An Introduction to the Theory of Functions.

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Introduction.

I should like it to be clear that I am not speaking on this subject as a specialist. I became interested in this theory as an aid to mathematical physics. Often one has the feeling of groping in the dark: one obtains particular results, perhaps by calculation, but a feeling remains that one has failed to grasp the significance of those results; something suggests that there is a quicker, a more powerful, a more illuminating way of dealing with the problem. I had this feeling in connection with differential equations. This feeling disappeared after I had read certain books on the theory of functions. Many people probably know that this theory exists: few realise how much one can do by its aid. The fault lies largely with the text-books which present the subject in a dead form.

It is clearly impossible to explain such a vast subject in fortyfive minutes. But I feel that boys who intend to be mathematicians ought to know something of this subject before they leave school for university: in particular they should have an idea of its methods, scope and possibilities. That I shall try to convey in part, this afternoon. If the subject is new to you, it is not important to follow the steps in each particular result. Rather, the object is that you should see the kind of result that is obtained, and the type of argument that is used. Particularly not e the little calculation used: the subject is such that, once grasped, it can be seen as a whole and kept in mind without any strain on the memory.

In many schools, far too much time is spent on particular problems so that pupils fail to appreciate the general structure of mathematics.

Modern mathematics consists essentially in <u>classification.</u> The older mathematics asked - solve this problem. Modern mathematics asks - what type of solution is to be expected, what methods are relevant? for a whole class of problems?

The theory of functions is a typical example of this very general approach.

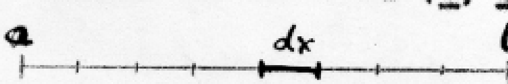
Analytic Functions.

The theory arises so soon as we start to discuss functions of a complex variable,

$$w = f(z), \quad \text{where } \underline{w} = \underline{u} + \underline{iv} \text{ and } \underline{z} = \underline{x} + \underline{iy} \qquad (I)$$

We obviously want to consider functions on which the operations of calculus - differentiation and integration can be carried out. A point immediately arises, which does not occur in the theory of real functions at all.

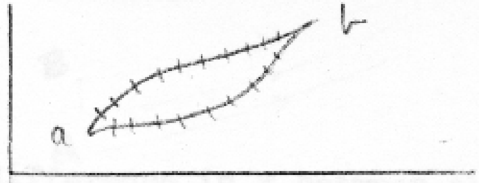
We can explain the integration of a real function $f(x)$ - without any attempt at rigour - as the sum of terms $f(x)dx$. For the definite integral, $\int_a^b f(x)dx$ we divide the interval ($\underline{a}$, $\underline{b}$) into parts dx.



multiply by $f(x)$, sum, and proceed to the limit. There is no ambiguity in regard to the path by which $\underline{x}$ goes from $\underline{a}$ to $\underline{b}$.

In the complex domain it is essentially different. Representing $\underline{z} = \underline{x} + \underline{iy}$ as is usual by the point $(\underline{x},\underline{y})$ on the Argand Diagram, $\underline{z}$ can pass from $\underline{a}$ to $\underline{b}$ by a great variety of different routes, and we may get entirely different values of

$\int_a^b f(z)\,dz$ for these different routes.



We are therefore led to impose a restriction on our function $f(\underline{z})$.

$f(z)$ is said to be <u>analytic in a given region</u> if it is defined there, and $\int_a^b f(z)\,dz$ is independent of the path joining $\underline{a}$ to $\underline{b}$, for paths which lie entirely inside that region.

This condition appears in textbooks as <u>Cauchy's Theorem</u>, but it is very rarely pointed out how fundamental this fact is. If we did not have this property, there could be no theory at all for the integration of complex functions.

II

Analytically, the condition is expressed by the equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \ldots\ldots \text{(II)}$$

*Symbols defined in I

which may be graphically expressed by saying that the rate of change in $\underline{u}$ in any direction $\underline{PQ}$ is the same as that of $\underline{v}$ in the direction $\underline{PR}$, 90° further round.

These conditions, known as the <u>Cauchy Riemann conditions</u> are satisfied by all the elementary functions - polynomials, exponentials, trigonometric functions etc., except at certain special points called <u>Singularities</u>. For example, $w = \frac{1}{z}$ satisfies these conditions at every point except $\underline{z} = 0$, where $\underline{w}$ itself is infinite.

