A complex variable \( w = f(z) \), \( z = u + iv \), \( f(z) = x + iy \)
is called analytic in a region if at each point of
the region the derived function \( f'(z) \) exists, not
depending on the direction of \( dx + idy \).

For a simple treatment it is helpful to make
certain assumptions about the continuity of the
partial derivatives (these assumptions can be reduced)
We require that \( du + idv = f'(x + iy)(dx + idy) \)
\[ \frac{f'(x + iy)}{dx + idy} = p + iq \]

Then \( du + idv = (p - i q)(dx + idy) \)
\[ = p dx - q dy + i (q dx + pdy) \]
\[ = p dx - q dy, \quad dv = q dx + pdy \]
\[ \frac{du}{dx} = p, \quad \frac{dv}{dx} = q, \quad \frac{dv}{dy} = -\frac{du}{dy} \]

Candy Riemann.

So \( \frac{\partial u}{\partial x} = p, \quad \frac{\partial v}{\partial y} = q \)

These are the equations satisfied by the
potential and stream function for an
incompressible plane \( 2 \) dimension with a
velocity potential. Current in a copper sheet may
be taken as an example of this.

The equations fail at points where current
enters or leaves. These correspond to singularities.

Stereographic projection.
If we suppose a sphere
placed with its centre all
or above the origin,
and points \( P \) of the
complex plane projected
to the corresponding point \( P' \), it is found that we
obtain an incompressible flow on the
sphere. This removes the special situation of \( 2 \).

Circles in the plane project to circles
on the sphere.
The equations for stereographic projection

deep $z = 0$ be taken

as the complex plane.

$x^2 + y^2 + z^2 = 1$ as the

sphere, $(1, 0, 0)$ as $N$.

$a + ib$ is represented by the point $(a, b, 0)$ on the

sphere, and $N$ with $P$. The point $N + bP$

has coordinates $(a \times b^2, 1-1)$. It moves from $N$ to $P$
in a straight line, as $r$ goes from 0 to 1.

It is on the sphere when $a^2 + b^2 + (1-r)^2 = 1$.

i.e. $t^2 (a^2 + b^2 - 1) - 2t = 0$, so $t = \frac{a^2 + b^2 - 1}{a^2 + b^2 + 1}$

The points then $\frac{2a}{a^2 + b^2 + 1}, \frac{2b}{a^2 + b^2 + 1}, \frac{a^2 + b^2 - 1}{a^2 + b^2 + 1}$

Inverse points on seen on the sphere.

Thus $\frac{a + i b}{a^2 + b^2} = a + i b$.

Then $\alpha = \frac{a}{a^2 + b^2}$, $\beta = -\frac{b}{a^2 + b^2}$.

The point corresponding to $a + i b$ on the sphere is

$\frac{2\alpha}{a^2 + b^2 + 1}, \frac{a + \beta}{a^2 + b^2 + 1}$

Thus $\frac{2\alpha}{a^2 + b^2 + 1} = \frac{2a}{a^2 + b^2 + 1}$, same as before.

$\frac{2\beta}{a^2 + b^2 + 1} = \frac{-2b/(a^2 + b^2)}{a^2 + b^2 + 1}$, minus previous

value.

$\frac{\alpha + \beta}{a^2 + b^2 + 1} = \frac{1}{a^2 + b^2 + 1} - \frac{1 - a^2 - b^2}{a^2 + b^2 + 1}$, minus previous value.
Thus, $y$ at $b$ maps to $(x, y, 3)$ on sphere $C_1$.
$x$ at $b$ maps to $(-x, -y, -3)$.

There are connected by a rotation of $180^\circ$ about $Ox$. Circle of convergence.

If $f(z)$ is analytic in a region containing the origin, and if $|z|$ is the nearest singularity, or is a pole, it is possible to expand $f(z)$ in an absolutely and uniformly convergent series $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

Provided $|z| < k$, in which a circle, centre 0, with nearest singularity on circumference.

For example $f(z) = \frac{1}{1-z}$ has only one singularity $z=1$.

$$f(z) = \frac{1}{1-z} = \frac{-1}{z-1} = -\frac{1}{1-z} \quad \text{(I)}$$

Convex for $|z| < 1$, except for $|z| > 1$.

If we want a series valid for $|z| > 1$, we put

$$f(z) = \frac{1}{1-z} = \frac{-1}{z-1} = \frac{-1}{1-z} \quad \text{(II)}$$

If $z = \frac{1}{2}$, $f(z) = -2 - 2z - 2z^2 - \ldots$.

If $z = \frac{1}{3}$, $f(z) = -3 - 3z - 3z^2 - \ldots$.

$\rho$ and corresponds to $z = \infty$. We call the above series the series about $z = \infty$.

If unmer $|z| < 1$.

$S$ is changed to $P$ by a rotation of $180^\circ$ about $Ox$.

Thus the upper hemisphere counts as the circle $|z| < 1$.

If corresponds to all $S$ in the hemisphere.

