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"Q. E. D."

A Series on
the
Teaching of Mathematics

6. ABSTRACT AND CONCRETE

by
Professor W.W. Sawyer

of

Wesleyan University
Connecticut

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ANNOUNCEMENTS FOR "Q.E.D." 6. 'ABSTRACT AND CONCRETE'

BY PROFESSOR W.W. SAWYER

OPENING:

Q.E.D.

We're broadcasting now the sixth programme in this series on the teaching of mathematics. "Abstract and Concrete". W.W. Sawyer, Professor of Mathematics at Wesleyan University, Connecticut, paid a short visit to this country last summer. He recorded for us then, a talk in which he reflected on these two aspects of mathematics, and on the way they come together in the teaching process.

"ABSTRACT AND CONCRETE"

CLOSING:

That recorded talk, "Abstract and Concrete", was given by W.W. Sawyer, Professor of Mathematics at Wesleyan University, Connecticut.

The next programme in this series will be broadcast a week today, when Kenneth Austwick will be talking about the use of "teaching machines" in the teaching of mathematics.

The phrase "the teaching of mathematics" can be read in two ways: you can put the accent on "teaching" or you can put it on "mathematics". So far, the emphasis in this series has been on teaching, and we've stressed the importance of <u>intuition</u> for effective communication. A mathematician might object, "Yes, you're teaching something, but is that something mathematics?" And he'd go on to stress the essentially abstract nature of mathematics.

Now, there undoubtedly are these two elements in mathematics, intuition and abstraction, and they're in some sense opposites. A teacher's continually trying to bring the subject down to earth, to make it intuitive. A mathematician's continually trying to purify the subject, to make it abstract. A good mathematician feels that teachers continually distort the subject, and a good teacher feels that mathematicians continually obscure the subject.

To teach mathematics successfully, you have to be aware both of the intuitive and the abstract aspects of mathematics, and somehow to unify these.

Why do we have to emphasise abstraction? The answer's simple. If we didn't emphasise abstraction people would think that mathematics dealt with actual objects in much the same way that physics does. But, in fact, mathematical questions, as a rule, can't be settled by direct appeal to experiment. To use a hackneyed example, Euclid's lines are supposed to have no width, and his points no size. No such objects can be found in the physical world. Euclid's geometry describes an imaginary world which resembles the actual world sufficiently for it to be a useful study for surveyors, carpenters and engineers.

A revolution in physics might change our ideas about the actual world. But it couldn't change our ideas about Euclid's geometry any more than biological research into the habits of wolves could change the story of Little Red Riding Hood.

One of the most profound and controversial abstractions of mathematics may meet us almost in the nursery. We've a book entitled, say, "Nursery Rhymes". On the cover is a picture of a boy happily reading a book called "Nursery Rhymes", on the cover of which there's, of course, a smaller boy reading a book called "Nursery Rhymes" .... and so on. Children are often intrigued by this, and see that the picture involves an unending sequence of smaller and smaller boys reading smaller and smaller books. Now, of course, that unending sequence doesn't exist in the physical world. Even if the artists is conscientious enough to use a microscope, he has to stop drawing some time. In the last resort, he would be halted by the atomic structure of matter: he can hardly draw a book smaller than a hydrogen atom. The unending sequence exists only in the imagination.

Infinity is something that we can never experience, and yet it's a central concept of mathematics. Our whole thinking is based on the assumption that there are infinitely many numbers, so that counting need never stop; that there are infinitely many fractions between 0 and 1, that there are infinitely many points on the circumference of a circle. Our imagination leads us naturally to the idea of infinity, but we have no guarantee that it's justified in so doing. Mathematicians aren't agreed as to the logical standing of infinity. In some centuries, mathematicians have taken all kinds of liberties with infinity.

In others they've burned their fingers with paradoxes, and have recoiled somewhat.

In the present century, many fruitful and interesting results have been obtained by mathematicians who handle infinity in an extremely bold and daring manner. But the advance is on the basis of faith and hope, not certainty. Some time in the future we may again run into paradoxes and have to retreat, saving as many theorems as we can.

Infinity in physics is as uncertain as infinity in mathematics. We don't know how infinity is related to time. No-one has yet resolved Kan't paradox about time past. Is time past to be thought of as finite? If so, what was happening before the beginning of time? Is time past infinite? If so, an eternity has already passed; how has time managed to reach us?

Nor do we understand how infinity is related to space. Is the universe finite or infinite? We don't know. Are there an infinite number of stars? We don't know. We don't even know whether the question makes sense or not. If infinity is a physical fact .. in the sense that the universe extends indefinitely far and ~~contains indefinitely far and~~ contains infinitely many stars ... we still have no way of knowing that it is so, for we can only observe and count a finite number of stars.

