A complex variable $w = f(z)$, $w = u + iv$, $z = x + iy$ is called analytic in a region if at each point of the region the derived function $f'(z)$ exists, not depending on the direction of $dx : dy$.

For a simple treatment it is helpful to make certain assumptions about the continuity of the partial derivatives (these assumptions can be reduced) ⁊.

We require then $du + idv = f'(x+iy)(dx + idy)$

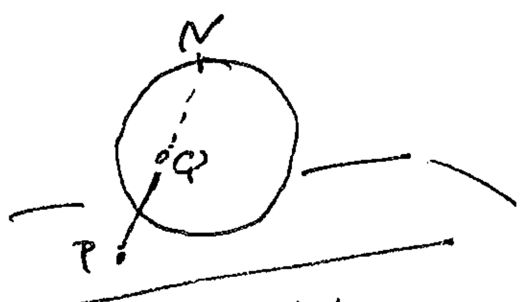
Let $f'(x+iy) = p + iq$.

$$\text{Then} \quad du + idv = (p+iq)(dx+idy)$$
$$= pdx - qdy + i(qdx + pdy)$$
$$\therefore \quad du = pdx - qdy, \quad dv = q\,dx + pdy$$
$$\therefore \quad \frac{\partial u}{\partial x} = p \quad \frac{\partial u}{\partial y} = -q \quad \frac{\partial v}{\partial x} = q \quad \frac{\partial v}{\partial y} = p$$
$$\text{So} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} : \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{Cauchy-Riemann.}$$

These are the equations satisfied by the potential and stream function for an incompressible flow in 2 dimensions with a velocity potential. Current in a copper sheet may be taken as an example of this.

The equations fail at point where current enters or leaves. These correspond to singularities.

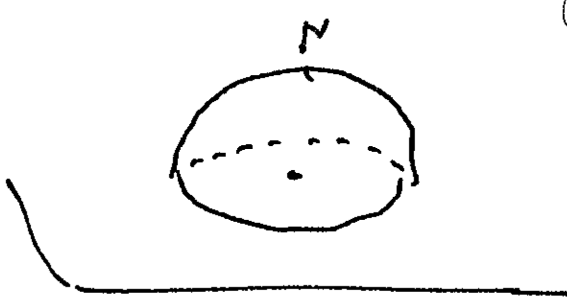
Stereographic projection.



If we suppose a sphere placed with its centre at or above the origin, and point P of the complex plane projected to the correspond- point Q, it is found that we obtain u & v for an incompressible flow on the sphere. This removes the special status of ∞, which now is seen at the point N.

Circles in the plane project to circles on the sphere.

The equations for stereographic projection



Let $z=0$ be taken as the complex plane, $x^2+y^2+z^2=1$ as the sphere, $(1,0,0)$ as N.

$a+ib$ is represented by the point $(a,b,0)=P$. We need to find where the line NP meets the sphere.

The vector $\underline{NP}$ is $(a,b,-1)$. The point $\underline{N}+t\underline{NP}$ has coordinates $(at, bt, 1-t)$. It moves from N to P, in a straight line, as t goes from 0 to 1.

It is on the sphere when $a^2t^2+b^2t^2+(1-t)^2=1$

i.e. $t^2(a^2+b^2+1)-2t=0$, so $t=\dfrac{2}{a^2+b^2+1}$

The point is then $\dfrac{2a}{a^2+b^2+1}, \dfrac{2b}{a^2+b^2+1}, \dfrac{a^2+b^2-1}{a^2+b^2+1}$

Inverse points as seen on the sphere.

$$\frac{1}{a+ib}=\frac{a-ib}{a^2+b^2}=\alpha+i\beta \text{ say}$$

Thus $\alpha=\dfrac{a}{a^2+b^2}$ $\quad \beta=-\dfrac{b}{a^2+b^2}$

The point corresponding to $\alpha+i\beta$ on the sphere is

$$\frac{2\alpha}{\alpha^2+\beta^2+1}, \frac{2\beta}{\alpha^2+\beta^2+1}, \frac{\alpha^2+\beta^2-1}{\alpha^2+\beta^2+1}$$

$$\alpha^2+\beta^2=\left(\frac{a}{a^2+b^2}\right)^2+\left(\frac{-b}{a^2+b^2}\right)^2=\frac{1}{a^2+b^2}$$

Thus $\dfrac{2\alpha}{\alpha^2+\beta^2+1}=\dfrac{\frac{2a}{a^2+b^2}}{\frac{1}{a^2+b^2}+1}=\dfrac{2a}{a^2+b^2+1}$ same as before.

$\dfrac{2\beta}{\alpha^2+\beta^2+1}=\dfrac{-2b/(a^2+b^2)}{\frac{1}{a^2+b^2}+1}=\dfrac{-2b}{a^2+b^2+1}$, minus previous value.

$\dfrac{\alpha^2+\beta^2-1}{\alpha^2+\beta^2+1}=\dfrac{\frac{1}{a^2+b^2}-1}{\frac{1}{a^2+b^2}+1}=\dfrac{1-a^2-b^2}{a^2+b^2+1}=-$ previous value.

Thus, if $a+ib$ maps to (x, y, z) on sphere
$\alpha + i\beta$ maps to $(x, -y, -z)$.
These are connected by a rotation of 180° about OX.

Circle of convergence.

If $f(z)$ is analytic in a region containing the origin, and $z=k$ is the nearest singularity, it is possible to expand $f(z)$ in an absolutely and uniformly convergent series $f(z) = \sum_0^\infty a_n z^n$ provided $|z| < k$, i.e. within a ~~cent~~ circle, centre O, with nearest singularity on circumference.

For example $f(z) = \dfrac{1}{1-z}$ has only one singularity $z=1$.

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots \qquad (\mathrm{I})$$

convergent for $|z| < 1$, divergent for $|z| > 1$.

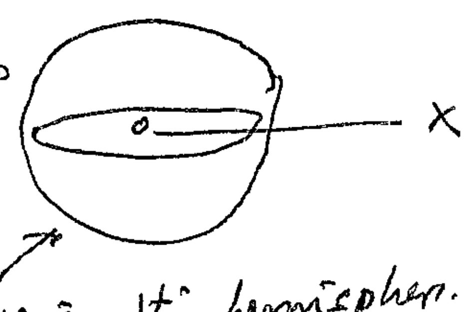
If we want a series valid for $|z| > 1$, we put

$$f(z) = \frac{1}{1-z} = \frac{-1}{z-1} = \frac{-1/z}{1-1/z}$$

$$= -\frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \dots\right]$$

$$= -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} \dots \dots \qquad (\mathrm{II})$$

If $\rho = \frac{1}{z}$ $f(z) = -\rho - \rho^2 - \rho^3 \dots$
$\rho = 0$ corresponds to $z = \infty$. We call the above series, the series about $z = \infty$. It converges for $|\rho| < 1$.



series II go here

z is changed to ρ by a rotation of 180° about OX.
Thus the upper hemisphere counts as the circle $|\rho| < 1$.
It corresponds to all the points outside $|z| = 1$

series I go in this hemisphere.

