a_i-a_k}{i+k+2}=0$$

We want $b_i - b_k$ to have factor $i+k+1$

$\therefore\ b(-k-1) = b(k)$

~~If $u = k+\frac{1}{2}$ $k = u - \frac{1}{2}$ $-k = -u+\frac{1}{2}$~~

Let $k = u+\alpha$ $\quad -k-1 = -u-\alpha-1$

$$b(u+\alpha) = b(-u-\alpha-1)$$

If $\alpha = -\frac{1}{2}$ $\qquad -k-1 \qquad k$

$$b(u-\tfrac{1}{2}) = b(-u-\tfrac{1}{2})$$

i.e. $b(u-\frac{1}{2})$ is unchanged by $u \to -u$

i.e. $b(u-\frac{1}{2})$ is even fn of u.

$b(k)$ — — . $k+\frac{1}{2}$.

Let $b(k) = p(k+\frac{1}{2})^4 + q(k+\frac{1}{2})^2$

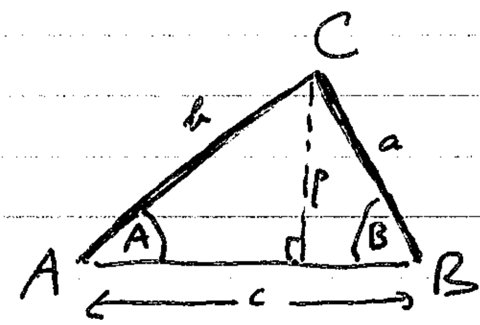
$$\frac{b(i)-b(k)}{i+k+1} = p\frac{[(i+\frac{1}{2})^4-(k+\frac{1}{2})^4]}{i+k+1} + q\frac{[(i+\frac{1}{2})^2-(k-\frac{1}{2})^2]}{i+k+1}$$

$$= p\frac{[(i+\frac{1}{2})^2-(k+\frac{1}{2})^2][(i+\frac{1}{2})^2+(k+\frac{1}{2})^2] + q(i+k+1)(i-k)}{i+k+1}$$

$$= p(i-k)[(i+\tfrac{1}{2})^2+(k+\tfrac{1}{2})^2] + q(i-k)$$

$$= p(i-k)(i^2+k^2+i+k+\tfrac{1}{2}) + q(i-k)$$

Trigonometry of General Triangle



$\sin A = \frac{p}{b}$

$p = b \sin A$.

$p = b \sin A$

$p = a \sin B$

$\therefore \; b \sin A = a \sin B$

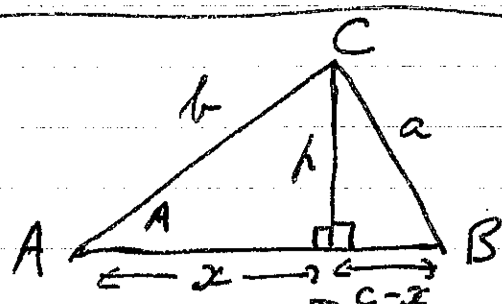
$\div ab$ $\qquad \frac{\sin A}{a} = \frac{\sin B}{b}$

$= \frac{\sin C}{c}$ by symmetry

Sine Rule.

Cosine Rule.

Two eqns required, as there are two unknowns p, x.



Pythagoras for ΔADC: $x^2 + h^2 = b^2$ ①

- - - - ΔBCD : $(c-x)^2 + h^2 = a^2$ ②

We can get an eqn. involving x alone.