Infinity, then, is not a concept corresponding to any object that we even have seen or are ever likely to see. It is an abstract concept. It is a concept suggested to our mind as the continuation of a certain pattern ... the pattern of the boys ob the book cover, or the patter of counting 1,2,3,4 .. and so on.

In mathematics we're often interested in continuing a pattern. Edwin Abbott in his famous book "Flatland" discussed a world of two dimensions only. Its people are like pieces of cardboard, free to slide about on the floor, but unable either to rise from the floor or even to conceive the possibility of so doing. After showing how this world appears to us in our space of three dimensions, Abbott argues by analogy; perhaps there could be a world of four dimensions that would stand in the same relation to us as we do to Flatland. His book merely suggests by analogy that four dimensions could exist. But we can do more than this. We can develop the theory of four dimensional space by strict reasoning. This illustrates well the importance of separating mathematical and physical concepts, abstract and intuitive. Someone who thought purely in physical terms might argue that a fourth dimension was quite impossible. The fourth dimension would have to be in a direction perpendicular to all to all the directions we know. How could such a thing exist? We gladly concede to such an opponent that the space we live in has only three dimensions. All we maintain is that it doesn't <u>have</u> to have three; no logical contradiction would arise if, like Flatland, it had only two; no logical contradiction would arise if it had four. We can produce a blueprint to show what a universe with four spatial dimensions would be like. For that matter, we could produce a flight simulator, that would teach us how to navigate an aeroplane in space of four dimensions.

Again, soneone whose thinking was excessively physical might refuse to believe in nagative numbers on the ground that you can't have a quantity less than nothing. Still more, such a person would refuse to believe in the square root of minus one.

In mathematical research there are times when you have to practise what theatregoers call "the suspension of disbelief". You say to yourself, "I know that space does not have a fourth dimension, but it looks interesting to investigate what would happen if it did."

This stage, where a mathematician seems to be talking nonsense, doesn't usually last very long. Sooner or later, someone comes along and shows that, by a shift in the point of view, you can give a perfectly logical account of the new development. To justify the idea of the fourth dimension, we show that we could build a calculating machine that'd answer any question you liked to ask about space of four dimensions. This machine would never contradict itself. It would tell a consistent story. This story may contain patterns which the scientist or engineer can use to understand some part of the actual world, But the pure mathematician isn't interested in that. The story told by the calculating machine specifies what he calls a mathematical structure, and he's interested in that for itself. Almost certainly, he won't think of four dimensional space as something that might have been the geometry of the physical universe. That kind of thought only exists in popular talks about mathematics! It's little used by the research worker in pure mathematics.

You'll notice the <u>change</u> as we move from the concrete, intuitive aspect of mathematics to the abstract, logical aspect. We first meet geometry as dealing with the shapes of actual objects, things we can see and touch. At this stage, it contains much that's vague and unclear, for it's not easy to analyse our perception of actual things.

To compensate for this, the material is thoroughly familiar. We know how objects behave; we're able to recognise, by a single act of mental vision, the reasonableness of certain statements. The last stage is quite different. The vagueness has gone. We now have a calculating machine, strictly following a prescribed procedure. We know that the machine will never contradict itself. The machine tells a story. If it's been constructed to tell the story of two dimensions, we find that its statements agree with our intuitive perception of drawings on paper. If it's constructed to tell the story of three dimensions, we notice a similar agreement with our experience of solid objects. If it is constructed to tell the story of four or five or six dimensions, we are confronted with entirely novel structures.

A mathematician, as I've said, is mainly interested in the structure itself. He's not interested in where it came from, or in what it can be used for. This is one of the penalties of specialisation. You can often meet a book or a university lecture that is admirable for the specialist and terrible for everybody else. Mathematically the presentation is excellent. It begins with certain axioms clearly stated and develops their consequences logically. The students follow every step, and yet they're bewildered. To use my earlier image of the calculating machine, it's as if the students had been told how to construct a machine, and shown what each part did to every other part, but they didn't know what the purpose of the machine was, or how anybody came to think of it.

Even for the specialist, such a presentation is poor. What a mathematician most wants to know about any piece of work is the idea that led ot it. Once this idea is grasped, the details fall into place. In fact, once you've got the central idea, you can usually work out the details for yourself.