The point outside $|z| = 1$

This we may think of the points outside a circle as being inside a circle, centre $Ox$. 
An important property of integrals is that

\[ \oint_C f(z) \, dz = 0 \]

for any closed path lying entirely in the region.

\[ 0 = \oint_C f(z) \, dz = \oint_{ABCD} f(z) \, dz + \oint_{DCA} f(z) \, dz \]

\[ = \int_{AB} f(z) \, dz + \int_{BC} f(z) \, dz + \int_{CD} f(z) \, dz + \int_{DA} f(z) \, dz \]

The value of the integral is not altered if the path is deformed, the end points remaining the same. It is assumed that the path does not pass any singularities.

If \( f(z) \) is analytic in the region between the two "cutted curves," then

\[ \oint_C f(z) \, dz = \oint_{A_1 B_1} f(z) \, dz + \oint_{B_1 C_1} f(z) \, dz + \oint_{C_1 D_1} f(z) \, dz + \oint_{D_1 A_1} f(z) \, dz \]

If \( f(z) \) is analytic near \( z = 0 \), then for a circle around origin,

\[ \oint_C f(z) \, dz \sim \oint_C \frac{k \, dz}{z} \]

\[ = \int_0^{2\pi} \frac{k \, e^{i\theta}}{e^{i\theta}} \, d\theta = k \cdot 2\pi i \]
Moment function

\[ f(t) = \int_a^b w(k) \, dk \quad \text{where} \quad w(k) \text{ is real, } > 0 \]

is called a moment function.

\[
f(t) = \int_a^b w(k) \, dk = \int_a^b w(k) \, dk \left( 1 + \frac{k^2}{3} + \frac{k^4}{3^2} + \cdots \right)
= \int_a^b w(t) \, dt \left( \frac{1}{3} + \frac{k^2}{3^2} + \frac{k^4}{3^2} + \cdots \right)
\]

Coefficient of \( \frac{1}{3} \) in \( \int_a^b w(t) \, dt \), the central mass.

Coefficient of \( \frac{1}{3} \) in \( \int_a^b w(t) \, dt \), the moment about origin.

Coefficient of \( \frac{1}{3} \) in \( \int_a^b f'(w(t)) \, dt \), the moment about origin.

Coefficient of \( \frac{1}{3} \) in \( \int_a^b f''(w(t)) \, dt \), the moment about origin.

Analytic nature of \( f(t) \)

1. If \( t \) is any point on the interval [a, b], \( f(t) \) is analytic at t.

Proof \( f(t) = \frac{d}{dt} \int_a^b w(k) \, dk \)

\[
= \int_a^b w(k) \, dk \cdot \frac{d}{dt} \left( \frac{k}{3} \right)
= \int_a^b w(k) \, dk \cdot \frac{k}{3}
\]

At \( t = 0 \) or on \( \int_a^b, \text{ } 1^2 - 1 \), the distance of \( t \) from point \( t \) has a minimum value of \( \frac{1}{2} \), so \( \frac{1}{2} \) will be finite.

For \( 0 \leq t \leq 1 \), and the integral will be satisfactory.

There will be a discontinuity across the curve \( (a, b) \), if \( t \) is real, \( 0 < t < 1 \), we shall get different values for \( f(t) \) depending on whether we approach \( t \) from below or above.
If $z$ is above the real axis, displacing the path to $P_1$ will not cause the integrand $w(t)$ to pass over any singularity, since $\beta - \gamma \neq 0$ in this region.

Thus \( f^+(\beta) = \int P_1 \frac{w(t)dt}{\beta - t} \), where $f^+(\beta)$ is the value of $f(\beta)$ found by continuation from above.

Similarly \( f^-(\beta) = \int P_2 \frac{w(t)dt}{\beta - t} \).

Now the difference of these is undefined around

\[
 f^+(\beta) - f^-(\beta) = \int C \frac{-w(t)At}{A - t} dt = 2\pi i w(\beta)
\]

the radius of the circle having $\beta$ interior.

Note: The above proof uses $w(t)$ being analytic. The theorem holds true in many cases where this is not so, for instance if $w(t)$ has poles, such as

This theorem is used if we know $f(\beta)$ and want to find $w(t)$ to five an
In the course of my work I came across the function \( \varphi(a) = \int_0^1 \frac{-\ln v}{a + v} \, dv \)

\[ -\ln v > 0 \text{ for } 0 < v < 1. \]

Suppose \( a > 0 \).

\[ \frac{1}{a} \int_0^1 \frac{-\ln v \, dv}{a + v} < \frac{1}{a} \int_0^1 -\ln v \, dv \]

\[ = -\frac{1}{a} \left[ -v \ln v + v \right]_0^1 \]

\[ = \frac{1}{a} \ln v \rightarrow 0 \text{ as } v \rightarrow 0, \text{ so that } \frac{1}{a} \ln v \rightarrow 0. \]

Thus \( \varphi(a) \) is finite when \( a > 0 \).

If \( a = 0 \)

\[ \int_0^1 \frac{-\ln v \, dv}{v} = -\frac{1}{2} (\ln v)^2 \bigg|_0^1 = +\infty. \]

\( \varphi(a) \rightarrow +\infty \) as \( a \rightarrow 0 \).

If \( a \) increases, clearly \( \frac{1}{a + v} \) decreases.