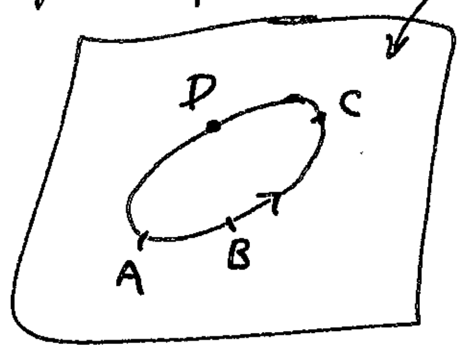
Thus we may think of the points outside a circle as being inside a circle, centre ∞.

An important property of integrals

$f(z)$ analytic in this region

$$\oint f(z)\,dz = 0$$

for any closed path lying entirely in the region

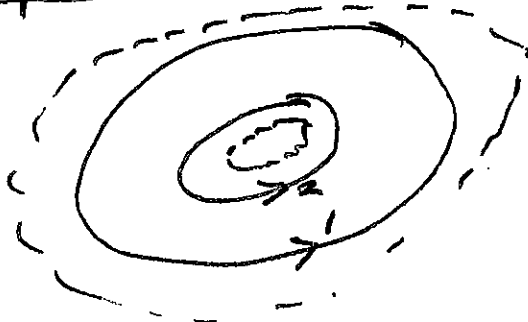


$$0 = \oint = \int_{ABCDA} = \int_{ABC} + \int_{CDA} = \int_{ABC} - \int_{ADC}$$

$$\therefore \int_{ABC} f(z)\,dz = \int_{ADC} f(z)\,dz$$

The value of the integral is not altered if the path is deformed, the end points staying the same, it is essential that the path does not pass over any singularity.

If $f(z)$ is analytic in the region between the two dotted curves, then



$$\int_{C_1} = \int_{C_2}.$$ Proof. Consider

If $f(z)$ ~~resembles~~ $\sim \frac{k}{z}$ near $z=0$, for C, a circle around origin, $\int_C f(z)\,dz \sim \int_C \frac{k}{z}\,dz$

$$= \int_0^{2\pi} \frac{k\, ire^{i\theta}\,d\theta}{re^{i\theta}} = k \cdot 2\pi i$$

Moment functions

$$f(z) = \int^{b} \frac{w(t)\,dt}{z-t} \quad \text{where } w(t) \text{ is real}, \geq 0$$

is called a moment function.

$$f(z) = \int_a^b \frac{w(t)\,dt}{z(1-\frac{t}{z})} = \int_a^b \frac{w(t)\,dt}{z}\left(1+\frac{t}{z}+\frac{t^2}{z^2}+\dots\right)$$

$$= \int_a^b w(t)\,dt\left(\frac{1}{z}+\frac{t}{z}+\frac{t^2}{z^2}+\dots\right)$$

Coefficient of $\frac{1}{z}$ is $\int_a^b w(t)\,dt$, the total mass

of $\frac{1}{z^2}$ is $\int_a^b t\,w(t)\,dt$ the moment about origin

of $\frac{t}{z^3}$ is $\int_a^b t^2 w(t)\,dt$ moment of inertia about origin

$\int^b t^n w(t)\,dt$ is known as the nth moment.

Analytic nature of $f(z)$

(1) If z is any point not on the interval $[a,b]$, $f(z)$ is analytic at z.

Proof $f'(z) = \frac{d}{dz}\int_a^b \frac{w(t)\,dt}{z-t}$

$$= \int_a^b w(t)\,dt \cdot \frac{\partial}{\partial z}(z-t)$$

$$= \int_a^b \frac{-w(t)}{(z-t)^2}\,dt.$$

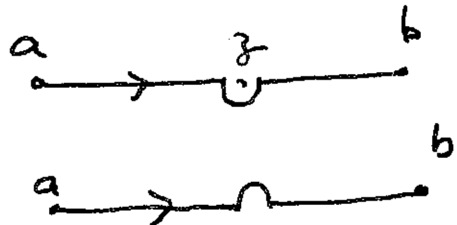
As z is not on $[a,b]$, $|z-t|$, the minimum distance of z from point t, has a greater than zero, so $\frac{1}{(z-t)^2}$ will be finite for $0 \leq t \leq 1$, and the integral will be satisfactory.

There will be a discontinuity across the cut (a,b). If z is real, $0 < z < t$, we shall get different values for $f(z)$ depending on whether we approach z from below or above.

If z is above the real axis, displacing the path to P_1 will not cause the integrand $\frac{w(t)}{z-t}$ to pass over any singularity, since $z-t \neq 0$ in this region.

P_1

P_2



Thus $f_+(z) = \int_{P_1} \frac{w(t)\,dt}{z-t}\,dt$, where $f_+(z)$ is the value of $f(z)$ found by continuation from above.

Similarly $f_-(z) = \int_{P_2} \frac{w(t)\,dt}{z-t}$

Now the difference of these is integral around 

So $f_+(z) - f_-(z) = \int_C \frac{w(t)\,dt}{z-t}$

$$= \int_C \frac{-w(t)\,dt}{t-z}$$

$f_+(z) - f_-(z) = -2\pi i\, w(z)$ the radius of the circle having tended to 0.

Note The above proof uses $w(t)$ being analytic. The theorem holds true in many cases where this is not so, for instance if $w(t)$ has graph such as

This theorem is used if we know $f(z)$ and want to find $w(t)$ to give it.

C14

In the course of my work I came across the function $\varphi(a) = \int_0^1 \frac{-\log v \, dv}{a+v}$

$-\ln v > 0$ for $0 < v < 1$.

Suppose $a > 0$. $\int_0^1 \frac{-\ln v \, dv}{a+v} < \frac{1}{a}\int_0^1 -\ln v \, dv$

$= -\frac{1}{a}\Big[v \ln v - v \Big]_0^1$

$v \ln v \to 0$ as $v \to 0$, so this $= \frac{1}{a}$.

Thus $\varphi(a)$ is finite when $a > 0$.

If $a = 0$ $\int_0^1 \frac{-\ln v \, dv}{v} = -\frac{1}{2}(\ln v)^2\Big]_0^1 = +\infty$.

$\varphi(a) \to +\infty$ as $a \to 0$.

If $\underline{a}$ increases, clearly $\frac{1}{a+v}$ decreases.

Hence $\varphi(a)$ decreases steadily as $\underline{a}$ goes from 0 to $+\infty$.