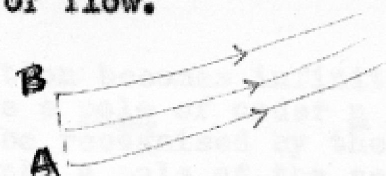
Functions may be:	SINGLE VALUED	as	$w = z^3$
	MANY VALUED	as	$w = \sqrt{z}$
INFINITELY	MANY VALUED	as	$w = \sin^{-1} z$

<u>Except where otherwise stated, functions will be assumed to be single valued.</u>

Physical Interpretation.

The equations (II) are immediately recognised by the physicist. They occur in the theory of gravitation, heat conduction, electrostatics, magnetism, electric currents and hydrodynamics., subjects all of which may be visualied as dealing with the ~~flow~~ flow of some fluid.

v is known as the stream function. The curves v = constant are the lines of flow.



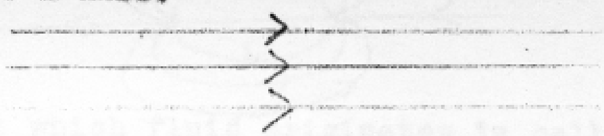
The difference in v between the points A and B is equal to the fluid crossing the line AB in one second.

u is called the Velocity Potential. (For the flow of heat u equals the temperature with a minus sign.) The curves u = constant cross v = constant at right angles. The velocity of the current downstream is equal to the rate of change of u.

Thus to any ~~analytic~~ analytic function there corresponds a way in which fluid can flow. But the converse is also true. To every such way in which ~~the~~ fluid can flow, there corresponds an an alytic function. (X See note)

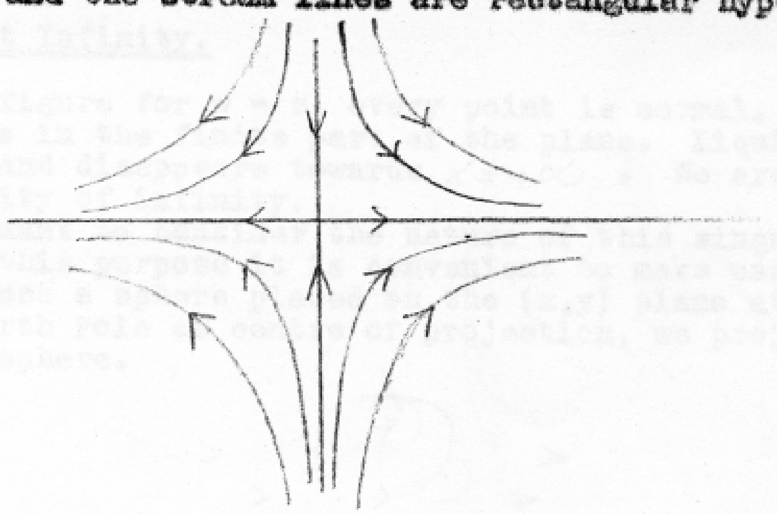
The simplest example of this is given by the function z itself. If $w = z$, we have to consider the real and imaginary parts of $x + iy$. The lines of flow are y = constant. Thus we are dealing with fluid which flows with uniform speed parallel to the x axis.

$w = z$



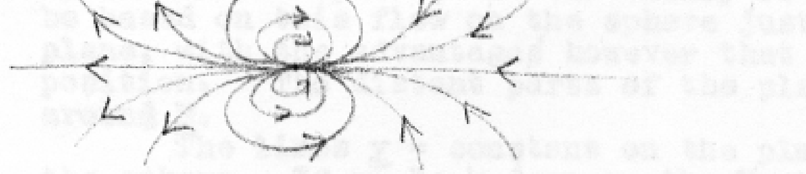
If we consider $w = z^2$, we have $u + iv = (x+iy)^2 = x^2 - y^2 + i2xy$

Thus $v = 2xy$ and the stream lines are rectangular hyperbolas,

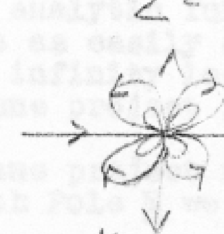


(X. The flow is tacitly assumed to be ~~irrational)~~ irrotational)

$W = \frac{1}{Z} = \frac{1}{X+iY} = \frac{X-iY}{X^2+Y^2}$ gives us the figure



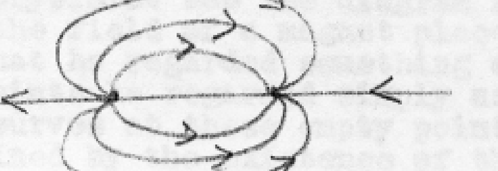
$W = \frac{1}{Z^2}$ gives



If a function becomes infinite like $\frac{K}{(z-a)^n}$ at a point a it is said to have a <u>pole</u> of order n at a. A pole of the first order may always be recognised by the "two petalled flower" which occurs in its graph: a pole of the second order always gives us a four petalled figure, as in $\frac{1}{z^2}$: and generally a pole of order n gives $2n$ petals.