Hence \( \varphi(a) \) decreases steadily as \( a \) goes from 0 to \( +\infty \).

\[ \frac{1}{a + v} = \frac{1}{a(1 + \frac{v}{a})} = \frac{1}{a} \left( 1 - \frac{v}{a} + \frac{v^2}{a^2} - \frac{v^3}{a^3} - \cdots \right) \]

\[ \varphi(a) = \int_0^1 -\ln v \, \frac{(-1)^n v^n}{a^{n+1}} \, dv \]

\[ \int_0^1 -v^n \ln v \, dv = \left[ -\frac{v^{n+1} \ln v}{n+1} \right]_0^1 + \int_0^1 \frac{v^{n+1}}{n+1} \, dv \]

\[ \int_0^1 \frac{v^{n+1}}{n+1} \, dv = \int_0^1 \frac{v^n \, dv}{n+1} \]

\[ = \left. \frac{v^{n+1}}{(n+1)^2} \right|_0^1 = \frac{1}{a^{n+1}} \]

Thus \( \varphi(a) = \sum_{n=0}^{\infty} \frac{(-1)^n v^n}{a^{n+1}} \).

This sense converges for \(|a| > 1\),

diverges for \(|a| < 1\).
It diverges everywhere inside the circle |a| = 1, and converges everywhere outside it. This indicates that there must be a singularity somewhere on |a| = 1.

For |a| < 1 is the singularity nearest to the origin, the series about 0 converge for |z| > |a|.

If |a| were less than 1, the region of convergence of \( \varphi(z) \) would have some part inside |a| = 1.

The graph of \( \varphi(z) \) for \( a > 0 \) is

There is no indication of a singularity at \( a = -1 \).

The singularity must be elsewhere.

Series valid for \( 0 < a < 1 \).

\[
\varphi(z) = \int_0^1 \frac{-\ln v \, dv}{a + v} = \ln (1 + a) - \ln a \frac{\ln}{1 + a}
\]

\[
\int_0^a \frac{-\ln u \, du}{1 + u} = \left[ \ln (1 + u) \right]_0^a = \ln (1 + a) - \ln 1 = \ln (1 + a)
\]

Thus part of \( \varphi(z) \) is

\[
-\ln a \ln (1 + a) (a)
\]

\[
\int_0^a \frac{-\ln u \, du}{1 + u} = \int_0^1 \frac{-\ln u \, du}{1 + u} + \int_1^a \frac{-\ln u \, du}{1 + u}
\]

\[
\int_0^1 \frac{-\ln u \, du}{1 + u} = \int_0^1 \frac{(-1)^n}{n+1} \, du = \sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^2} (b)
\]
\[
\ln \left( \frac{\sqrt{a}}{1 + \sqrt{a}} \right) = \ln \frac{a}{1 + a}
\]

\[
\int \frac{a}{1 + a} \left( -\frac{1}{w^2} \right) \, dw = \int a \frac{dw}{w(1+w)}
\]

\[
= \int a \frac{1}{w} \left( 1 - \frac{1}{1+w} \right) \, dw
\]

\[
= \int a \frac{1}{w} \, dw - \frac{1}{2} \left( \ln a \right)^2
\]

\[
\int a \frac{1}{w} \, dw = \ln a
\]

\[
\int -\frac{1}{2} \left( \ln a \right)^2 = -\frac{1}{2} \left( \ln a \right)^2
\]

Finally, \[
\int \frac{a}{1 + a} \, dw = \int -\frac{1}{2} \ln a \, dw
\]

\[
= \ln a \left( \ln a \right) - \frac{1}{2} \left( \ln a \right)^2
\]

\[
= \ln a \left( \ln a \right) - \frac{1}{2} \left( \ln a \right)^2
\]

\[
= \ln a \left( \ln a \right) + \frac{\xi}{0} \left( \frac{a}{1-a} \right) + \frac{\xi}{0} \left( \frac{a}{(n+1)^2} \right)
\]

\[
= \ln a \left( \ln a \right) + \frac{\xi}{0} \left( \frac{a}{1-a} \right) + \frac{\xi}{0} \left( \frac{a}{(n+1)^2} \right)
\]

\[
= \ln a \left( \ln a \right) + \frac{\xi}{0} \left( \frac{a}{1-a} \right) + \frac{\xi}{0} \left( \frac{a}{(n+1)^2} \right)
\]

In (a) we have \(-\ln a \left( \ln a \right) + \left( \ln a \right)^2\).

After cancelling, we find

\[
\varphi(a) = \frac{1}{2} \left( \ln a \right)^2 + \frac{\xi}{0} \left( \frac{a}{(n+1)^2} \right)
\]

\[
= \frac{1}{2} \left( \ln a \right)^2 + \frac{\xi}{0} \left( \frac{a}{(n+1)^2} \right)
\]
For $|a| > 1$, $q(a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} a^{n+1}$.

How does this series behave if we take $a$ on the unit circle?

There is a theorem that if $\sum c_n$ is absolutely convergent (i.e., $\sum |c_n| < \infty$) then $\sum c_n$ is convergent.