$\frac{1}{a+v} = \frac{1}{a(1+\frac{v}{a})} = \frac{1}{a}\left(1 - \frac{v}{a} + \frac{v^2}{a^2} - \frac{v^3}{a^3} \cdots\right)$

$\varphi(a) = \int_0^1 -\ln v . \sum_0^\infty \frac{(-1)^n v^n}{a^{n+1}}$

$\int_0^1 -v^n \ln v \, dv = \left[\frac{-v^{n+1}\ln v}{n+1}\right]_0^1 + \int_0^1 \frac{v^{n+1}}{n+1} \frac{1}{v} dv$

$\Big[\ \Big]_0^1 = 0$ $\int_0^1 \frac{v^{n+1}}{n+1} . \frac{1}{v} dv = \int_0^1 \frac{v^n \, dv}{n+1}$

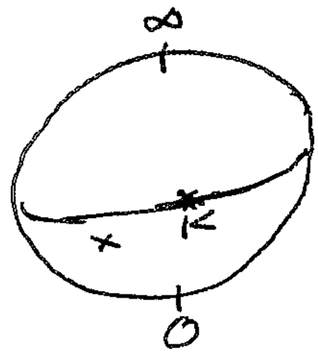
$= \frac{v^{n+1}}{(n+1)^2}\Big]_0^1 = \frac{1}{(n+1)^2}$

Thus $\varphi(a) = \sum_0^\infty \frac{(-1)^n}{(n+1)^2 a^{n+1}}$ (1)

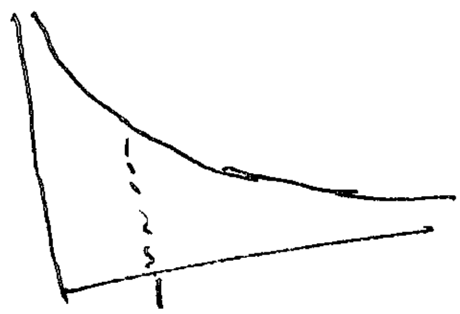
This series converges for $|a| > 1$, diverges f $|a| < 1$.

It diverges everywhere inside the circle $|a| = 1$ and converges everywhere outside it. This indicates that there must be a singularity somewhere on $|a| = 1$.

For if k is the singularity furthest from the origin, the series about ∞ converges for $|z| > |k|$. If $|k|$ were less than 1, the region of convergence of series (1) would have some part inside $|a| = 1$.



The graph of $\varphi(a)$ for $a > 0$ is



There is no indication of a singularity at $a = 1$. The singularity must be elsewhere.

Series valid for $0 < a < 1$.

$$\varphi(a) = \int_0^1 \frac{-\ln v \, dv}{a + v} \qquad \text{let } v = au$$

$$\varphi(a) = \int_0^{\frac{1}{a}} \frac{-\ln(au) \cdot a\, du}{a + au} = \int_0^{\frac{1}{a}} \frac{-\ln a - \ln u}{1+u}\, du$$

$$\int_0^{\frac{1}{a}} \frac{du}{1+u} = \Big[\ln(1+u)\Big]_0^{\frac{1}{a}} = \ln\left(1 + \frac{1}{a}\right)$$

Thus part of $\varphi(a)$ is $\underline{\underline{-\ln a \ln\left(1 + \frac{1}{a}\right)}}$ (α)

$$\int_0^{1/a} \frac{-\ln u\, du}{1+u} = \int_0^1 \frac{-\ln u\, du}{1+u} + \int_1^{1/a} \frac{-\ln u\, du}{1+u}.$$

$$\int_0^1 \frac{-\ln u\, du}{1+u} = \int_0^1 -\ln u \cdot \sum_0^\infty (-u)^n du = \sum_0^\infty \frac{(-1)^n}{(n+1)^2} \quad (\beta)$$

In $\int_1^{1/a} -\frac{\ln u\,du}{1+u}$ put $u = \frac{1}{w}$. 016

$$\int = \int_{w=1}^{a} \frac{\ln w}{1+\frac{1}{w}}\left(\frac{-dw}{w^2}\right) = \int_a^1 \frac{\ln w\,dw}{w(1+w)}$$

$$= \int_a^1 \ln w \left[\frac{1}{w} - \frac{1}{1+w}\right] dw$$

$$\cancel{\int_a^1 \frac{\ln w\,dw}{1+w}} = \int_a^1 \frac{\ln w\,dw}{w} = \left[\tfrac{1}{2}(\ln w)^2\right]_a^1 = -\tfrac{1}{2}(\ln a)^2 \quad (\gamma)$$

Finally $$\int_a^1 \frac{-\ln w\,dw}{1+w} = \int_a^1 -\ln w \sum_0^\infty (-w)^n\,dw$$

$$= \sum_0^\infty (-1)^{n+1} \int_a^1 w^n \ln w\,dw$$

$$= \sum_0^\infty (-1)^{n+1} \left[\frac{w^{n+1}}{n+1}\ln w - \frac{w^{n+1}}{(n+1)^2}\right]_a^1$$

$$= \sum_0^\infty (-1)^{n+1} \left\{-\frac{a^{n+1}\ln a}{n+1} - \frac{1-a^{n+1}}{(n+1)^2}\right\}$$

$$= \ln a \sum_0^\infty \frac{(-1)^n a^{n+1}}{n+1} + \sum_0^\infty \frac{(-1)^n(1-a^{n+1})}{(n+1)^2}$$

$$= \ln a \ln(1+a) + \sum_0^\infty \frac{(-1)^n}{(n+1)^2} + \sum_0^\infty \frac{(-a)^{n+1}}{(n+1)^2} \quad (\delta)$$

In (α) we have $-\ln a \ln(1+a) + (\ln a)^2$.
After cancelling we find

$$\varphi(a) = \tfrac{1}{2}(\ln a)^2 + 2\sum \frac{(-1)^n}{(n+1)^2} + \sum \frac{(-a)^{n+1}}{(n+1)^2}$$

For $|a|>1$ $\quad \varphi(a) = \sum_0^\infty \frac{(-1)^n}{(n+1)^2 a^{n+1}}$

How does this series behave if we place $\underline{a}$ on the unit circle?

There is a theorem that, if $\sum c_n$ is absolutely convergent (i.e $\sum |c_n|$ convergent) then $\sum c_n$ is convergent.