② − ①. $\quad (c-x)^2 - x^2 = a^2 - b^2$

$(c^2 + x^2 - 2cx) - x^2 = a^2 - b^2$

$c^2 - 2cx = a^2 - b^2$

Solve for x. $+2cx$ $\quad c^2 = a^2 - b^2 + 2cx$

Add $-a^2 + b^2$

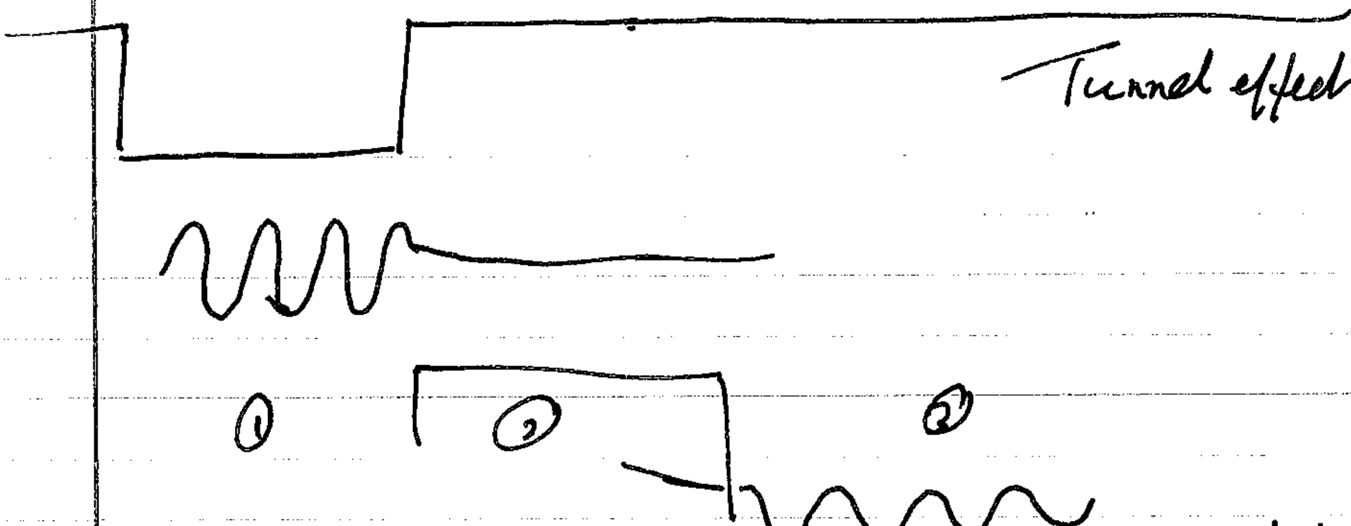
$c^2 - a^2 + b^2 = 2cx$

$\div 2c$ $\qquad x = \frac{c^2 + b^2 - a^2}{2c}$

$\cos A = \frac{x}{b}$ $\qquad \therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc}$.

Note symmetry
a singled out
b, c enter in the same way.

① Tunnel effect. T_7

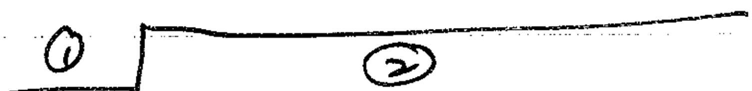


What worried me in this discussion was that we have a real exponential Be^{-x} ~~and~~ in ②.
ψ, ψ' both real at junction of ② and ③ so a real sine wave would join onto them.
But if $\psi = P \sin(\omega x + \varepsilon)$
$|\psi|^2$, the probability, has maxima and minima.

Which it should not have. $Ke^{i\mu x}$ is the accepted wave function for electron moving freely $|\psi|^2 = K^2$ constant.

For an electron trapped in an interval, $\psi = \sin x$ is accepted. see e.g. Yarwood, Atomic & Nuclear Physics §6.11.

What I had overlooked in the brief argument above is that we can eliminate the term e^x in region ②, if it extends to infinity

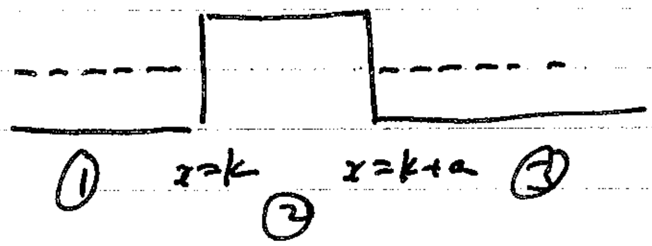


$Ae^x + Be^{-x}$
$A = 0$ if ② extends indefinitely.

No need to assume $A=0$, if barrier terminates. Our only misgiving is — will not e^x give much too large a wave emerging?

Fresh treatment of limited barrier.

$$\psi'' + \psi = 0 \quad \text{in } (1)$$
$$\psi'' - \psi = 0 \quad \text{in } (2)$$
$$\psi'' + \psi = 0 \quad \text{in } (3)$$



We suppose
$$\psi = \cos x \quad \text{in } (1)$$
$$\psi = Ae^{x} + Be^{-x} \quad \text{in } (2)$$
$$\psi = Ce^{ix} \quad \text{in } (3)$$

Continuity of ψ and ψ'

(i) $\cos k = Ae^{k} + Be^{-k}$ (iii) $Ae^{k+a} + Be^{-k-a} = Ce^{i(k+a)}$

(ii) $-\sin k = Ae^{k} - Be^{-k}$ (iv) $Ae^{k+a} - Be^{-k-a} = iCe^{i(k+a)}$

From (i) and (ii) $Ae^{k} = \frac{1}{2}(\cos k - \sin k)$ $Be^{-k} = \frac{1}{2}(\cos k + \sin k)$

From (iii) and (iv), eliminating C,
$$Ae^{k+a} - Be^{-k-a} = i\left[Ae^{k+a} + Be^{-k-a}\right]$$
$$Ae^{k+a}(1-i) = Be^{-k-a}(1+i)$$
satisfied if $Ae^{k+a} = \rho(1+i)$ $e^{-k-a}B = \rho(1-i)$

Substitute in (i)
$$\cos k = \rho e^{-a}(1+i) + \rho e^{a}(1-i)$$
$$\therefore \rho = \frac{\cos k}{e^{-a}(1+i) + e^{a}(1-i)}$$
$$\therefore A = \frac{(1+i)e^{-k-a}\cos k}{e^{-a}(1+i) + e^{a}(1-i)} \qquad B = \frac{(1-i)e^{k+a}\cos k}{e^{-a}(1+i) + e^{a}(1-i)}$$

(iii) $\rightarrow$ $C = \left(Ae^{k+a} + Be^{-k-a}\right)e^{-i(k+a)}$
$$= \left[\rho(1+i) + \rho(1-i)\right]e^{-i(k+a)} = 2\rho e^{-i(k+a)}$$
$$\therefore C = \frac{2\cos k\, e^{-i(k+a)}}{e^{-a}(1+i) + e^{a}(1-i)}$$

e^{a} may be large, but occurs only in denominator.