The same kind of bad teaching is sometimes found at the other end of the scale, in the teaching of young children. I mean the mistake of rote learning, of trying to turn children, as quickly as possible, into little calculating machines; of teaching them rules without showing them how they could arrive at these rules for themselves. This is the sort of thing Miss Biggs wages war on ... and quite rightly, for the result is that children almost invariably use the wrong rule. Some time ago, I was told of a girl who had to measure out some dangerous drug. The amount required was one-sixth of some unit. But she didn't have a measure for one-sixth. However, she had measures for one half and one quarter. She thought, "six plus two is four. So I'll measure out one-half and one-quarter and the sum of these should be one-sixth". The patient, I believe, survived. He was lucky. I didn't have this story at first hand, but I see no reason for disbelieving it. Thousands of students make this kind of mistake in exams.

The remarkable thing is that this girl would probably have understood more about arithmetic if she'd never learnt it at all. It's clear to a fairly young child that he'll get less cake if the cake is shared out among four children than if it's shared between two; he'll get even less if it's shared among six. That's to say, he knows that one sixth of a cake is less than a quarter, much less than one half, and that it would be entirely ridiculous to think that you got one-sixth by putting together one half and one quarter.

He'd have no doubt which to choose, if he were offered one-sixth of a cake, or three-quarters.

The depressing thing about arithmetic, badly taught, is that it destroys a child's intellect, and to some extent, his integrity. Before they're taught arithmetic, children won't give their assent to utter nonsense: afterwards, they will. Instead of looking at things and thinking about them, they make wild guesses in the hope of pleasing a teacher or an examiner.

In the past two or three years, I've been doing some experimental teaching with American boys and girls. These were intelligent children, quite bright enough to get into grammar schools if they lived here. They'd been taught arithmetic by rote, and it had nearly destroyed their power of looking at evidence. Their picture of an arithmetic lesson was something like this: the teacher asks a question. This question should remind you of some rule. You apply the rule and get an answer. You'll notice that this prodedure is entirely concerned with words; the question is words, the rule is words, the answer is words. Vision doesn't come into it at any stage.

My own idea of a mathematics lesson is rather different. The teacher asks a question. The pupils try to think what the question means. They perhaps draw a picture of the situation involved. They think about this situation. They arrive at a solution. Finally, this solution is put into words, as the answer.

I found it very hard work indeed to get my experimental class to do this. Here's just one example. At the beginning of algebra you want pupils to recognise that 3x added to 4x gives you 7x.

Now this is not at all an unreasonable doctrine. <u>x</u> stands for any number you like to think of. Suppose you think of a dozen. Then my statement implies that three dozen added to four dozen gives seven dozen, which strikes me as very reasonable. But my pupils wouldn't have it. They wrote on the blackboard three times twelve plus four times twelve, and they argued, "You've added the three to the four, so you ought to add the twelve to the twelve as well." They were thinking about figures written on paper, and playing some mysterious game with them. They weren't thinking about the meaning of the figures at all.

There's a great contrast when children use a system, such as Cuisenaire or Stem's, in which they learn to associate numbers with the lengths of coloured rods. Recently I visited a school where quite young children were doing this:

The teacher said, "Show what you know about six", and put a six inch rod on the table. A boy said immediately, "That's easy, two threes", and picked out two three inch bars and matched them against the length of the six. Another child put out six one inch bars to form a total length of six inches. I asked a girl, "How many twos do you think there would be in six?" She answered, "About three", and placed three two inch bars to show that she was right. I liked that word "about" in her answer. I don't think **she** had any doubt that there were three twos in six. It expressed rather that she was making a judgement; she felt that three of the two inch blocks would make up the six inches; no doubt she remembered having made this experiment many times before. She was giving the answer, "Three twos are six", as something she had thought out for herself, not as something learnt parrotwise.

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This, I am sure, is the essential thing in the teaching of mathematics. A mathematician, above all, is a person who thinks for himself. However far you may go in mathematics, you will always find that you're weighing evidence. Mathematical education should not be a succession of arbitrary rules. Rather, the pupil's attention should be drawn to certain evidence, and he should be invited to think about it. If he is to think about it, it must be in a form he can appreciate, it must reach his imagination, it must be intuitive. We needn't be worried by the argument that "mathematics is abstract". In the early teaching of mathematics, there is no danger of making the subject too concrete. The danger is rather that the subject gets so far from the concrete that it comes to mean nothing at all.

The beauty of the approach through evidence is that it is never final. If the pupil has learnt to appreciate evidence he is always ready to make a further step forward. The teacher must produce evidence showing the necessity for such a step. This can be done, for example, when a student needs to pass from elementary mathematics to more abstract and sophisticated branches. The student's knowledge of concrete, elementary mathematics will be a help when this time comes. There is no paradox here. After all, humanity's belief in the value of abstractions was reached as the result of concrete experience. Abstract and concrete are not simply opposites. Like mind and body, they're Siamese twins.