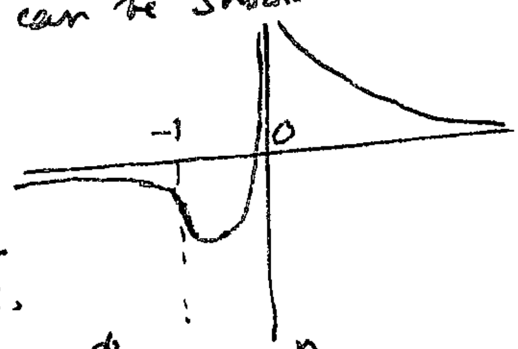
The absolute series corresponding to $\varphi(a)$ is

$$\sum_0^\infty \frac{1}{(n+1)^2 |a|^{n+1}}.$$

If $\underline{a}$ is on unit circle, $|a|=1$, and we have $\sum_0^\infty \frac{1}{(n+1)^2}$, which is convergent. It is in fact $\pi^2/6$. So $\varphi(a)$ is convergent everywhere on $|a|=1$. How then does it manage to have a singularity on $|a|=1$?

The clue was given by the graph for $a<0$.

The graph of $\varphi(a)$ can be shown to be,



and the tangent at $a=-1$ is vertical.

In fact $\varphi(a) = \sum_0^\infty \frac{(-1)^n}{(n+1)^2 a^{n+1}}$

$$\varphi'(a) = \sum_0^\infty \frac{(-1)^n}{(n+1)^2}\left[-(n+1)a^{-n-2}\right]$$

$$= \sum_0^\infty \frac{(-1)^{n+1}}{(n+1)a^{n+2}}$$

$$\log\left(1+\frac{1}{a}\right) = \frac{1}{a} - \frac{1}{2a^2} + \frac{1}{3a^3} - \frac{1}{4a^4} \cdots$$

$$= \sum_0^\infty \frac{(-1)^n}{(n+1)a^{n+1}}$$

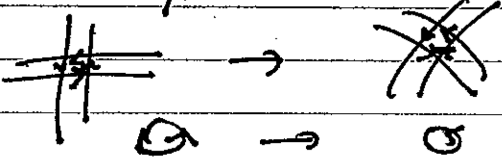
$\therefore \varphi'(a) = -\frac{1}{a}\log\left(1+\frac{1}{a}\right)$ which is $-\infty$ for $a=-1$.

We are concerned with functions, $\mathbb{C} \to \mathbb{C}$, for which $f'(z)$ exists, i.e for $w = f(z)$, $\frac{dw}{dz} = f'(z)$

Consider a particular point z, and variations dz from it. Then $dw = f'(z)\,dz$. Here $f'(z)$ is some particular complex number. Multiplication by it produces a rotation and change of scale, i.e it leaves angles and ratios unchanged. Such a transformation is called <u>conformal</u>. As $dw = f'(z)\,dz$ holds only for differentials, this means that the geometry is preserved only in the limit as we considered neighbourhoods of z and w whose size $\to 0$. This in particular means that the angle between the tangents to two curves, at a place where they cross, is not altered.

Note that multiplication produces a turning: it cannot produce a turning over. Thus, if $z = x + iy$, $w = x - iy$ the function $z \to w$ cannot be analytic.

Always



direction of rotation preserved

Transformation, $w = \frac{az+b}{cz+d}$, maps conformally 1-1 the entire plane z to the entire plane w. It preserves the essential behaviour of functions at each point, and can give a very useful way of putting some situation in an easily understood form.

We suppose, of course, $ad - bc \neq 0$.

Such a transformation can be the combined effect of simpler transformations. For example

$$\frac{12z+25}{3z+6} = 4 + \frac{1}{3z+6} = 4 + \frac{1}{3(z+2)}$$

Thus $z^* = \dfrac{12z+25}{3z+6}$ can be achieved by the following sequence of steps.

$$z_1 = z + 2 \qquad (1)$$
$$z_2 = 3z_1 \qquad (2)$$
$$z_3 = 1/z_2 \qquad (3)$$
$$z^* = z_3 + 4. \qquad (4)$$

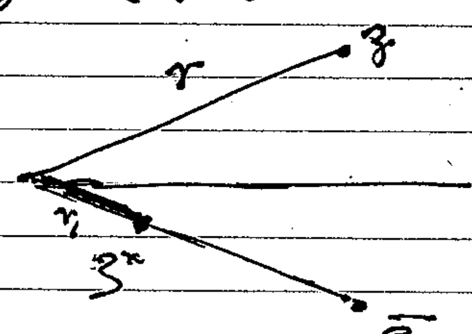
Such a decomposition is always possible.

(1) and (4) are simply translations.

(2) is a change of scale.

(3) is a new kind of transformation.

If $z^* = \dfrac{1}{z}$ and z is (r, θ) then z^* is $(\frac{1}{r}, -\theta)$



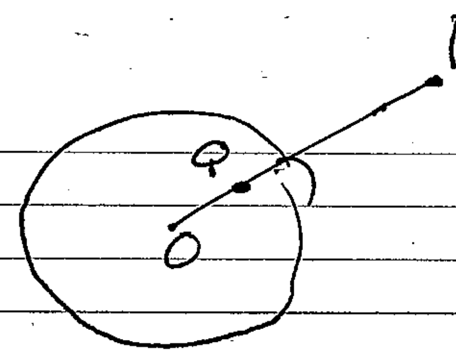
If $z = x + iy$, $\bar{z} = x - iy$, reflection in the real axis. $\bar{z}$ is at distance r from the origin, z^* at distance $\frac{1}{r}$.

$\bar{z} \to z^*$ is known as <u>inversion in the unit circle.</u>

C 3.

C20

Given any circle, radius R, centre O, inversion in it sends P to Q where OPQ is straight and $OP.OQ = R^2$.

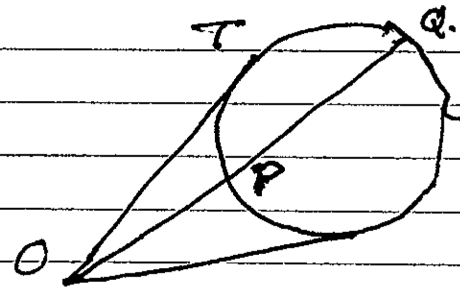


This operation has much in common with reflection. If $P \to Q$, then $Q \to P$. If the radius R becomes very large, it could be mistaken for reflection.

It reverses sense of rotation.

$z^* = 1/z$ is analytic: it is equivalent to 2 operations, each of which reverses sense of rotation. It could not be equivalent to a single such operation.

<u>Theorem</u> If we have a circle C, and if OT is a tangent to it, inversion of C in the circle, centre O, radius OT, makes $C \to C$.



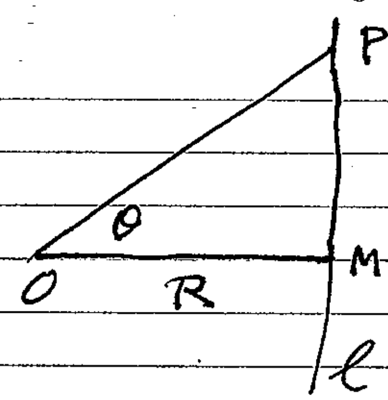
<u>Proof</u> $OP.OQ = OT^2$ is a known theorem.