The absolute series corresponding to $q(a)$ is
\[
\sum_{n=0}^{\infty} \frac{1}{(n+1)^2 |a|^{n+1}}.
\]

If $a$ is on the unit circle, $|a| = 1$, and we have $\sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$, which is convergent. It is in fact $\pi^2/6$. So $q(a)$ is convergent everywhere on $|a| = 1$. How then does it manage to have a singularity on $|a| = 1$?

The clue was given by the graph for $a < 0$.

The graph of $q(a)$ can be shown as follows:

and the tangent at $a = -1$ is verified.

In fact $q(a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} a^{n+1}$

\[
q'(a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} \left[ -(n+1) a^{n} \right]
\]

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)} a^{n-1}
\]

\[
\log(1+\frac{1}{a}) = \frac{1}{a} - \frac{1}{2a^2} + \frac{1}{3a^3} - \frac{1}{4a^4} \ldots
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)} a^{n+1}
\]

\[
\therefore q'(a) = -\frac{1}{a} \log(1+\frac{1}{a}) \text{ which is } -\infty \text{ for } a = -1.
\]
We are concerned with functions $f : \mathbb{C} \to \mathbb{C}$, for which $f'(z)$ exists, i.e. for $w = f(z)$, \[
\frac{dw}{dz} = f'(z)\]

Consider a particular point $z$, and variations $dz$ from it. Then $\frac{dw}{dz} = f'(z)\,dz$. Here $f'(z)$ is some particular complex number. Multiplication by it produces a rotation and change of scale, i.e. it leaves angles and ratios unchanged. Such a transformation is called conformal. No $\frac{dw}{dz} = f'(z)\,dz$ holds only for differentiable. This means that the geometry is preserved only in the limit as we consider neighborhoods of $z$ and $w$ close $z \to w$. This in particular means that the angle between the tangents to two curves, at a point where they cross, is not altered.

Note that multiplication produces a turning; it cannot produce a turning over. Thus, if $z = x + iy$, $w = x - iy$ the function $z \to w$ cannot be analytic.

Always \[
\begin{array}{c}
\text{direction of} \\
\text{rotation preserved}
\end{array}
\]

Transformation $w = \frac{az + b}{cz + d}$ maps conformally 1-1 the entire plane $z$ to the entire plane $w$. It preserves the essential behaviour of functions at each point, and can give a very useful way of putting some structure in an easily understood form. We suppose, of course, $ad - bc \neq 0$. 
Such a transformation can be the combined effect of simpler transformations. For example

$$\frac{12x + 2y}{3x + 6} = 4 + \frac{1}{3x + 6} = 4 + \frac{1}{3(3+2)}$$

Thus, $3^x = \frac{12x + 2y}{3x + 6}$ can be achieved by the following sequence of steps.

1. $z_1 = 8 + 2$ 
2. $z_2 = 3z_1$ 
3. $z_3 = \frac{1}{3z_2}$ 
4. $3^x = z_3 + 4$

Such a decomposition is always possible.  

1 and 4 are simply translations.  

2 is a change of scale.  

3 is a new kind of transformation.

If $3^x = \frac{1}{3}$ and $3$ is $(1,0)$ then $3^x$ is $\left(\frac{1}{3}, -\frac{1}{3}\right)$

If $z = x + iy$, $\overline{z} = x - iy$

Reflection in the real axis.  

If $z$ is at distance $r$ from the origin, $z^*$ or $\overline{z}$ is at distance $\frac{1}{r}$.

$\overline{z} \rightarrow z^*$ is known as inversion in the unit circle.
Given any circle, radius $R$, centre $O$, inversion is
it sends $P$ to $Q$,
where $OPQ$ is straight
and $OP \cdot OQ = R^2$.

This operation has much in common with reflection. If $P \rightarrow Q$, then $Q \rightarrow P$. If the radius $R$ become very large, it could be mistaken for reflection.

If reverses sense of rotation $O \rightarrow Q$,
$g^2 = 1 \frac{1}{g}$ is analytic: it is equivalent to
2 operations, each of which reverses
sense of rotation. It could not be equivalent
To a single such operation.

Theorem. If we have a
circle $C$, and if $OT$
is a tangent to it,
inversion of $C$ in the
circle, centre $O$,
radius $OT$, makes
$C \rightarrow C$.

Then $OP \cdot OQ = OT^2$ is a known theorem.

Changing the radius of the circle of inversion only produces a change of scale, so $C$
inscribed in any circle, centre $O$, goes
to a circle.
Problem: \( l \) is a line at perpendicular distance \( R \) from a point \( O \).

What do we get if we
insert \( l \) in the circle
centre \( O \), radius \( R \)?

Suggestion: find \( OQ \)
when \( \angle POQ \) is \( \theta \).
\[ w = \frac{a_2 + b}{c_2 + d} \quad \text{where } ad - bc \neq 0. \]

Just one value of \( w \) corresponds to a given value of \( b \).
\[
cw^2 + wd = a_2 + b \quad \Rightarrow \quad b = \frac{wd - b}{cw + a}.
\]

This gives a single value for \( b \) when \( w \) is known.

Question: What goes wrong with this argument if \( ad - bc = 0 \)?