$e^{\frac{1}{z}}$ gives a much more complicated singularity at $z = o$, having ∞ petals there. Any singularity of this type is called an <u>essential singularity.</u>

Any physicist looking at the figure for $\frac{1}{z}$ would immediately say "I know what this is. It is a picture of the lines of force of a magnet". A magnet corresponds to the flow of fluid in which is produced at one point and made to dissapear from a second point very close to the first.



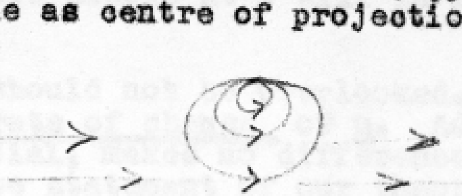
A point at which fluid originates is called a <u>source;</u> a point where it disappears is called a <u>sink</u>. Any point at which sources, or sinks or both exist corresponds to a singularity of the analytic function, and vice versa.

<u>Singularities at Infinity.</u>

In the figure for $w = z$, every point is normal. There are no sources or sinks in the finite part of the plane. Liquid appears from $X = -\infty$ and disappears towards $X = +\infty$. We are dealing with a singularity of infinity.

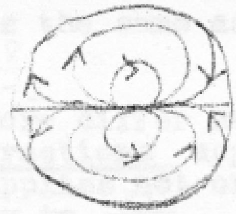
We now want to consider the nature of this singularity at infinity. For this purpose it is convenient to make use of a projection.

We suppose a sphere placed on the (x,y) plane at the origin. and with the North Pole as centre of projection, we project from the plane into the sphere.



In this way, to the flow of fluid on the plane there corresponds a flow on the sphere, and the theory of analytic functions can be based on this flow on the sphere just as easily as for the plane, with the advantage however that infinity loses its special position. The distant parts of the plane project into the region around N.

The lines y = constant on the plane project into circles on the sphere. If we look down on the North Pole N we shall see this figure.



So z behaves near N exactly like $\frac{1}{z}$ near 0.
In the same way we may verify that z^2 behaves near N like $\frac{1}{z^2}$ near 0.

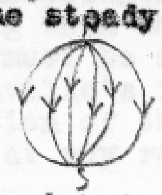
We say z has a simple pole at infinity.
z^2 has a pole of second order at infinity.

Classification of Functions by their Singularities.

When our physicist saw the diagram for $w = \frac{1}{z}$ he described it as "the field of a magnet placed at the origin." You will notice that he regarded something as existing at the singularit all the regular points he regarded simply as empty space. The behaviour of the curves at these empty points he regarded as completely determined by the existence of the magnet at the origin.

This suggests a corresponding theorem for pure mathematics: the Behaviour of an analytic function is completely determined by its behaviour near its singularities.

If, for example, we were told that current flowed in at the North Pole of our sphere, and out again at the South Pole, we should not doubt that the steady flow would be down the meridians.



If now we allow the sink to move up from the South Pole, until it is close to N, we shall still have a determinate problem - though one not so simple to solve, and the flow corresponding to this dipole at N will in fact reproduce the diagram for $w = z$.

One point should not be overlooked. The velocity of flow is determined by the rate of change of u. Adding a constant to u the velocity potential, makes no difference to the way the fluid flows. The precise statement of our theorem is thus:

If two functions have the same behaviour at their singularities they are identical, except perhaps for a constant.

An illustration of this - the function $\frac{z^3}{(z-1)(z-2)}$ behaves

like z at ∞

like $-\frac{1}{z-1}$ at $z=1$

like $\frac{8}{z-2}$ at $z=2$

its singularities are thus the same as those of

$$z - \frac{1}{z-1} + \frac{8}{z-2}$$

from which it can, therefore differ only by a constant. <u>Thus Resolution into Partial Fractions</u> appears quite naturally.

But this method applies not only to polynomials. We can, for instance apply to $\frac{\pi^2}{\sin^2 \pi z}$. This function becomes infinite at z = o where it behaves like $\frac{1}{z^2}$ and it behaves similarly at $z = \pm 1, \pm 2, \pm 3, \ldots$

There is an essential singularity at infinity. In fact,

$$\frac{\pi^2}{\sin^2 \pi z} = \frac{1}{z^2} + \frac{1}{(z-1)^2} + \frac{1}{(z-2)^2} + \cdots + \frac{1}{(z+1)^2} + \frac{1}{(z+2)^2} + \cdots$$

a result which may be regarded as an extensionsof Partial Fractions.