Changing the radius of the circle of inversion only produces a change of scale, so C inverted in any circle, centre O, goes to a circle.

C4 (C21)

Problem ℓ is a line at perpendicular distance R from a point O. What do we get if we invert ℓ in the circle centre O, radius R?

Suggestion: Find OQ when $\angle POM$ is θ.



$$w = \frac{az+b}{cz+d} \quad \text{where } ad - bc \neq 0.$$

Just one value of w corresponds to a given value of z.

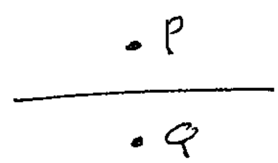
$cwz + wd = az + b \quad \therefore \; z = \dfrac{wd - b}{-cw + a}$.

This gives a single value for z when w is known

[Question. What goes wrong with this argument if $ad - bc = 0$?]

"Kreisverwandschaft."

<u>Inversion</u> Reflection in a line is $P \to Q$, $Q \to P$



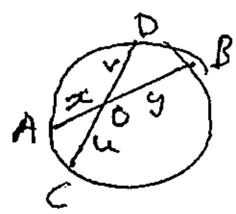
Inversion may be thought of as reflection in a circle.

$OP.OQ = R^2.$

Important property of a circle

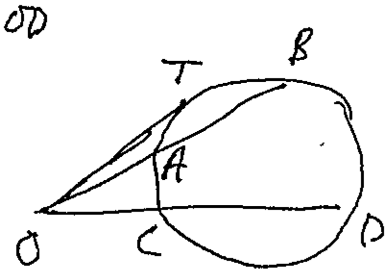
$x.y = u.v$

(lengths)



$OA.OB = OC.OD$

This still holds if O is outside the circle.



For the tangent OT, $A \to T$, $B \to T$.

So $OA.OB = OC.OD = OT^2$.

If we invert with respect to the circle, centre O, radius OT $A \to B$, $B \to A$

So the circle TBDCA reflects to itself.

If we invert with respect to a circle, centre O, but a different radius, we have similar effect with a change of scale: the circle goes to a different circle.

There is an exception, if O lies on the circle BDCA.

1) $f(z) = f(x+iy)$ is analytic in a region if there is a continuous derivative $f'(z)$, independent of direction in which z is changing.

$$df(z) = f'(z)\,dz.$$

Let $f(x+iy) = u+iv$. $\quad f'(z) = p+iq$

$$du + i\,dv = (p+iq)(dx+i\,dy)$$
$$= p\,dx - q\,dy + i(q\,dx + p\,dy)$$

$du = p\,dx - q\,dy \quad \therefore \frac{\partial u}{\partial x} = p \quad \frac{\partial u}{\partial y} = -q.$

$dv = q\,dx + p\,dy \quad \therefore \frac{\partial v}{\partial x} = q \quad \frac{\partial v}{\partial y} = p.$

Thus $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Cauchy Riemann.

These derivatives must exist and be continuous.

2) Geometrical meaning of Cauchy Riemann equations.

Suppose a point (x,y) starts at $(x_0\ y_0)$ and moves a distance s at angle θ to OX.

Then $x = x_0 + s\cos\theta \qquad y = y_0 + s\cdot\sin\theta$

Rate of change of $u(x,y)$ w.r.t. s is then

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta.$$

Suppose v changes as a result of x, y, moving in direction $\theta + \frac{\pi}{2}$. $\quad \cos(\theta+\frac{\pi}{2}) = -\sin\theta$, $\sin(\theta+\frac{\pi}{2}) = \cos\theta$

$$\frac{dv}{ds} = -\frac{\partial v}{\partial x}\sin\theta + \frac{\partial v}{\partial y}\cos\theta$$
$$= \frac{\partial u}{\partial y}\sin\theta + \frac{\partial u}{\partial x}\cos\theta \quad \text{if C-R hold.}$$

Thus rate of change of u in direction θ

= rate of change of v in direction $\theta + \frac{\pi}{2}$.

In particular, if u is constant for $\angle\theta$, v is constant for $\angle\theta + \frac{\pi}{2}$. The curves $u = c$, $v = k$ cross at right angles.

Integration in 2 dimensions.

To define $\int X(x,y)\,dx + Y(x,y)\,dy$ along a given curve, we suppose the curve given by $x = x(t)$, $y = y(t)$ $0 \le t \le 1$.

Then $X(x,y) = X(x(t), y(t))$ a fn of t.

$$dx = \frac{dx}{dt}\,dt = x'(t)\,dt.$$

We define the integral as

$$\int_0^1 X(x(t), y(t))\,x'(t) + Y(x(t), y(t))\,y'(t)\,dt.$$

Suppose for example we require $\int y\,dx + 2x\,dy$ along the line joining $(1,1)$ to $(3,4)$.

$x = 1+2t$, $y = 1+3t$ takes x, y from $(1,1)$ to $(3,4)$ as t goes from 0 to 1.

$$\int = \int_0^1 (1+3t)2 + (2+4t)3 \,.\, dt$$

$$= \int_0^1 8 + 18t\,dt = \left. 8t + 9t^2 \right]_0^1 = 17.$$

We would get a different result if we went from $(1,1)$ to $(3,1)$ and then $(3,1)$ to $(3,4)$

In first part $x = 1+2t$, $y = 1$ $x' = 2$, $y' = 0$

We have $\int_0^1 2 + 4t.\,dt = \left. 2t + 2t^2 \right]_0^1 = 4.$

In the second part $x = 3$, $y = 1+3t$. $x' = 0$ $y' = 3$

$$\int_0^1 6.3\,dt = \left. 18t \right]_0^1 = 18. \qquad \int = 22.$$

$\int X\,dx + Y\,dy$ is an expression for work done. This example corresponds to a non conservative system. Work could be obtained from it by following an appropriate loop.

The result will be independent of the path if $X\frac{dx}{dt}+Y\frac{dy}{dt}\equiv\frac{d}{dt}f(x,y)$ for some $f(x,y)$ For $\int_0^1\frac{d}{dt}f(x,y)dt=f(x_1,y_1)-f(x_0,y_0)$.

Now $\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$

This case arises if there exists $f(x,y)$:-

$X=\frac{\partial f}{\partial x}$ $\quad Y=\frac{\partial f}{\partial y}$.

We suppose X and Y have continuous derivatives.

Then $\frac{\partial}{\partial y}X=\frac{\partial}{\partial y}\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\frac{\partial}{\partial y}f=\frac{\partial}{\partial x}Y$

$\frac{\partial X}{\partial y}=\frac{\partial Y}{\partial x}$ is the usual condition for $Xdx+Ydy$ to be an exact differential.