As $|e^{-i(k+a)}| = 1$ the exponential does not matter here.

$\varphi = Ce^{ix}$ $\Psi = Ce^{ix}e^{-iEt/\hbar}$

In wave mechanics, the classical quantities are replaced by operators. If Ω is some operator representing a quantity, a situation in which Ω has the definite value $\underline{a}$ will have $\Omega\Psi = a\Psi$, i.e. Ψ is an eigenfunction for Ω.

For example, energy has the operator $i\hbar\frac{\partial}{\partial t}$. An electron moving freely has a definite energy, for

$$i\hbar\frac{\partial\Psi}{\partial t} = i\hbar\left[-\frac{iE}{\hbar}.Ce^{ix}e^{-iEt/\hbar}\right]$$

$$= E.Ce^{ix}e^{-iEt/\hbar} = E\Psi.$$

Momentum, p_x, is given by $-i\hbar\frac{\partial}{\partial x}$. This also has Ψ as an eigenfunction

$$-i\hbar\frac{\partial\Psi}{\partial x} = -i\hbar.\, iCe^{ix}e^{-iEt/\hbar} = \hbar\Psi$$

so p_x has the definite value $\hbar$.

$$E = \tfrac{1}{2}M\dot{x}^2 \quad p_x = M\dot{x} \text{ so } E = \frac{1}{2}\frac{p_x^2}{M}$$

$$E = \frac{\hbar^2}{2M} = \frac{h^2}{8\pi^2 M} \qquad \hbar = \frac{h}{2\pi}$$

We chose energy so the actual to make $(E-V)\frac{8\pi^2 M}{h^2} = 1$. As $V=0$, $E = \frac{h^2}{8\pi^2 M}$ in agreement with the above.

(4)

T/10

In finding p_x we can neglect time factor, for $\frac{\partial}{\partial x}$ leaves it unchanged.

$\psi_+ = e^{\alpha i x}$ and $\psi_- = e^{-\alpha i x}$ are interesting.

$$-i\hbar\frac{\partial}{\partial x}\psi_+ = -i\hbar . \alpha i e^{i\alpha x} = \alpha\hbar e^{\alpha i x} = \alpha\hbar\,\psi_+ .$$

So this represents a particle with $p_x = \alpha\hbar$

$$-i\hbar\frac{\partial}{\partial x}\psi_- = -i\hbar(-\alpha i e^{-\alpha i x}) = -\hbar\alpha e^{-\alpha i x}$$

So ψ_- has momentum $-\alpha\hbar$.

Thus ψ_+ represents motion to right and ψ_- to the left.

It is reasonable that only ψ_+ appears in region (3), for electrons emerging from barrier.

In region (1) we may have $\psi = \cos x$.

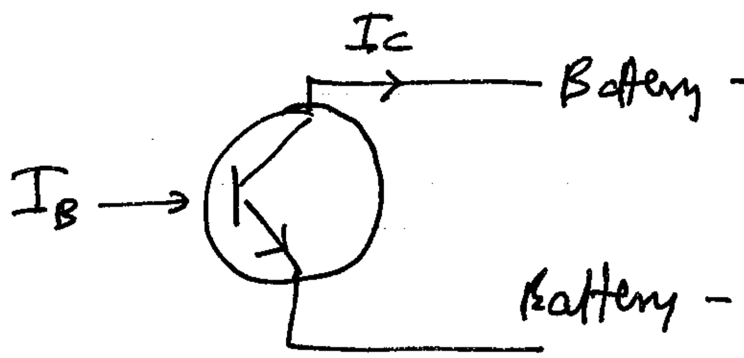
$$\psi = \tfrac{1}{2}(e^{ix} + e^{-ix})$$

superposition of wave to right and to left.

T1

T II

Transistor.



NPN transistor.

I_B Base current

I_c Collector current.

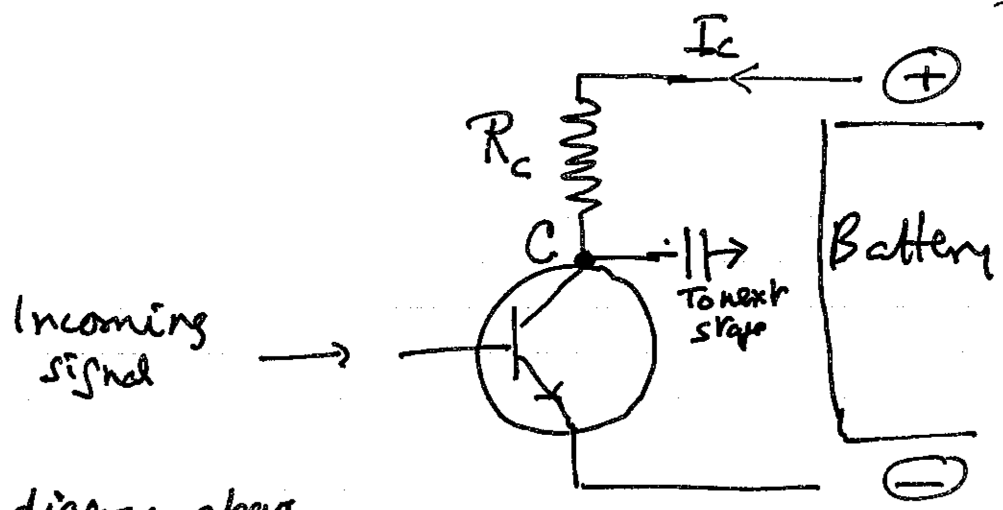
To design a transistor radio, a fair amount of knowledge is required. However, an understanding of how a transistor behaves requires only a simple picture of what is happening.