"Kreisverwandtschaft."

**Inversion**

Reflection in a line is

\[ \frac{P}{\omega Q}, \omega \rightarrow \omega \]

Inversion may be thought of as reflection in a circle.

\[ \text{Important property of a circle} \]

\[ x \cdot y = u \cdot v \quad \text{(length)} \]

\[ OA \cdot OB = OC \cdot OD \]

This still holds if \( O \) is outside the circle.

For the point \( O \rightarrow T \),

\[ A \rightarrow T, \quad B \rightarrow T. \]

So \[ OA \cdot OB = OC \cdot OD = R^2. \]

If we invert with respect to the circle, centre \( O \),

radii \( OT \rightarrow A \rightarrow B, \quad B \rightarrow H \)

So the circle \( TBCD \) reflects to itself.

If we invert with respect to a circle, centred,

at a different radius, we have similar effect with a change of scale: the circle goes
to a different circle.

There is an exception. If \( O \) lies on the circle \( BCD \).
1) \( f(z) = f(x+iy) \) is analytic in a region if there is a continuous derivative \( f'(z) \), independent of direction at which \( z \) is changed.

\[
\frac{df}{dz} = f'(z) \frac{dz}{d\theta}.
\]

Let \( f(x+iy) = u+iv \). Then \( f'(z) = p+iq \)

\[
u = \frac{du}{dx} = (p+iq)(dx+idy) = p \, dx - q \, dy + i(p \, dx + q \, dy),
\]

\[
\frac{du}{dx} = p, \quad \frac{du}{dy} = -q.
\]

Thus \( \frac{du}{dx} = \frac{3}{2}, \quad \frac{du}{dy} = -3 \). Cauhy-Riemann.

These derivatives must exist and be continuous.

2) Geometrical meaning of Cauchy-Riemann equations.

Suppose a point \( (x,y) \) starts at \( (x_0, y_0) \) and moves a distance 5 at angle \( \theta \) to the Ox.

Then \( x = x_0 + 5 \cos \theta \), \( y = y_0 + 5 \sin \theta \).

Rate of change of \( u(x,y) \) as \( x, y \) change is

\[
\frac{du}{ds} = \frac{dx}{ds} \frac{du}{dx} + \frac{dy}{ds} \frac{du}{dy} = 5 \cos \theta \cdot \frac{p}{2} + 5 \sin \theta \cdot \frac{-q}{3}.
\]

Suppose \( v \) changes as a result of \( x, y \), moving in direction \( \theta + \frac{\pi}{2} \). \( \theta + \frac{\pi}{2} \). \( \frac{dv}{d\theta} = -5 \sin \theta \cdot \frac{p}{2} + 5 \cos \theta \cdot \frac{q}{3} \)

\[
\frac{dv}{d\theta} = \frac{dv}{ds} \frac{ds}{d\theta} = \frac{dv}{ds} \cos \theta + \frac{dv}{ds} \sin \theta \]

Thus rate of change of \( u \) in direction \( \theta \)

\[
= \text{rate of change of } v \text{ in direction } \theta + \frac{\pi}{2}.
\]

In particular, \( u \) is constant for \( \theta \), \( v \) is constant for \( \theta + \frac{\pi}{2} \). The curves \( u = c \), \( v = k \)

cross at right angles.
Integration in 2 dimensions.

To define \( \int [x(x,y) \, dx + y(x,y) \, dy] \) along a given curve, we suppose the curve given by
\[ x = x(t), \quad y = y(t), \quad 0 \leq t \leq 1. \]
Then \( x(x,y) = x(x(t), y(t)) \) a function of \( t \).
\[
\frac{dx}{dt} = \frac{dx}{dt} = x'(t) \, dt.
\]
We define the integral as
\[
\int_a^b x'(t) \, dt.
\]
Suppose for example we require \( \int [y \, dx + 2x \, dy] \) along the line joining \((1,1)\) to \((3,4)\).
\[
x = 1 + 2t, \quad y = 1 + 3t \quad \text{takes } x, y \text{ from } (1,1) \text{ to } (3,4) \text{ a } t \text{ goes from } 0 \text{ to } 1.
\]
\[
\int_0^1 (1 + 3t) \, dt = \frac{1 + 9t^2}{2} \bigg|_0^1 = 17.
\]
We would get a different result if we went from \((1,1)\) to \((3,1)\) and then \((3,1)\) to \((3,4)\)
in first part \( x = 1 + 2t, \quad y = 1, \quad x' = 2, \quad y' = 0 \)
We have \( \int_0^1 (2 + 4t) \, dt = 2t + 2t^2 \bigg|_0^1 = 4 \).
In the second part \( x = 3, \quad y = 1 + 3t, \quad x' = 0, \quad y' = 3 \)
\[
\int_0^1 6.3 \, dt = 18t \bigg|_0^1 = 18.
\]
\[
\int y \, dx + y \, dy = \text{an expression for work along the path. This example corresponds to a non-conservative system where work could be obtained from it by following an appropriate loop.}
\]
The result will be independent of the path if
\[ X \frac{dy}{dx} + Y \frac{dx}{dy} = \frac{df}{dt} f(x, y) \text{ for some } f(x, y) \text{ for } \int \frac{df}{dt} f(x, y) dy = f(x, y) - f(x_0, y_0). \]

Now
\[ \frac{df}{dt} = \frac{df}{dx} \frac{dx}{dt} + \frac{df}{dy} \frac{dy}{dt}. \]

This case arises if there exists \( f(x, y) = \)
\[ X = \frac{\partial f}{\partial x}, \quad Y = \frac{\partial f}{\partial y}. \]

We suppose \( X \) and \( Y \) have continuous derivatives.