[Psaggio. Appendix A. $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$ if f is continuous, and has continuous first and second derivatives.]

If $\frac{\partial X}{\partial y}=\frac{\partial Y}{\partial x}$ we can show that there is an f with the required property. $\int Xdx+Ydy$ is then independent of the path between given end points, provided $\frac{\partial X}{\partial y}=\frac{\partial Y}{\partial x}$ holds throughout the region. Stokes theorem.

$$\oint Xdx+Ydy=\iint\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\,.\,dxdy.$$

If $f(x+iy)$ satisfies the Cauchy-Riemann conditions, $\int_a^b f(z)\,dz$ does not depend on the path connecting given a, b.

Proof If $f(x+iy) = u + iv$.

$$\int (u+iv)(dx + i\,dy) = \int u\,dx - v\,dy + i\int v\,dx + u\,dy.$$

$u\,dx - v\,dy$ is exact if $\frac{\partial}{\partial y}(u) = \frac{\partial}{\partial x}(-v)$

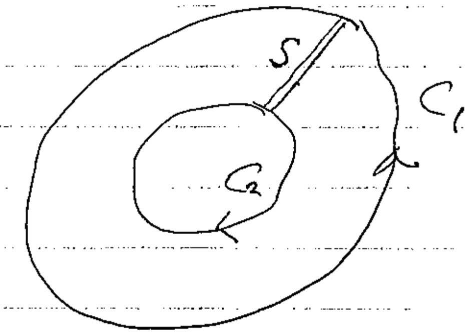
i.e. $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ This is a C-R condition.

$v\,dx + u\,dy$ is exact if $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$, the other C-R condition. So we have

Cauchy's Theorem.

If $f(z)$ is analytic in some region, $\oint f(z)\,dz = 0$ for any smooth closed curve in that region, and for any curve consisting of a finite number of smooth curves.

Corollary If C_1 and C_2 are two closed curves, C_1 outside C_2, and a region in which $f(z)$ is analytic contains C_1 and C_2.



then $\oint_{C_1} = \oint_{C_2}$

Proof Consider region bounded by C_2, C_1 and strip S. This is bounded by C_1, S, C_2, S. The two parts on S cancel and we have $\int_{C_1 - C_2} f(z)\,dz = 0$.

Residue Theorem. $f(z)$ is analytic on a region containing curve C. A point $\underline{a}$ is inside C. Consider $\oint_C \frac{f(z)\,dz}{z-a}$. By previous theorem this equals value for a small circle, centre $\underline{a}$. On this circle $f(z) \doteq f(a)$. In the limit we have $f(a)\oint \frac{dz}{z-a}$, integral around small circle.

On the small circle $z = a + re^{i\theta}$ where r is radius of circle, $0 \le \theta \le 2\pi$.

$$dz = re^{i\theta} i\,d\theta.$$

$$\oint \frac{dz}{z-a} = \int_0^{2\pi} \frac{re^{i\theta} i\,d\theta}{re^{i\theta}} = \int_0^{2\pi} i\,d\theta = 2\pi i$$

$$\therefore \quad 2\pi i\, f(a) = \oint_C \frac{f(z)\,dz}{z-a}$$

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)\,dz}{z-a}$$

Given $f(z)$ on the boundary, this gives an explicit solution for values of $f(z)$ inside the curve.

C28

$I = \int x^{-3/2}(1-x)^{-3/2}\,dx$ Let $y^2 = x - x^2$

If $y = tx$ $t^2x^2 = x - x^2$

$x = \dfrac{1}{1+t^2}$

$1-x = \dfrac{t^2}{1+t^2}$

$dx = \dfrac{-2t}{(1+t^2)^2}$

$t^2 = \dfrac{x}{1-x}$

$t = \sqrt{\dfrac{x}{1-x}}$

$$I = \int \left[\frac{t^2}{(1+t^2)^2}\right]^{-3/2} \frac{-2t}{(1+t^2)^2}\,dt$$

$$= \int \frac{(1+t^2)^3}{t^3}\,\frac{-2t}{(1+t^2)^2}\,dt$$

$$= -2\int \frac{1+t^2}{t^2}\,dt$$

$$= -2\left[t - \frac{1}{t}\right]$$

$$= 2\left[\sqrt{\frac{1-x}{x}} - \sqrt{\frac{x}{1-x}}\right]$$

$$= \frac{2}{\sqrt{x(1-x)}}\left[1-x-x\right]$$

$$= \frac{2(1-2x)}{\sqrt{x(1-x)}}$$

Check $\dfrac{d}{dx}(2-4x)x^{-1/2}(1-x)^{-\frac{1}{2}}$

$= \cancel{-4x^{\frac{1}{2}}}$ $\ln I = \ln 2 + \ln(1-2x) - \frac{1}{2}\ln x - \frac{1}{2}\ln(1-x)$

$$\frac{I'}{I} = \frac{-2}{1-2x} - \frac{1}{2x} + \frac{1}{2(1-x)}$$

$$= \frac{-4x(1-x) - (1-x)(1-2x) + x(1-2x)}{2x(1-x)(1-2x)}$$

Numer $= -4x + 4x^2$
$\quad -1 + 3x - 2x^2 \quad = 1.$
$\quad + x - 2x^2$

$$I' = \frac{I}{2x(1-x)(1-2x)} = \frac{\cancel{2}(\cancel{1-2x})}{\sqrt{x(1-x)}}\cdot\frac{1}{\cancel{2}x(1-x)(\cancel{1-2x})}$$

$$= \frac{1}{\sqrt{x^3(1-x)^3}}$$

$$w(z) = \frac{2(1-2z)}{\sqrt{z(1-z)}}$$

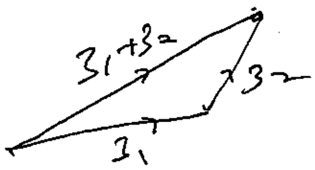
If z is real between 0 and 1

w is real and falls from $+\infty$ to $-\infty$.

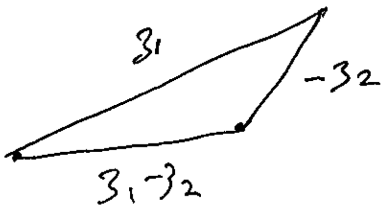
z real, > 1. z is pure imaginary.

z real < 0 z is pure imaginary

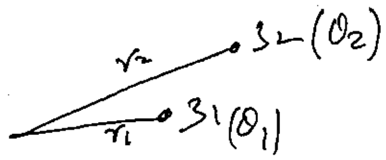
$|z|$ is length of OP, P being point that represents z.