If we turn a tap on a hose, a small turn of the tap may produce a large increase in the pressure of the emerging jet. In a transistor, the current going into the base plays a role similar to that of the tap. A change in I_B produces a much larger change in I_c. The energy of course is provided by the battery. The base current controls the rate at which this energy is released.

I obtained the following rough values with rather crude apparatus.

I_B (micro amps)	I_c (micro amps)
10	40
20	100
30	190
40	300
50	400
60	550
70	700
80	850
90	950

It will be seen that, for $I_B \geqslant 30$, an increase of 10 μA in I_B produces a change between 100 μA and 150 μA in I_c. The divisions on the scales were too few to permit accurate readings. The effect however is clear.



In the diagram above,
the collector C was connected directly to the battery. However, in a radio receiver this is not so. If C is connected to the battery directly, its voltage will remain constant. It will not be possible to pass on to the next stage of the receiver information about the changing current I_c. Accordingly, we always have a resistor, R_c, between C and the battery. The current, I_c, will produce a voltage drop, $R_c I_c$, between the ends of this resistor. If the battery has the constant voltage, E, the collector voltage, V_c is given by

$$V_c = E - R_c I_c.$$

This is shown in the following experimental results

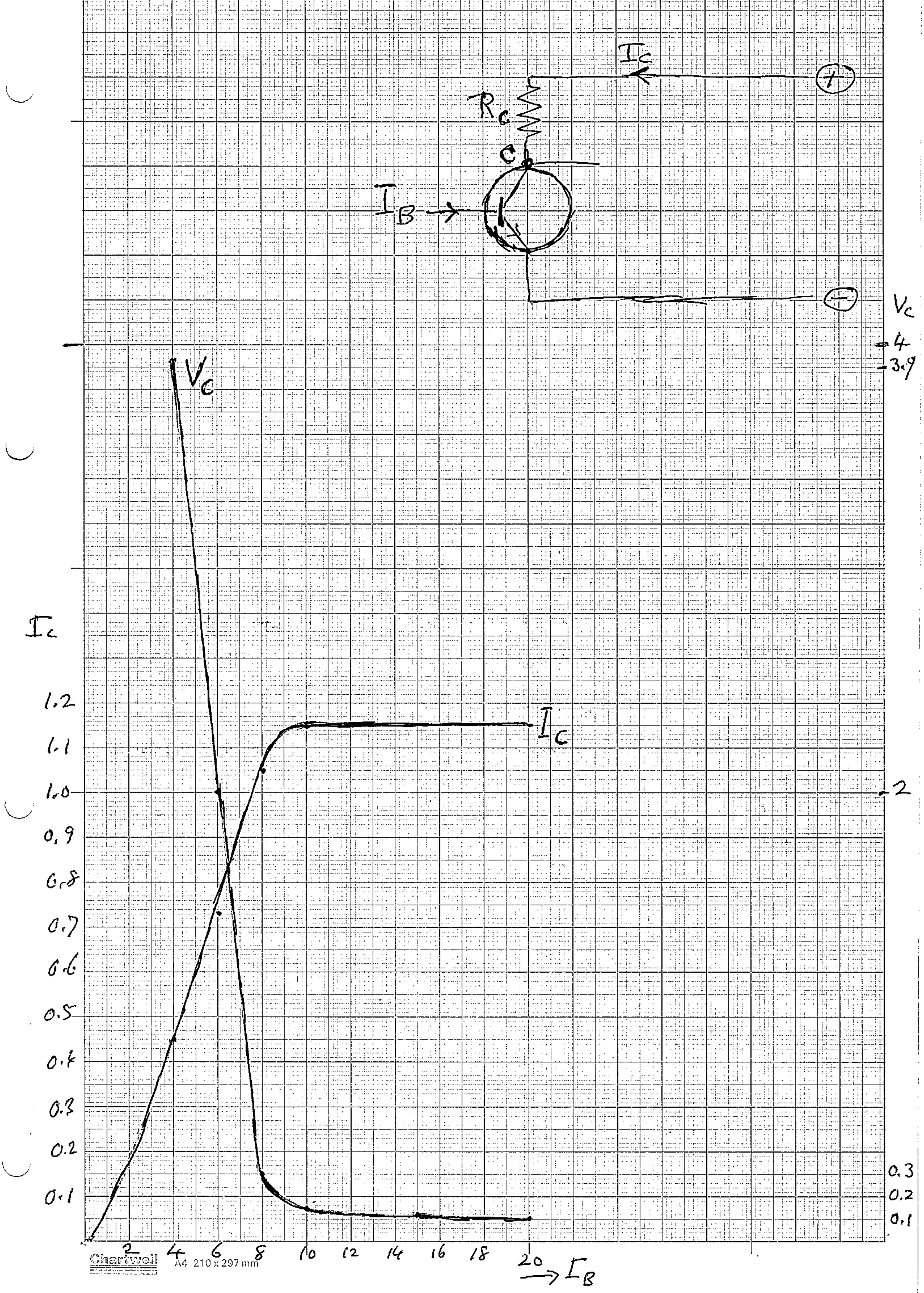
* I_B	I_c (mA)	V_c
4	0.45	3.93
6	0.73	2.02
8	1.05	0.3
10	1.15	0.155
15	1.18	0.113
20	1.18	0.103

The effect of the falling V_c is that the current I_c fails to rise even for quite large increases in I_B.