Then
\[ \frac{\partial X}{\partial y} \frac{\partial f}{\partial x} + \frac{\partial Y}{\partial x} \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}. \]

\( \frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x} \) is the usual condition for \( Xdx + Ydy \)

To be an exact differential,
\[ \left[ \text{Picard, Appendix A.} \right] \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \]
\[ \therefore f \text{ is continuous, and has continuous } \]
\[ \text{first and second derivatives.} \]

If
\[ \frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x} \frac{\partial f}{\partial x}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \]

we can show that there is an \( f \) with the required property. It \( Xdx + Ydy \)

is then independent of the path between given end points, provided \( \frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x} \) holds throughout its

region. Stokes' Theorem gives
\[ \oint Xdx + Ydy = \int \left( \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) dxdy. \]
If \( f(x + iy) \) satisfies the Cauchy-Riemann conditions, \( \int f(x) \, dx \) does not depend on the path.

\[ \int \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} f(x + iy) \right) \, dx = 0. \]

Cauchy's Theorem.

If \( f(z) \) is analytic in some region \( R \),
for every smooth closed curve \( C \) in that region, and for any curve \( C' \) consisting of a finite number of smooth curves,

\[ \int_C f(z) \, dz = 0. \]

Corollary.

If \( C_1 \) and \( C_2 \) are two closed curves, \( C_1 \) inside \( C_2 \), and \( R \) is a region in which \( f(z) \) is analytic, containing \( C_1 \) and \( C_2 \),
then \( \oint_{C_1} f(z) \, dz = \oint_{C_2} f(z) \, dz \).

Proof.

Consider region bounded by \( C_2, C_1, S, C_2 \), and \( S \).
This is bounded by \( C_1, S, C_2, S \). The two parts
on \( S \) cancel and we have \( \oint_{C_1} f(z) \, dz = 0 \).

Residue Theorem.

If \( f(z) \) is analytic on a region
containing curve \( C \), at point \( a \) inside \( C \),
then
\[ \int_C \frac{f(z)}{z-a} \, dz = 2\pi i \text{ Res}(f, a). \]

This equals \( 2\pi i \text{ Res}(f, a) \).

On the circle \( f(z) \neq f(a) \). In the limit we have
\[ \int_{C} \frac{f(z)}{z-a} \, dz \rightarrow \int_{C} \frac{f(a)}{z-a} \, dz = 2\pi i \text{ Res}(f, a). \]
On the small circle \( \gamma = a + re^{i\theta} \) where \( r \leq \text{radius of circle}, \ 0 \leq \theta \leq 2\pi \).

\[
\oint_{\gamma} \frac{dz}{z-a} = \int_0^{2\pi} \frac{r e^{i\theta} \, i \, d\theta}{r e^{i\theta}} = \int_0^{2\pi} i \, d\theta = 2\pi i
\]

\[
2\pi i \, f(a) = \oint_{\gamma} \frac{f(z) \, dz}{z-a}
\]

\[
f(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z) \, dz}{z-a}
\]

Given \( f(z) \) on the boundary, this gives an explicit solution for values of \( f(z) \) inside the curve.
\[ I = \int \frac{x^{3/2}}{(1-x^{3/2})} \, dx \quad \text{if } y = b_x \quad t^2 = x - x^2 \]

\[ I = \left( \int \frac{t^2}{(1+t^3)^2} \right) \frac{-2t}{t^3} \, dt \]

\[ = \int \frac{1 + t^2}{t^2} \, dt \]

\[ = -2 \left[ t - \frac{1}{t} \right] \]

\[ = 2 \left[ \sqrt{\frac{1-x}{x}} - \sqrt{\frac{x}{1-x}} \right] \]

\[ = \frac{2}{\sqrt{x(1-x)}} \left[ (1-x) - x \right] \]

\[ = \frac{2(1-2x)}{\sqrt{x(1-x)}} \]

\[ \frac{d}{dx} (2-2x) x^{-1/2} (1-x) \]

\[ = \ln \frac{2}{1-x} \quad \text{for } I \]

\[ I' = \frac{2}{1-x} - 2 \frac{1}{x(1-x)} \]

\[ = \frac{2x(1-x)}{x(1-x)(1-2x)} \]

\[ \text{Max} = -4x + 4x^2 \]

\[ -1 + 3x - 2x^2 = 0 \]

\[ I' = \frac{I}{2x(1-x)(1-2x)} = \frac{2x(1-2x)}{2x(1-x)^2} \]

\[ \boxed{I = \frac{1}{\sqrt{x^3(1-x)^3}}} \]
\[ z_0(s) = \frac{2(1 - 2s)}{\sqrt{3} \cdot (1 - 3s^2)} \]