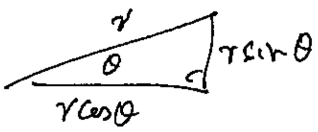
$|z_1+z_2| \le |z_1| + |z_2|$



$$\begin{matrix} |z_1| - |z_2| \\ |z_2| - |z_1| \end{matrix} \le |z_1 - z_2| \le |z_1| + |z_2|$$



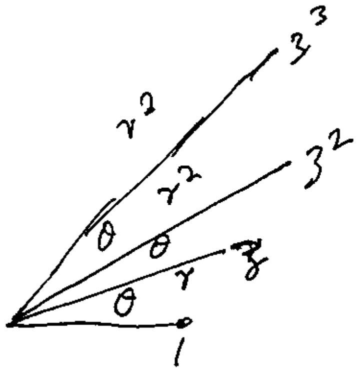
$z_1 z_2 \quad r_1 r_2 : \theta_1 + \theta_2$



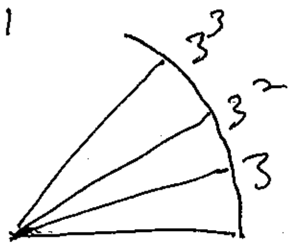
$r(\cos\theta + i\sin\theta)$ is number at r, θ.

$$r_1(\cos\alpha + i\sin\alpha) \cdot r_2(\cos\beta + i\sin\beta)$$
$$= r_1 r_2\left[(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)\right]$$
$$= r_1 r_2\left[\cos(\alpha+\beta) + i\sin(\alpha+\beta)\right]$$

Powers of a number



Special case $r=1$



Root extraction.

$z \quad r, \theta \qquad z^n \quad r^n, n\theta$

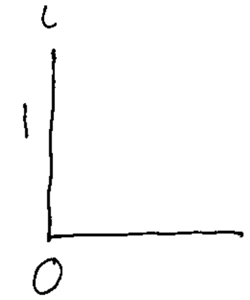
If $z^n = w$ where w is at R, Θ

$r^n = R \qquad n\theta = \Theta$

$r = \sqrt[n]{R} \qquad \theta = \Theta / n$

Particularly simple case, $R = 1$. Then $r = 1$.

Find $\sqrt{i}$.



$\sqrt{i}$ will be at distance 1, angle $\pi/4$ (45^0)

$$\sqrt{i} = \cos 45^0 + i \sin 45^0 = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \frac{1+i}{\sqrt{2}}$$

Check $\left(\frac{1+i}{\sqrt{2}}\right)^2 = \frac{1-1+2i}{2} = i$.

Above we gave one value ~~of~~ for n^{th} root of R, Θ

There should be n roots, as eqn $z^n = w$ is of degree n.

R, Θ is the same point as $R, \Theta + 2s\pi$

So $\theta = \dfrac{\Theta + 2s\pi}{n}$ will do

$= \dfrac{\Theta}{n} + s\left(\dfrac{2\pi}{n}\right)$

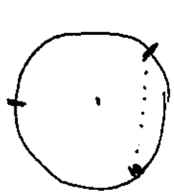
$s = 0, 1, 2 \ldots n-1$ give different positions.

$z^3 = -1$. -1 is at distance 1, angle π

So z is at $1, \dfrac{\pi + 2s\pi}{3}$

$\dfrac{180^0 + 2s.180^0}{3}$

$= 60^0 + s. 120^0$



Application to Differential Equations.

$$\frac{d^2y}{dx^2} + y = 0$$

Stock method. Try $y = e^{mx}$. Then $\frac{d^2y}{dx^2} = m^2 e^{mx}$

so $m^2 + 1 = 0$

$m = i$ or $-i$

Solution e^{ix}, e^{-ix}

$$e^{ix} = \cos x + i\sin x$$
$$e^{-ix} = \cos x - i\sin x$$

General solution

$$y = P(\cos x + i\sin x) + Q(\cos x - i\sin x)$$
$$= (P+Q)\cos x + i(P-Q)\sin x$$
$$= A\cos x + B\sin x.$$

If $w^2 = z$, there are two values of w corresponding to each value of z. Sometimes we can write $w = \pm\sqrt{z}$, when we do not care particularly which one is meant. There are other times when it is important to know which root we are dealing with.

Square roots are particularly simple in polar co-ordinates. If z is at (r, θ), $\sqrt{z}$ is at $(\sqrt{r}, \theta/2)$. This gives 2 solutions, because adding $360°$ to the angle for z makes no difference, but it adds $180°$ for w, which changes by a factor of -1. If we take $r = 1$, the diagrams are simple.

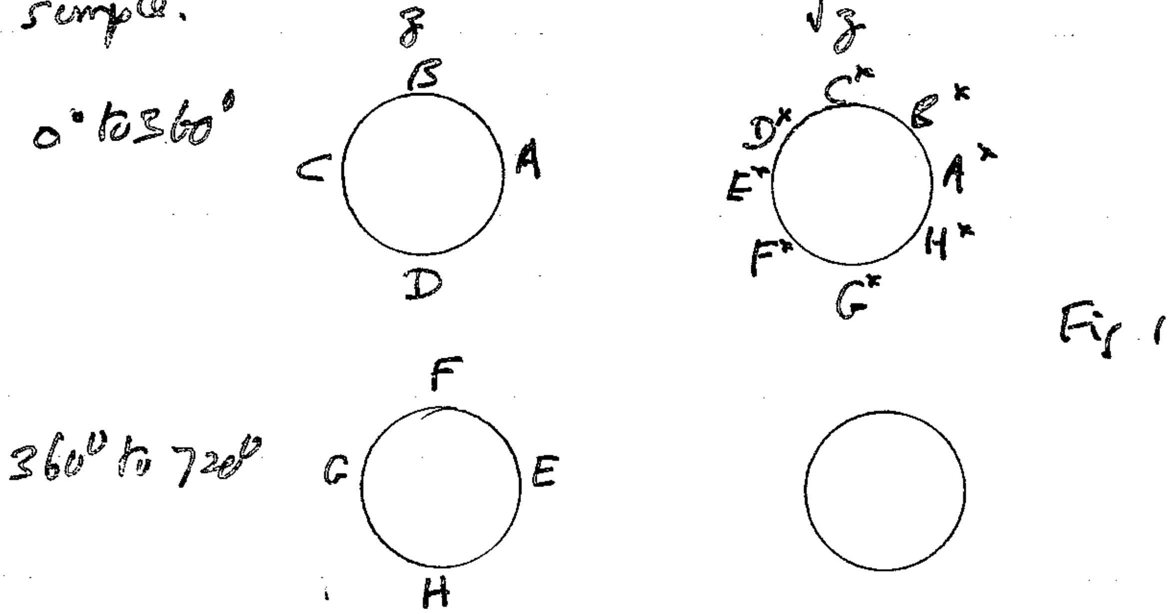


Fig 1

As we go round the circle for z, we start at A for $z = 1$, with $w = \sqrt{z}$ at A*, $\sqrt{z} = 1$. As we continue around the circle, $\sqrt{z}$ always changing continuously, we get to E, $z = 1$, and $w = \sqrt{z}$ is now at E*, -1.