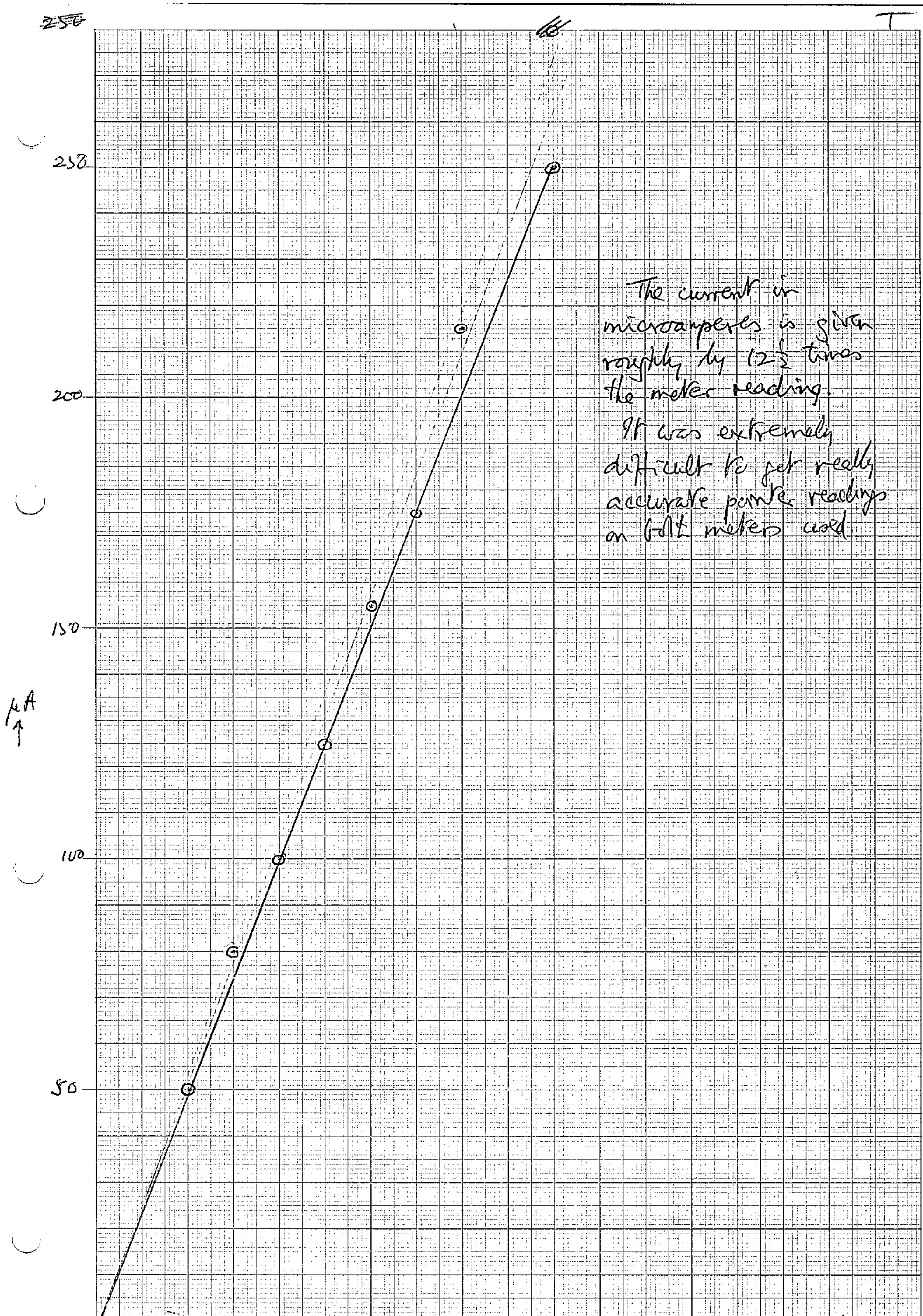
* Microammeter with shunt. Microamps are about 12 times the numbers shown here.

The readings are far from accurate, but show the general effect.

T13



I_c
R_c
C
I_B
+
−
V_c
4
3.9
V_c
I_c
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
I_c
2
0.3
0.2
0.1
2
4
6
8
10
12
14
16
18
20
$\rightarrow I_B$
Chartwell
A4 210 x 297 mm



The current in microamperes is given roughly by $12\frac{1}{2}$ times the meter reading.

It was extremely difficult to get really accurate pointer readings on both meters used.