If \( s \) is real between 0 and 1

- \( w \) is real and goes from \( +\infty \) to \( -\infty \)
- \( s \) real, \( s > 1 \), \( s \) is pure imaginary
- \( s \) real, \( s < 0 \), \( s \) is pure imaginary
\( |z| \) is length of \( \theta \), \( P \) being point that represents \( z \).

\[
|z_1 - z_2| \leq |z_1| + |z_2|
\]

\[
|z_1 - z_2| \leq |z_1| + |z_2|
\]

\[
|z_1| - |z_2| \leq |z_1 - z_2| = |z_1| + |z_2|
\]

\[
\eta = 3\angle(\theta)
\]

\[
\eta \in 3\angle(\theta)
\]

\[
\eta (\cos \theta + i \sin \theta) = \text{number of } \eta \in \theta.
\]

\[
\eta_1 (\cos \alpha + i \sin \alpha), \eta_2 (\cos \beta + i \sin \beta)
\]

\[
= \eta_1 [\cos (\alpha + \beta) + i \sin (\alpha + \beta)]
\]

\[
= \eta_1 [\cos (\alpha + \beta) + i \sin (\alpha + \beta)]
\]

\[
\text{Power of a number}
\]

\[
\eta^3
\]

\[
\text{Special case } \eta = 1
\]
Root extraction.

3 \, r \, \theta = 3^n \, r^n \, \theta^n

If \, 3^n = w \, where \, w \, is \, at \, R, \, \Theta

r^n = R \quad \theta^n = \Theta

r = \sqrt[3]{R} \quad \theta = \Theta/n

Particularly simple case, R = 1. Then r = 1.

Find \sqrt[i]{i}.

\[ \sqrt[i]{i} = \cos \frac{45^\circ}{i} + i \sin \frac{45^\circ}{i} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \frac{1+i}{\sqrt{2}} \]

Check \left( \frac{1+i}{\sqrt{2}} \right)^2 = \frac{1-1+i}{2} = i.

Above we gave one value of \, f \, for \, n^{th} \, root \, of \, R, \, \Theta

There should be \, n \, roots, \, \therefore \, eqn \, 3^n = w \, is \, of \, degree \, n.

R, \Theta \, in \, the \, same \, power \, as \, R, \Theta + 2\pi

so \, \theta = \frac{\Theta + 2\pi}{n} \, will \, do

\[ \theta = \Theta + \frac{2\pi}{n} \]

s = 0, 1, 2... \, \therefore -1 \, give \, different \, positions.

3^3 = -1. \, -1 \, is \, at \, distance \, 1, \, angle \, \pi

so \, 3 \, is \, at \, \frac{\pi}{3} + \frac{2\pi}{3}

\[ 180^\circ + 2 \times 120^\circ \]

\[ = 60^\circ + 5 \times 120^\circ \]

\[ 180^\circ + 2 \times 120^\circ \]

\[ = 60^\circ + 5 \times 120^\circ \]
Application to Differential Equation.

\[ \frac{d^2 y}{dx^2} + y = 0 \]

Stock method. Try \( y = e^{mx} \). Then \( \frac{dy}{dx} = me^{mx} \).

So \( m^2 + 1 = 0 \).

\( m = i \) or \(-i\).

Solve: \( e^{ix}, e^{-ix} \).

\[ e^{ix} = \cos x + i \sin x \]

\[ e^{-ix} = \cos x - i \sin x \]

General solution

\[ y = P(\cos x + i \sin x) + Q(\cos x - i \sin x) \]

\[ = (P + Q) \cos x + i(P - Q) \sin x \]

\[ = A \cos x + B \sin x. \]
If \( w^2 = 3 \), there are two values of \( w \) corresponding to each value of \( z \). Sometimes we can write \( w = \pm \sqrt{3} \), where we do not care particularly which one is meant. There are other times when it is important to know which root we are dealing with.

Square roots are particularly simple in polar coordinates. If \( z = a + bi \) \((r, \theta)\), \( \sqrt{z} \) is at \((\sqrt{r}, \theta/2)\). This gives 2 solutions, because adding 360° to the angle for \( z \) makes no difference, but it adds 180° for \( \sqrt{z} \), which changes by a factor of -1. If we take \( r = 1 \), the diagrams are simple.

\[
\begin{array}{c}
0° \text{ to } 360° \\
C \quad D \\
E \quad A \\
B \\
\end{array}
\]

\[
\begin{array}{c}
260° \text{ to } 720° \\
C \\
E \\
F \\
H \\
\end{array}
\]

As we go round the circle for \( z \), we start at \( A \)
for \( \theta = 0 \), with \( w = \sqrt{3} \) or \( A^* \), \( \sqrt{3} = 1 \). As we continue around the circle, \( \sqrt{3} \) always changing continuously, we get to \( E \), \( \theta = 1 \), and \( w = \sqrt{3} \) in
next step \( E^* \), -1.