The only difficulty that can arise in such cases is that the series may diverge. This difficulty can always be overcome by a particular device of which the details can be found in any text-book under the heading <u>Mittag-Leffler Theorem</u>.

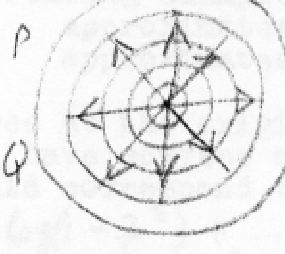
The simplest example of the specification of a function is <u>Lionville's Theorem</u> which states that any single valued function which has no singularities at all is a constant. This a very reasonable result. A stream can only start at a source: but every source is a singularity, so there can be no beginning and no end of any stream.

A stream cannot return into itself, as in a whirlpool, if the potential is single valued. Only one possibility remains - the fluid is at rest, and the potential is constant over the whole sphere. It is not surprising then to be told that if a function is free from singularities, it must be a constant. It is a pity that many text-books present Lionville's Theorem simply as an analytical theorem, without any indication of this graphical approach which, if not a rigorous proof, is at any rate an excellent aid to memory.

<u>Logarithmic Singularities</u>

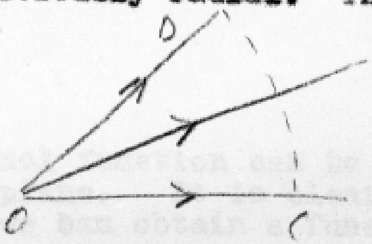
In all the examples we have so far considered, the singularities have involved sources and sinks very close together, like the North and South poles of a magnet. It is natural to ask what function arises when the two"magnetic" poles are separated, and we have a source or a sink on its own. Let us consider for example, what happens if fluid is generated at unit rate at the origin, and flows out uniformly in all directions.

First, it is clear that this flow cannot correspond to a single valued function.



We have seen that the change in the stream function, v between any two points, measures the rate at which fluid crosses the ~~lin~~ linejoining them. If we consider a path APQB which goes round the origin, we see $v(B) - v(A)$ = fluid crossing APQB in a second = I. So that we return to the point A with v increased by unity above its original value there. Each time we go round the origin, v will increase by I. v, and hence ~~u~~ +iv also, is therefore infinitely many valued.

The flow of the fluid is obviously radial. The fluid crossing CD in each second is $\theta/2\pi$, where $\theta = \angle DOC$. We may thus take $v = \theta/2\pi$ as the stream function. (It will be noted that this expression increases by1 for each journey round the origin).



Since unit quantity of fluid emerges from O each second, and the fluid is incompressible, unit quantity must cross any circle with centre O each second. The radial velocity at distance r must accordingly be $\frac{1}{2\pi r}$. u being the velocity potential, $\frac{du}{dr} = \frac{1}{2\pi r}$

Accordingly $u = \frac{1}{2\pi} \log r$

Thus our function $u + iv$ is $\frac{1}{2\pi} \log r + \frac{i\theta}{2\pi} = \frac{1}{2\pi}\{\log r + i\theta\} = \frac{1}{2\pi} \log z$
The function log z thus corresponds to a source of strength 2π at the origin, with an equal sink at infinity.

This procedure shows how closely allied log z is to z^{-n}. Students first become aware of this relationship between z^{-n} and log z when they are learning to integrate. $\int z^{-n} dz$ is usually a power of z, but is a logarithmic function when n = 1.

Although log z is many valued, the flow of fluid to which it corresponds is quite definite, and there is no ~~xxxx~~ difficulty in determining logarithmic functions from their behaviour near singularities.

Thus log $(x^3 - x)$ has a singularity - a source of strength 2π at every point where $x^3 - x = 0$, i.e. at -1, 0, +1.
The same is true of log(z+1) + log z + log(z -1).
These functions can only differ by a constant, and are in fact equal. Taking antilogarithms, we obtain $x^3 - x = (x+1)x(x-1)$

The fact that any polynomial can be resolved into factors corresponds to the fact that the logarithm of any polynomail represents the flow of fluid from a number of separate sources.