ACI T15

H. Cotton. Applied Electricity

p180. Purely inductive circuit.

voltage $v = E \sin \omega t = L\, di/dt$.

$$\therefore \quad i = -\frac{E}{\omega L}\cos \omega t = \frac{E}{\omega L}\sin\left(\omega t - \frac{\pi}{2}\right) \quad ①$$

A purely inductive circuit does not dissipate any power.

Current lags behind voltage by 90°.

p182 Purely capacitive circuit.

C

$$\left.\begin{aligned} v &= \frac{q}{C} = E\sin\omega t \\ q &= EC\sin\omega t \end{aligned}\right\} ②$$

$$i = \frac{dq}{dt} = EC\omega\cos\omega t$$
$$= EC\omega\sin\left(\omega t + \frac{\pi}{2}\right)$$

Current leads voltage by 90°.

at time $t + \frac{\pi}{2\omega}$, ① shows $v = E\sin\left(\omega t + \frac{\pi}{2}\right)$

$$i = \frac{E}{\omega L}\sin\omega t$$

~~Let~~ If $i = i_0 \sin\omega t$, $E = \frac{\omega L i}{\sin\omega t} = \omega L i_0$

and $v = \omega L i_0 \sin\left(\omega t + \frac{\pi}{2}\right)$

when $i = i_0 \sin\omega t$

In ②, at time $t - \frac{\pi}{2\omega}$ $\quad v = E\sin\left(\omega t - \frac{\pi}{2}\right)$

$$i = EC\omega\sin\omega t.$$

If $i = i_0\sin\omega t$ $\quad EC\omega = i_0 \quad E = i_0/C\omega$

$$v = \frac{i_0}{\omega C}\sin\left(\omega t - \frac{\pi}{2}\right)$$

If a circuit contains resistance, inductance and capacity, we can think of these as being in series. In fact, resistance is distributed within the inductance coil, but the series picture seems to give correct results.

The same current flows through R and L and into C. Call it $i_0 \sin \omega t$. Then the voltage drops are summed in

$$R i_0 \sin \omega t + \omega L i_0 \sin(\omega t + \tfrac{\pi}{2}) + \frac{i_0}{\omega C} \sin(\omega t - \tfrac{\pi}{2})$$

$$= \mathcal{I}\left[R i_0 e^{j\omega t} + \omega L i_0 e^{j\omega t + j\frac{\pi}{2}} + \frac{i_0}{\omega C} e^{j\omega t - j\frac{\pi}{2}} \right]$$

Now $e^{j\pi/2} = (\cos + j \sin)\frac{\pi}{2} = j$.

$e^{-j\pi/2} = 1/j = -j$.

Thus voltage drop across circuit is

$$\mathcal{I}\left[R i_0 e^{j\omega t} + \omega L j e^{j\omega t} - \frac{i_0}{\omega C} e^{j\omega t} \right]$$

$$= \mathcal{I}\left[\left(R + \omega L j + \frac{1}{\omega C j}\right) i_0 e^{j\omega t} \right]$$

As $\mathcal{I}\, i_0 e^{j\omega t}$ is the current, this looks like Ohm's Law, since volts $= R \times$ current.

Using the $\mathcal{I}(\ )$ convention, we can work just as if the inductance had resistance $\omega L j$ and the capacitor $\frac{1}{\omega C j}$.

The term *impedance* is used for this generalized resistance

The flow of currents in a network is determined by Kirchoff's two laws.

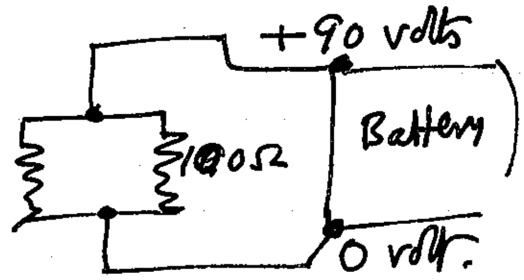
(1) Ohm's Law. A current i amps can flow through a resistance R only if there is a voltage difference Ri between the ends of the resistance.

(2) As much current flows out of a junction as flows into it.

Given these two principles, algebra will decide what happens.

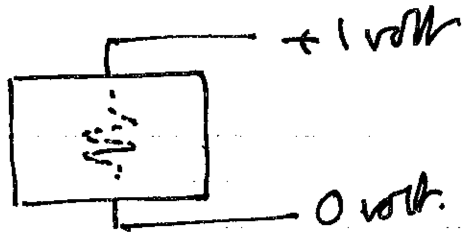
Problem 1.

What current flows in each resistor? What current does the battery supply?

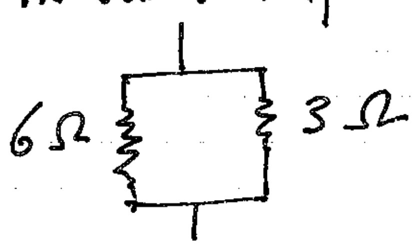


Problem 2

A box is believed to contain a single resister. 1 volt is applied to its terminals and the current that flows is measured.



On opening the box it is found to contain



What single resistor did they calculate was in the box?

Problem 3 A sensitive microammeter must not carry a current exceeding 50 μA. Its resistance is 950 Ω. What shunt must be put across it, if it is to be used for currents up to 1 mA?