Thus the z values proceed as

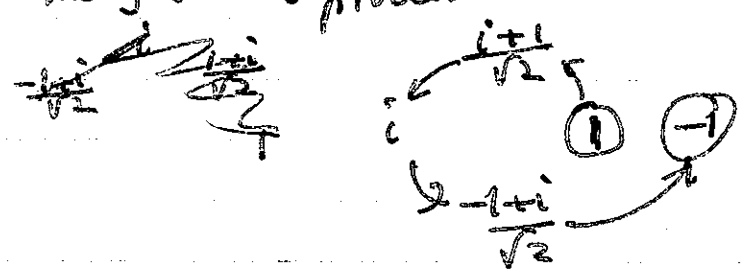


Fig 2

If z goes round the unit circle, with $\sqrt{z}$ changing in a continuous manner (this is an essential point of the theory), the effect of a complete circuit is to change one value of $\sqrt{z}$ to the other.

Riemann introduced the important idea of the Riemann surface, each point of which can be fixed by the values of z and w. Further, a small change in z and w must lead to a point not far away.

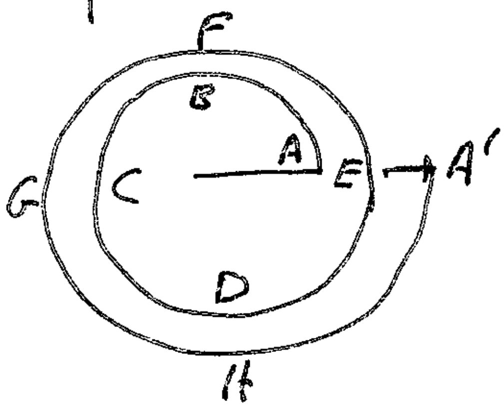


Fig 3

If we arrange points by their z-values, we obtain the curious shape in Figure 3. As a rotation of $720°$ brings us back to $w = +1$, the point A' in Fig. 3 must coincide with A. This seems strange. The point is that $z = x+iy$, $w = u+iv$ involves 4 numbers (x, y, u, v) and so that diagram should be in 4 dimensions, and naturally when we try to show this in 3 dimensions, we run into difficulties. Note that these do not appear if we arrange things corresponding to $\sqrt{z}$ as in the 4th diagram of Figure 1.

$z=-1, w=i$

$z=-i, w=(-1+i)/\sqrt{2}$ $\quad$ $z=i, w=(1+i)/\sqrt{2}$

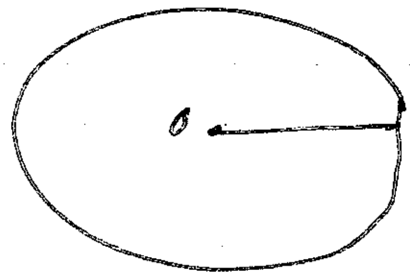
$z=1, w=-1$ $\quad$ $z=1, w=1$

$z=i, w=-(1+i)/\sqrt{2}$ $\quad$ $z=-i, w=\frac{1-i}{\sqrt{2}}$.

$z=-1, w=-i$

Exercise. In a similar way, consider the mappings and Riemann surface for $w = \log z$, $z = e^w$.

If we want to define a function for $w = \sqrt{z}$, it is necessary to have a cut, along the positive real axis for instance, to prevent z making a loop around the origin and so changing one $\sqrt{z}$ to the other.



If $z = 1$ has $\sqrt{z} = 1$ on the upper side of the cut, $\sqrt{z} = -1$ for $z = 1$ on the lower side of the cut.

If $z = (r, \theta)$, $\sqrt{z} = (\sqrt{r}, \theta/2)$ it being understood that θ goes from 0 to 2π so $\theta/2$ goes from 0 to π, i.e. w stays in the upper half plane, for $w = r(\cos\theta/2 + i\sin\theta/2)$, and as $0 < \theta < 2\pi$, $0 < \frac{\theta}{2} < \pi$, so $\sin\frac{\theta}{2} > 0$ and thus $\Im(w) > 0$ ~~in the~~ for z in the cut plane.

If we consider z moving along a path, and this path crosses the cut then $\sqrt{z}$ must be regarded as having the value that is not given by the description above. This consideration is important in the work on $w = \sin^{-1} z$, defined as $\int_0^z \frac{dt}{\sqrt{1-t^2}}$.

In these symbols the Maxwell eqns are

$$(1)\quad 0 = \cdot + \frac{\partial K_{12}}{\partial x_2} + \frac{\partial K_{13}}{\partial x_3} + \frac{\partial K_{14}}{\partial x_4}$$

$$0 = \frac{\partial K_{21}}{\partial x_1} \quad \cdot \quad \frac{\partial K_{23}}{\partial x_3} + \frac{\partial K_{24}}{\partial x_4}$$

$$0 = \frac{\partial K_{31}}{\partial x_1} \quad \frac{\partial K_{32}}{\partial x_2} \quad \cdot \quad \frac{\partial K_{34}}{\partial x_4}$$

$$0 = \frac{\partial K_{41}}{\partial x_1} + \frac{\partial K_{42}}{\partial x_2} + \frac{\partial K_{43}}{\partial x_3} \quad \cdot$$

$$0 = \frac{\partial K_{ij}}{\partial x^j}$$

This is a tensor equation, that is accordingly valid in any system.

$$K_{ij} = \frac{\partial S_i}{\partial x_j} - \frac{\partial S_j}{\partial x_i}$$

$$\frac{\partial K_{ij}}{\partial x^k} = \frac{\partial^2 S_i}{\partial x^j \partial x^k} - \frac{\partial^2 S_j}{\partial x^k \partial x^i}$$

The second term is obtained from the first by cyclic permutation $(i\,j\,k)$

Hence $\dfrac{\partial K_{ij}}{\partial x^k} + \dfrac{\partial K_{jk}}{\partial x^i} + \dfrac{\partial K_{ki}}{\partial x^j} = 0.$

If two of i, j, k are equal, say $i = j$, LHS becomes

$$0 + \frac{\partial K_{ik}}{\partial x^i} + \frac{\partial K_{ki}}{\partial x^i} = \frac{\partial}{\partial x^i}(K_{ik} + K_{ki}) = \frac{\partial}{\partial x^i}\, 0.$$

We get substantial results only when i, j, k distinct. There are 4 ways of choosing 3 distinct numbers from 1, 2, 3, 4.