A more interesting example is $\log \sin \pi z$
Near $z = 0$ $\sin \pi z$ approximates to πz
log $\sin \pi z$ approximates to $\log \pi + \log z$

We thus have a source at the origin. The periodicity of $\sin \pi z$ shows that we shall have sources of equal strength at $\pm 1, \pm 2, \pm 3,$ but such a flow would correspond to the function

$$\log z + \log(1 - z^2) + \log\left(1 - \frac{z^2}{4}\right) + \dots \log\left(1 - \frac{z^2}{n^2}\right) \dots\dots$$
$$= \log z(1 - z^2)\left(1 - \frac{z^2}{1^2}\right)\left(1 - \frac{z^2}{2^2}\right)\left(1 - \frac{z^2}{3^2}\right)\dots\dots$$

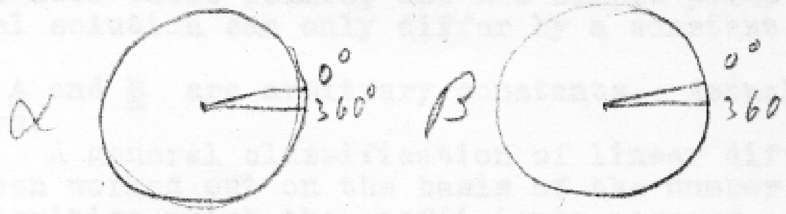
These two functions in fact differ only by a constant, Accordingly $\sin \pi z = \pi z\left(1 - \frac{z^2}{1^2}\right)\left(1 - \frac{z^2}{2^2}\right)\left(1 - \frac{z^2}{3^2}\right)$

This well known representation of $\sin \pi z$ as an infinite product thus appears as a natural extension of the resolution of a polynomial into its factors.
The same procedure, with certain refinements not now discussed can be applied quite generally.

Algebraic Functions.

We hace seen that any rational function can be obtained by considering the flow of fluid on a plane. It is clear that some extension will be necessary before we ban obtain a function such as $\sqrt{z}$, which is double-valued.

We first construct a special surface, as follows. We take two planes, α and β.



and make a straight cut in each. We now place α on top of β and connect- say with fine wires- the points at 0° on α to those at 360° on β, and those at 360° on α to those at 0° on β
Fluid can now flow on α from β around to 360° then down to β, around β from 0° to 360°, then up again to 0° on α
where the fluid originally started.
This surface is known as the Riemann surface of $\sqrt{z}$. If we put sources and sinks in any way on this surface, the resulting flow will always represent some function of $\sqrt{z}$. The values of u and v on the upper surface will give one value of this function

those on the lower surface, the other.

The crossover from the upper to the lower surface occurs along a line joining 0 to ∞. 0 and ∞ are called Branch Points of the function $\sqrt{z}$.

In the same way, a representation can be found for any algebraic function, by considering the flow of fluid on a suitably constructed surface.

Differential Equations.

An important application of the theory is to linear differential equations. This application depends upon theorems such as:

If $w'' = pw' + qw$, the solutions w have singularities only at the points where p or q have singularities.

That is to say, the singularities of the solution w can be detected by inspection of the coefficients in the differential equations p, q. The behaviour of w near these singularities is easy to determine. But once we know how w behaves near its singularities, we know all about w as we have seen earlier.

For example, consider the equation

$$z(z-1)w'' + (4z-2)w' + 2w = 0$$

If we divide by $z(z-1)$ so that the coefficient of w'' becomes unity, the resulting coefficients of w' and w are infinite only for $z = 0$ and $z = 1$.

It is in general also necessary to consider the point $z = \infty$. In this particular case ∞ is not a singularity. w can accordingly have singularities only at $z = 0$ and $z = 1$. By means of approximations it is easily shown that w is single valued near these points, and has simple poles there. So the general solution can only differ by a constant from $w = \frac{A}{z} + \frac{B}{z-1}$, where A and B are arbitrary constants. Actually the constant is zero.

A general classification of linear differential equations has been worked out on the basis of the number and type of singularities which the coefficients possess. An important example is the Hypergeometric Differential Equation, which has 3 singularities of a simple type.

References.

The literature in English gives the results of the theory of functions, but too often, in my experience omits all reference to the underlying ideas.

Klein's "Lectures on the Development of Mathematics in the XIX Century" gives, in Chapter 6, an account of how Riemann was led by his interest in physics to his basic ideas. These ideas are also explained in detail in a paper on "Algebraic Functions and ~~xxx~~ their Integrals" in Volume III of Klein's Collected Works.
An excellent explanation of Riemann's methods is given in the self-contained Section 3 of Hurwitz-Courant "Funktionentheorie".
More advanced but well worth study is Klein's "Lectures on the Hypergeometric Function". These works are all in German. Some effort should be made to popularise these ideas in the English language.