Chap 1. §2. Computable functions and partially computable functions 18

To do calculations we must have signs for numbers. $S_1 = 1$. The number n is shown as $\underbrace{1 \cdots 1}_{n}$

$$S_i^n = \underbrace{S_i S_i \cdots S_i}_{n}$$

Def 2.1 With n we associate tape expression $\bar{n} = 1^{n+1}$

Def 2.2 With $(n_1\ n_2 \cdots n_k)$ we associate tape expression $\overline{(n_1\ n_2 \cdots n_k)} = \bar{n}_1 B \bar{n}_2 B \cdots B \bar{n}_k$

Example $\overline{(2\ 3\ 0)} = 111B1111B1$

Def 2.3 M any expression $\langle M \rangle$ = number of times 1 occurs in M.

$\langle PQ \rangle = \langle P \rangle + \langle Q \rangle$

Note $\langle \overline{m-1} \rangle = m$

Def. 2.4 Z a Turing machine. For each n we associate with Z a fn of n variable $\Psi_Z^{(n)}(x_1\ x_2 \cdots x_n)$ as follows:-

(1) If $\exists$ a computation $Z, \alpha_1 \cdots \alpha_p$ we put

$$\Psi_Z^{(n)}(m_1\ m_2 \cdots m_n) = \langle \alpha_p \rangle = \langle Res_Z(\alpha_1) \rangle$$

(2) otherwise undefined.

Example Z finds $m_1 + m_2$.

$Z =$
$q_1\ 1\ B\ q_1$
$q_1\ B\ R\ q_2$
$q_2\ 1\ R\ q_2$
$q_2\ B\ R\ q_3$
$q_3\ 1\ B\ q_3$

$\alpha_1 = q_1 1 1^{m_1} B 1 1^{m_2}$
$\to q_1 B 1^{m_1} B 1 1^{m_2}$
$\to B q_2 1^{m_1} B 1 1^{m_2}$
$\vdots$
$\to B 1^{m_1} q_2 B 1 1^{m_2}$
$\to B 1^{m_1} B q_3 1 1^{m_2}$
$\to B 1^{m_1} B q_3 B 1^{m_2}$

terminates because there is no instruction $q_3 B$!

$$\Psi_Z^{(2)}(m_1\ m_2) = \langle B 1^{m_1} B q_3 1^{m_2} \rangle = m_1 + m_2.$$

Def 2.5 $f(x_1 \cdots x_n)$ is computable if to partially computable if $\exists Z$:-

$$f(x_1 \cdots x_n) = \Psi_Z^{(n)}(x_1 \cdots x_n).$$

If f is a total function (defined for all ordered n-tuples), we say f is computable.

For a fn only partially computable, the machine may go on for ever without finding an answer, or telling us there is none.

Martin Davis Computability & Unsolvability. (McGraw Hill 1958). ② T_{19}

Expression. Finite sequence (perhaps ∅) of symbols chosen from $q_1\ q_2\ q_3 \ldots S_1\ S_2\ S_3 \ldots$ L R.

Quadruple Expression
$$\begin{cases} (1)\ q_i\, S_j\, S_k\, q_e \\ (2)\ q_i\, S_j\, R\, q_e \\ (3)\ q_i\, S_j\, L\, q_e \end{cases}$$
[4] $q_i\, S_j\, q_k\, q_e$.

(1) In state q_i, seeing S_j, replace by S_k, go to state q_e

(2) go right, state q_e.

(3) go left, state q_e.

A Turing machine is a set of quadruples, no pair having same pair $q_i S_j$ at start. So instruction unambiguous.

$\{q_i\}$ internal configurations

$\{S_j\}$ alphabet.

Simple Turing machine uses (1), (2), (3) only.

Symbol S_0 is blank, B.
S_1 1

Instantaneous description, does not contain R or L, contains just one q, which is not at extreme right.

Example $S_0 S_1 q_3 S_2 S_3 S_4$.

The tape is $S_0 S_1 \underset{\uparrow}{S_2} S_3 S_4$

The machine is in state q_3, and looking at S_2

Def. An instantaneous description, α is called terminal if $\sim\exists\beta :- \alpha \to \beta$.

Def. A computation is a sequence of instantaneous descriptions $\alpha_1\ \alpha_2 \ldots \alpha_p$ (Z) :- $1 \le i < p$, $\alpha_i \to \alpha_{i+1}$ and α_p is terminal. Then $\alpha_p = \mathrm{Res}_Z(\alpha_1)$

α_p is the resultant of α_1 w.r.t. Z

Illustration.

$$Z = \begin{cases} q_0 S_0 R q_1 & q_1 S_2 R q_1 & q_1 S_5 R q_1 \\ q_1 S_3 S_5 q_2 & q_2 S_8 L q_3 & q_3 S_0 S_5 q_3 \\ q_3 S_2 S_5 q_3 & q_3 S_3 S_5 q_3 & \end{cases}$$

If $\alpha_1 = S_2 q_1 S_0 S_5 S_2$

$\alpha_1 \rightarrow S_2 S_0 q_1 S_5 S_2$

$\rightarrow S_2 S_0 S_5 q_1 S_2$

$\rightarrow S_2 S_0 S_5 S_2 q_1 S_0$

$\rightarrow S_2 S_0 S_5 S_2 S_0 q_1 S_0$

$Res_Z(S_2 q_1 S_0 S_5 S_2)$ undefined.