Thus the 3 values proceed as

\[
\begin{array}{c}
1 \quad 2 \quad 3 \\
\sqrt{3} \quad -1 \quad i\sqrt{3} \\
\end{array}
\]
If I go round the unit circle, with  \( e^{i\theta} \) changing in a continuous manner (this is an essential point of the theory), the effect of a complete circuit in \( e^{i\theta} \) changes once value of \( \theta \) to its other.

Riemann introduced the important idea of the Riemann surface, each point of which can be fixed by the values of \( z \) and \( w \). Further, a small change in \( z \) and \( w \) must lead to a point not far away.

If we arrange points by their \( z \)-values, we obtain the curious shape in Figure 3. At a revolution of \( \frac{2\pi}{3} \) again, we have \( w = -1 \) and \( z = \sqrt{1 - i} \). The point \( A' \) in Figure 3 must coincide with \( A \). This seems strange. The point \( 1, 0 \) occurs in values of \( z \) and so that diagram should be in 4 dimensions, and naturally when 6 try to show this in 3 dimensions, we run into difficulties. Note that these do not appear if we arrange things corresponding to \( z \) as in the \( n \)-th diagram of Fig. 4.

\[
\begin{align*}
  z &= -1, w = (-1+i)/\sqrt{2} \\
  z &= i, w = (1+i)/\sqrt{2} \\
  z &= i, w = -(1+i)/\sqrt{2} \\
  z &= -i, w = -(-1+i)/\sqrt{2} \\
  z &= -1, w = -i \\
  z &= -i, w = -(-1+i)/\sqrt{2} \\
  z &= i, w = (1+i)/\sqrt{2} \\
  z &= 1, w = -i \\
  z &= i, w = (1+i)/\sqrt{2} \\
  z &= -i, w = -(-1+i)/\sqrt{2} \\
 \end{align*}
\]
Exercise. In a similar way, consider the mappings and Riemann surface for $w = \log z, \; z = e^w$.

If we want to define a function for $w = \sqrt[3]{z}$, it is necessary to have a cut, along the positive real axis for instance, to prevent making a loop around the origin and so changing one $\sqrt[3]{z}$ to the other.

If $\theta = 1$ has $\sqrt[3]{1} = 1$ on the upper side of the cut, $\sqrt[3]{1} = -1$ for $\theta = 1$ on the lower side of the cut.

If $\theta = (\pi, 0), \; \sqrt[3]{z} = (\sqrt[3]{r}, \; \theta/2)$.

It being understood that $\theta$ goes from $0$ to $2\pi$, so $\theta/2$ goes from $0$ to $\pi$, i.e., $w$ stays in the upper half plane, for $w = \sqrt[3]{r} (\cos \theta/2 + i \sin \theta/2)$, and as $0 < \theta < 2\pi, \; 0 < \frac{\theta}{2} < \pi, \; \text{so } \sin \frac{\theta}{2} > 0$ and hence $\theta(w) > 0$ i.e., $w$ in the cut plane.

If we consider $\theta$ moving along a path, and this path crosses the cut, then $\sqrt[3]{z}$ must be regarded as having the value that is not given by the description above. This consideration is important in the work on $w = \sin^{-1} z$, defined as $\int_0^{\sin^{-1} z} \frac{dt}{\sqrt{1-t^2}}$. 
In these symbols the Maxwell equations are

\[ 0 = \frac{\partial K_{ij}}{\partial x^i} + \frac{\partial K_{ik}}{\partial x^j} + \frac{\partial K_{jk}}{\partial x^i} \]

\[ 0 = \frac{\partial K_{ij}}{\partial x^i} + \frac{\partial K_{ik}}{\partial x^j} + \frac{\partial K_{jk}}{\partial x^i} \]

\[ 0 = \frac{\partial K_{ij}}{\partial x^i} + \frac{\partial K_{kj}}{\partial x^j} + \frac{\partial K_{jk}}{\partial x^i} \]

\[ 0 = \frac{\partial K_{ij}}{\partial x^j} + \frac{\partial K_{kj}}{\partial x^i} + \frac{\partial K_{ji}}{\partial x^j} \]

This is a tensor equation, that is accordingly valid in any system.

\[ K_{ij} = \frac{\partial s_i}{\partial x^j} - \frac{\partial s_j}{\partial x^i} \]

\[ \frac{\partial K_{ij}}{\partial x^k} = \frac{\partial^2 s_i}{\partial x^j \partial x^k} - \frac{\partial^2 s_j}{\partial x^i \partial x^k} \]

The second term is obtained from the first by cyclic permutation \((ijk)\)

Hence \[ \frac{\partial K_{ij}}{\partial x^i} + \frac{\partial K_{ik}}{\partial x^j} + \frac{\partial K_{jk}}{\partial x^i} = 0. \]

If two of \(i, j, k\) are equal, say \(i = j\), LHS becomes

\[ 0 + \frac{\partial K_{ik}}{\partial x^i} + \frac{\partial K_{ik}}{\partial x^i} = \frac{2}{\partial x^i} (K_{ik} + K_{ik}) = \frac{2}{\partial x^i} (2) = 0. \]

We get substantial results only when \(i, j, k\) distinct.

There are 6 ways of choosing 2 distinct letter numbers from 1, 2, 3, 4.