S_0 or B is symbol for blank. The characters are on an infinite strip with blanks extending indefinitely on either side.

Def. 2.1 With each number n we associate tape expression $\bar{n} = 1^{n+1} = \underbrace{1 - - - - - 1}_{n+1}$

Def 2.2 With k tuple we associate $\overline{(n_1, n_2 \ldots n_k)} = \bar{n}_1 B \bar{n}_2 B \ldots B \bar{n}_k$.

Def 2.3 Output. $\langle M \rangle$ is no. of times 1 occurs in M.

Then $\langle 11 B S_4 q_3 \rangle = 2$ Note $\langle \overline{m-1} \rangle = m$.

Def 2.4 Z a Turing machine. For each n, we associate with Z a function $\varphi^{(n)}(x_1 \ldots x_n)$ as follows:- Let $\alpha_1 = q_1 \overline{(m_1 \ldots m_n)}$

(1) If there is a computation of Z, $\alpha_1 \ldots \alpha_p$ $\quad \varphi_Z^{(n)}(m_1 \ldots m_n) = \langle \alpha_p \rangle$

$= \langle Res_Z(\alpha_1) \rangle$

(2) If no such computation, i.e. $Res_Z(\alpha_1)$ undefined $\varphi_Z^{(n)}$ is undefined.

$\varphi_Z^{(1)}(x)$ written $\varphi_Z(x)$.

Def 2.5 $f(x_1 \ldots x_n)$ is called partially computable if $\exists Z$:- $f(x_1 \ldots x_n) = \Psi_Z^{(n)}(x_1 \ldots x_n)$ for any $x_1 \ldots x_n$ in domain of f T21

Computable if $f(x_1 \ldots x_n)$ is total function (defined for all n-tuples).

Examples follow. If $f(x,y) = x - y$ defined only for $x \geq y$. Partially computable

§4. Relatively computable functions.

Th^y of computability in a special case.

A is set of numbers

Def. 4.1 $\alpha \underset{A}{\rightarrow} \beta\ (Z)$. If $\alpha = Pq_iS_jQ$

Z contains $q_iS_jq_kq_l$

(1) If $\langle\alpha\rangle \in A$ $\beta = Pq_kS_jQ$

(2) If $\langle\alpha\rangle \notin A$ $\beta = Pq_lS_jQ$.

Thus the next operation is switched to q_k is $\langle\alpha\rangle$ in A, to q_l otherwise.

Def. 4.2 $\alpha = Pq_iS_iQ$. α is final w.r.t Z if Z has no ~~sq~~ A quadruple with initial symbols q_iS_i

Theorem 4.1 If Z is simple, then α is final w.r.t Z iff it is terminal w.r.t Z iff

Theorem 4.2 α is final w.r.t Z

(1) α is terminal w.r.t Z

(2) No matter what set A is chosen $\nexists \beta$:- $\alpha \underset{A}{\rightarrow} \beta\ (Z)$.

Def 4.3 An A-computation of Z is finite sequence $\alpha_1\ \alpha_2 \ldots \alpha_p$:- for $1 \leq i < p$

$\alpha_i \rightarrow \alpha_{i+1}(Z)$ or $\alpha_i \underset{A}{\rightarrow} \alpha_{i+1}\ (Z)$

and α_p is final.

Then $\alpha_p = \mathrm{Res}_Z^A(\alpha_1)$ and α_p is A-resultant of α_1 w.r.t. Z

If Z is a simple Turing machine, then the definition of 4.3 is independent of A and an A computation in this sense is simply a computation in the sense of Def 1.9. T 22

[For $\alpha \underset{A}{\rightarrow} \beta(2)$ can never hold if Z is a simple Turing machine. For, by 4.1, $\alpha \underset{A}{\rightarrow} \beta(2)$ requires Z to contain $q_i S_j q_k q_l$, which a simple machine does not.]

Def 4.4. Z Turing machine. For each n we associate with Z a function $\Psi^{(n)}_{Z;A}(x_1 \dots x_n)$ [generally depending on A], as follows:

For each $(m_1 \dots m_n)$ we set $\alpha_1 = q_1(m_1, m_2 \dots m_n)$

(1) If ~~∃ A computation~~ ∃ an A computation of Z, $\alpha_1\ \alpha_2 \dots \alpha_p$ $\Psi^{(n)}_{Z:A}(m_1 \dots m_n) = \langle \alpha_p \rangle = \langle \mathrm{Res}_Z{}^A(\alpha_1) \rangle$

(2) If not, Ψ is undefined.

$\Psi_Z{}^A(x)$ means $\Psi^{(1)}_{Z:A}(x)$.

If Z is simple, by remark above

$$\Psi^{(n)}_{Z:A}(x_1 \dots x_n) = \Psi^{(n)}_Z(x_1 \dots x_n).$$

Def 4.5 $f(x_1 \dots x_n)$ is <u>partially A-computable</u> if ∃ Turing machine Z :- $f(x_1 \dots x_n) = \Psi^{(n)}_{Z:A}(x_1 \dots x_n)$

If so we say Z A-computes f.

If in addition f is <u>total fn</u>, f is called <u>A-computable</u>.